Cospectral complements, cospectral vertices

Cospectral complements

Is there a relation between $\phi(X,t)$ and $\phi(\bar{X},t)$?

If X is regular, yes.

$$\underset{k}{\theta \in ev(X)}, \ \theta \neq \text{valency} \Rightarrow \underset{n-1-k}{-\theta - 1 \in ev(\bar{X})}$$

In general

$$\det(tI - A(\bar{X})) = \det(tI - J + I + A)$$

$$= \det((t+1)I + A - J)$$

$$= \underbrace{\det((t+1)I + A)}_{(-1)^n \phi(X, -t-1)} \det(I - (\cdots)^{-1} \overset{\mathbf{1}\mathbf{1}^T}{J})$$

$$\det(I - z\mathbf{1}^T) = \det(I - \mathbf{1}^T z)$$

Therefore

$$(-1)^n \frac{\phi(\bar{X}, t)}{\phi(X, -t-1)} = 1 - \mathbf{1}^T((t+1)I + A)^{-1}\mathbf{1}$$

$\sum t^n \mathbf{1}^T A^n \mathbf{1}$

Now

$$\mathbf{1}^T (I - tA)^{-1} \mathbf{1}$$

is the generating function for all walks in X.

Lemma Assume X and Y are similar. Then $\bar{X}$ & $\bar{Y}$ are similar if & only if

$$\mathbf{1}^T A(X)^m \mathbf{1} = \mathbf{1}^T A(Y)^m \mathbf{1}$$

Remark If X is k-regular on n vertices, the number of walks on X with length m is nk^m.

$$\mathbf{1}^T\left((t+1)I + A\right)^{-1}\mathbf{1} = \sum_r \frac{\mathbf{1}^T E_r \mathbf{1}}{t+1+\theta_r}$$

$\mathbf{1}^T E_r \mathbf{1} \neq 0$
$\to$ main evals

Corollary If X & Y are cospectral, then $\bar{X}$ and $\bar{Y}$ are cospectral if and only if

$$\mathbf{1}^T E_r(X)\mathbf{1} = \mathbf{1}^T E_r(Y)\mathbf{1} \qquad \forall r$$

(If X is k-regular, $E_r(X)\mathbf{1} = 0$ unless $\theta_r = k$.)

Cospectral vertices

Vertices u & v in X are **cospectral** if $X \setminus u$ and $X \setminus v$ are cospectral; equivalently if $\phi(X \setminus u, t) = \phi(X \setminus v, t)$.

Examples:

