Periodic quantum walks

The state of a quantum system is given by a density matrix, a positive semidefinite matrix with trace one. (For example, if $\|x\|=1$, take xx^*.)

The evolution of the continuous walk on a graph X is described by the unitary matrix

$$U(t) = \exp(itA(X)), \qquad (t \geq 0)$$

If the initial state of the walk is D, then $D(t)$, the state at time t, is $U(t)DU(-t)$.

If we have the spectral decomposition

$$A = \sum_r \theta_r E_r$$

then

$$U(t) = \sum_r e^{it\theta_r} E_r$$

and

$$D(t) = \sum_{r,s} e^{it(\theta_r - \theta_s)} E_r D E_s$$

Note: $D = \sum_{r,s} E_r D E_s$. The eigenvalue support of D is the set

$$\{(\theta_r, \theta_s) : E_r D E_s \neq 0\}.$$

If $E_r D E_s = 0$, then $E_s D E_r = 0$. Also, if $E_r D E_s \neq 0$ then $E_r D^{1/2} D^{1/2} E_s \neq 0$. So $E_r D E_r = E_r D^{1/2} D^{1/2} E_r \neq 0$.

The eigenvalue graph relative to D has the eigenvalues θ_r such that $E_r D E_r \neq 0$ as vertices. Its edges are the pairs in the eigenvalue support of D.

A density matrix is a pure state if its rank is one.

Lemma If D is a pure state, its eigenvalue graph has just one connected component, a clique.

Proof If $D = zz^*$ then $E_r D E_s = E_r z (E_s z)^*$ and so $E_r D E_s \neq 0$ if & only if both $E_r z$ & $E_s z$ are not zero.

Lemma Let Γ be the Galois group of the splitting field of $\phi(X,t)$. If D is rational, Γ is isomorphic to a subgroup of the automorphism group of the eigenvalue graph of D. $\square$

[8-12-21]

The ratio condition

A state D is periodic if for some $t \neq 0$, $D(t) = D$.

As $D(t) = U(t) D U(-t)$ we see that D is periodic if & only if D and $U(t)$ commute.

Theorem Suppose D is a periodic state for the walk on X. If D is real and

$$(\theta_r, \theta_s), \quad (\theta_k, \theta_\ell)$$

lie in the eigenvalue support of D and $k \neq \ell$,

$$\frac{\theta_r - \theta_s}{\theta_k - \theta_\ell} \in \mathbb{Q}$$

Ratio Condition

Proof We have $D = D(t) = \sum_{r,s} e^{it(\theta_r-\theta_s)} E_r D E_s$

By hypothesis, D is a real matrix. Further, the idempotents E_r are real. As

$$\begin{aligned} E_k D E_l &= \sum_{r,s} e^{it(\theta_r-\theta_s)} E_k E_r D E_s E_l \\ &= e^{it(\theta_k-\theta_l)} E_k D E_l \end{aligned}$$

$(\theta_k, \theta_l) \in$ e/val support

and therefore $e^{it(\theta_k-\theta_l)}$ is real. Hence $e^{it(\theta_k-\theta_l)} = \pm 1$.

So $e^{2it(\theta_k-\theta_l)} = 1$ and $t(\theta_k-\theta_l) = m_{k,l}\pi$.

The theorem follows.

There is a second case where the ratio condition holds.

Theorem If D_0 is algebraic and $D_1 = D_0(t)$ is algebraic, the ratio condition holds.

Proof

$$D_1 = \sum_{r,s} e^{it(\theta_r - \theta_s)} E_r D E_s$$

Since the eigenvalues of X are algebraic, the spectral idempotents are algebraic. Therefore $e^{it(\theta_r - \theta_s)}$ is algebraic.