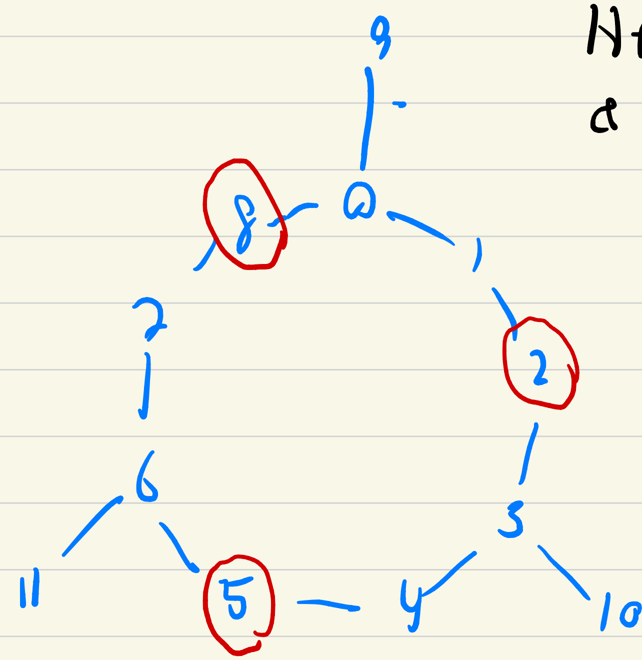
(a) u, v in the same orbit of $Aut(X)$

(b) any two vertices in a strongly regular graph

Construction

Herndon & Ellzey

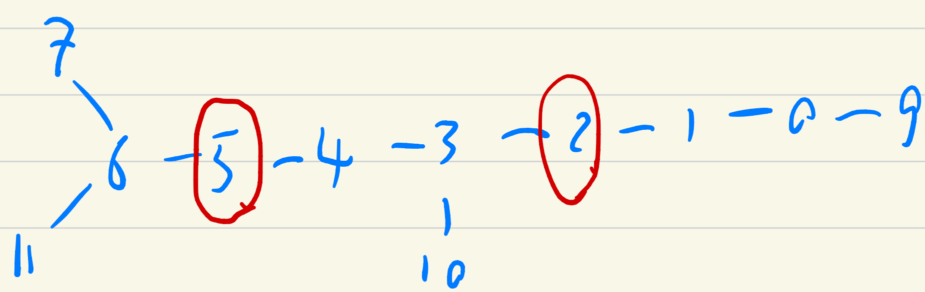
X



$Y \setminus 5 \cong Y \setminus 2$

$\Rightarrow$ 2 & 5 are cospectral in Y

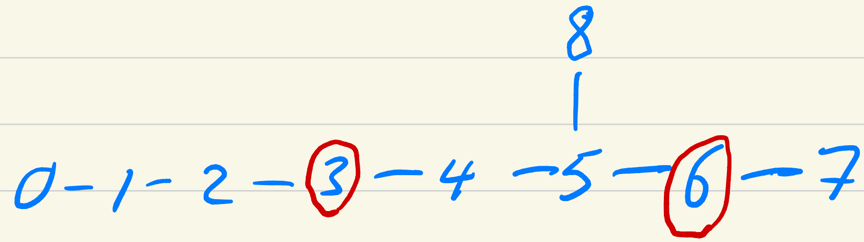
Y



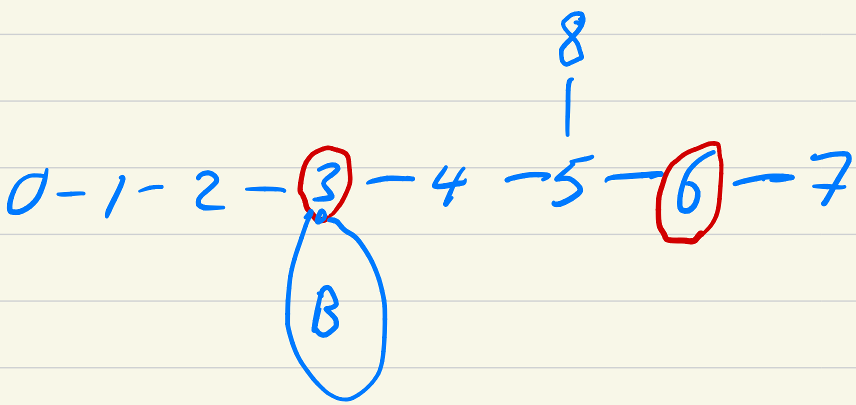
2 & 5 are pseudosimilar

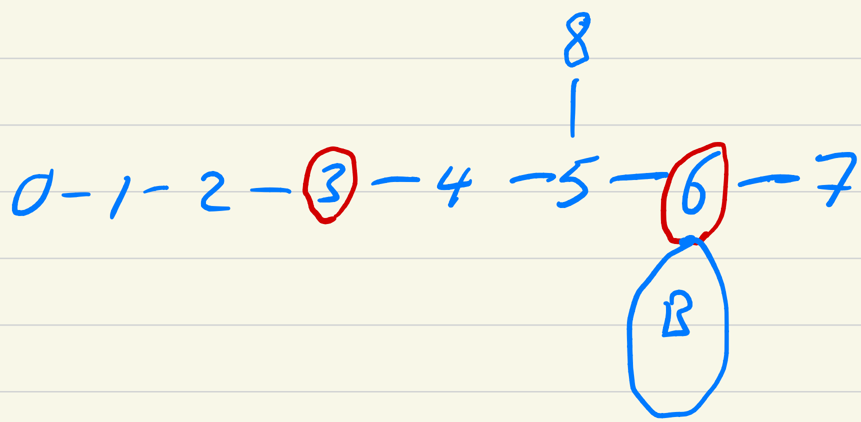
Schwenk's tree

S



$S \setminus 3$ & $S \setminus 6$ are cospectral, as are these trees:





(apply the 1-sum identity)

c) McKay's tree

M_3 M_6 $M \setminus 3$ $M \setminus 6$

$\bar{M}_3$ $\bar{M}_6$ $\overline{M \setminus 3}$ $\overline{M \setminus 6}$

$L(M_3)$ $L(M_6)$

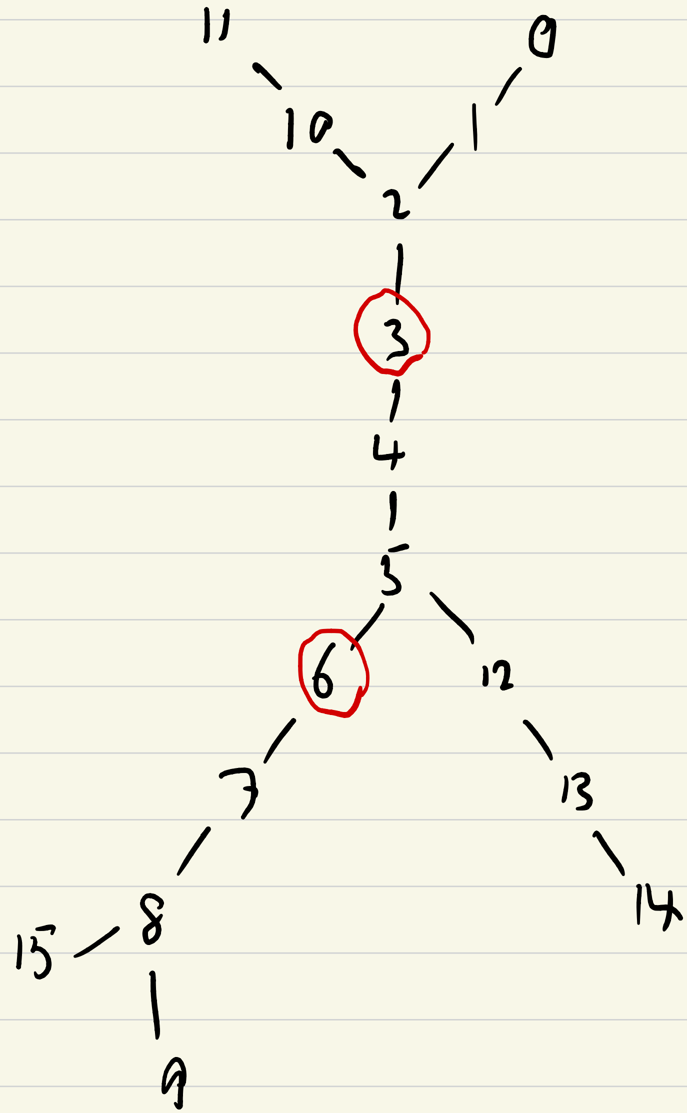
$L(\bar{M}_3)$ $L(\bar{M}_6)$

$\overline{L(M_3)}$ $\overline{L(M_6)}$

$\overline{L(\bar{M}_3)}$ $\overline{L(\bar{M}_6)}$

$D(M_3)$ $D(M_6)$

$D(\bar{M}_3)$ $D(\bar{M}_6)$



2010-21

Characterizing cospectral vertices

NO LECTURE

MONDAY OCTOBER 25

Since

$$((tI-A)^{-1})_{a,a} = \frac{\phi(X\setminus a, t)}{\phi(X,t)} = \sum_r \frac{(E_r)_{a,a}}{t-\theta_r}$$

we have:

Lemma TFAE:

(a) vertices a & b in X are cospectral

(b) $\phi(X\setminus a, t) = \phi(X\setminus b, t)$

(c) $(A^m)_{a,a} = (A^m)_{b,b} \quad \forall m \geq 0$

(d) $(E_r)_{a,a} = (E_r)_{b,b} \quad \forall r$

(e) The modules $\langle e_a - e_b \rangle_A$ and $\langle e_a + e_b \rangle_A$ are orthogonal.