Hence

$$\left(e^{ib(\theta_k-\theta_l)}\right)^{\frac{\theta_r-\theta_s}{\theta_k-\theta_l}} = e^{ib(\theta_r-\theta_s)}$$

and so $\alpha^{\beta} = \gamma$ with α, β, γ algebraic

Gelfond-Schneider: If $\alpha \neq 0, 1$ and β is algebraic and α^{β} is algebraic, β is rational

Using the ratio condition

We have $D_1 = \sum_{r,s} e^{it(\theta_r - \theta_s)} E_r D_0 E_s$. Here the idempotents E_r are algebraic. If D_0 is rational and $\gamma \in \Gamma$, then

$$(E_r D_0 E_s)^\gamma = E_r^\gamma D_0 E_s^\gamma$$

and E_r^γ, E_s^γ are spectral idempotents of X.
Therefore the eigenvalue support of D_0 is invariant under the Galois group of the splitting field of $\phi(X,t)$

The eigenvalue support of the vertex a in X is the set of poles of

$$\frac{\phi(X\setminus a,t)}{\phi(X,t)};$$

equivalently, it is the set of zeros of

$$\phi(X,t)/\gcd(\phi(X,t),\phi(X\setminus a,t))$$

Theorem Suppose the continuous walk on X with initial state D is periodic. If D is rational, then all eigenvalues in the eigenvalue support of D lie in a quadratic extension of the rationals.

Proof Assume (θ_k, θ_ℓ) lies in the eigenvalue support S of D & $\theta_k \neq \theta_\ell$. If $(\theta_r, \theta_s) \in S$, then

$$\frac{\theta_r - \theta_s}{\theta_k - \theta_\ell} \in \mathbb{Q} \qquad (*)$$

We prove that $(\theta_k - \theta_\ell)^{|S|} \in \mathbb{Z}$.

From (*):

$$\prod_{(\theta_r,\theta_s)} \frac{\theta_r-\theta_s}{\theta_k-\theta_\ell} = \alpha \in \mathbb{Q}$$

The product $\prod_{(\theta_r,\theta_s)} (\theta_r-\theta_s)$ is invariant under Γ, therefore it is an integer. Therefore $\alpha(\theta_k-\theta_\ell)^{|S|} \in \mathbb{Z}$ and $(\theta_k-\theta_\ell)^{|S|} \in \mathbb{Q}$.

But $(\theta_k - \theta_\ell)^{|S|}$ is an algebraic integer and a rational number. Therefore it is an integer.

$\psi(t) = t^d + a_1 t^{d-1} + \dots + a_d$, $\theta = p/q \rightarrow p^d + q(a_1 p^{d-1} + \dots + a_d q^{d-1})$
min poly of θ

Set $n = |S|$. Then $\theta_k - \theta_\ell$ is an n-th root of an integer. Since $\theta_k - \theta_\ell \in \mathbb{R}$, it can only be a square root. If $\theta_k - \theta_\ell = \sqrt{\Delta}$, then $\theta_r - \theta_s \in \mathbb{Q}(\sqrt{\Delta})$.

So $\theta_r - \theta_0 = m_r \sqrt{\Delta}$ $(r = 1, \dots)$ and since $\sum_r \theta_r \in \mathbb{Z}$ it follows that $\theta_0 \in \mathbb{Q}(\sqrt{\Delta})$. $\square$

Corollary If ϑ_r, ϑ_s lie in the eigenvalue support of D and $\vartheta_r \neq \vartheta_s$, then $|\vartheta_r - \vartheta_s| \geq 1$.

Lemma Assume $D = e_a e_a^T$ and let ρ be the covering radius of a. Then the size of the eigenvalue support is at least $\rho + 1$.

Lemma The covering radius of the vertex a is less than the size of the eigenvalue support of a.

Proof (1) the eigenvalue support of a is the set of nonzero vectors $E_r e_a$.

(2) The vectors $E_r e_a$ each lie in $\operatorname{span}\{A^k e_a : k \geq 0\}$, the form a basis for this subspace.

(3) If ρ is the covering radius of a, the vectors $e_a, Ae_a, \ldots, A^{\rho} e_a$ are linearly independent vectors in $\operatorname{span}\{A^k e_a : k \geq 0\}$.

(The sequence of sets $(A+I)^{\ell} e_a$ $(0 \leq \ell \leq \rho)$ is strictly increasing.) $\square$

Theorem The number of connected graphs that contain a periodic vertex & have maximum valency at most k:

Proof

Suppose k is the maximum valency of X. Then all eigenvalues of X lie in the interval $[-k,k]$, whence $|S| \leq 2k+1$ and therefore the covering radius of a is at most $2k$. Hence $|V(X)|$ is bounded by a function of k.

Question Can we replace "maximum valency at most k" by "average valency at most k"?