(f) There is an orthogonal matrix Q such that $QA = AQ$, $Q^2 = I$ and $Qe_a = e_b$

Proofs:

(e) $(e_a - e_b)^T A^k (e_a + e_b) = \underbrace{e_a^T A^k e_a - e_b^T A^k e_b}_{=0} + \underbrace{e_a^T A^k e_b - e_b^T A^k e_a}_{=0}$

$= 0$

$A^k A^l$

(f) $U(+) := \langle e_a + e_b \rangle_A$, $U(-) := \langle e_a - e_b \rangle_A$

$$U(0) := (U(+) + U(-))^{\perp}$$

Define L to act as -1 on $U(-)$, as 1 on $U(+)$ and $U(0)$. Then $L^2 = I$, L is orthogonal and

$$\left.\begin{aligned} L(e_a + e_b) &= e_a + e_b \\ L(e_a - e_b) &= e_b - e_a \end{aligned}\right\} \Rightarrow L(2e_a) = 2e_b$$

Extended adjacency algebras

Assume A is $n \times n$. We have been working with $\mathbb{F}[A]$, the adjacency algebra over $\mathbb{F}$.

We now assume $\mathbb{F} = \mathbb{R}$, and we choose a symmetric rank-1 matrix H (of order $n \times n$).

hh^T

$h = e_a$

We refer to $\langle A, H \rangle$ as an extended adjacency algebra.

In general, $\langle A, H \rangle$ is not commutative,

but it is $*$-closed.

Why does this matter?

Lemma If $\mathcal{A}$ is $*$-closed and $\mathcal{U}$ is $\mathcal{A}$-invariant, so is $\mathcal{U}^\perp$.

Proof. If $A \in \mathcal{A}$ and $\mathcal{U}$ is A-invariant, $\mathcal{U}^\perp$ is A^*-invariant. So if $\mathcal{U}$ is $\mathcal{A}$-invariant, so is $\mathcal{U}^\perp$.

Theorem If $\mathcal{A}$ is $*$-closed, then $\mathbb{C}^n$ is an orthogonal direct sum of simple $\mathcal{A}$-modules.

We want the decomposition of $\mathbb{R}^n$ into simple modules for $\mathcal{A} = \langle A, hh^* \rangle$.

Lemma $\langle h \rangle_A$ is a simple $\mathcal{A}$-module.

Proof. Set $U = \langle h \rangle_A$ and suppose U_1 is a proper submodule of U. Then if $u \in U_1$,

$$hh^* u = (h^* u) h$$

So either $h \in U_1$ or $U_1 \leq h^\perp$.

In the first case, $A^k h \in U_1$ for all k & therefore $U_1 = U$. In the second case, $U_1 \leq U^\perp$ and so $U_1 = \langle 0 \rangle$.

Corollary $\langle h \rangle_A$ is the only simple $\mathcal{A}$-module that contains h, any other simple module lies in $h^\perp$ and hence it is the intersection of $h^\perp$ with an eigenspace of A.

So hh^* acts as zero on $(\langle h \rangle_A)^\perp$.

Problem Describe the action of $\langle A, hh^* \rangle$ on $\langle h \rangle_A$.

Assume A has spectral decomposition

$$A = \sum_r \theta_r E_r$$

$E_r h \quad E_s h$

Lemma The non-zero vectors $E_r h$ form an orthogonal basis for $\langle h \rangle_A$.

The set $\{\theta_r : E_r h \neq 0\}$ is the eigenvalue support of h. Its size is equal to $\dim(\langle h \rangle_A)$.

Theorem Let $d = \dim(\langle h \rangle_A)$. The matrices

$$E_r h h^T E_s \qquad (\theta_r, \theta_s \in \operatorname{esupp}(h))$$

generate an algebra of dimension d^2.

so isomorphic to $\mathrm{Mat}_{d \times d}(\mathbb{R})$

Proof. Distinct matrices $E_r h h^T E_s$ are trace-orthogonal, and there are d^2 of them.