Controllable graphs

Lemma $A(X)$ has no eigenvector orthogonal to h if and only if

$$\langle A, hh^* \rangle = \mathrm{Mat}_{n\times n}(\mathbb{C})$$

$e_a e_a^T$ $\qquad$ e_a

The pair (X, h) is **controllable** if $\langle A, hh^T \rangle = \mathrm{Mat}_{n\times n}(\mathbb{R})$.

The graph X is **controllable** if $\langle A, J \rangle = \mathrm{Mat}_{n\times n}(\mathbb{R})$.

Theorem [O'Rourke & Touri] Almost all graphs are controllable. For almost all graphs, all the pairs $(A, e_a e_a^T)$ $(a \in V(X))$ are controllable.

Lemma X is controllable $\Leftrightarrow$ $\phi(\bar{X}, t)$ and $\phi(X, -t-1)$ are coprime. The vertex a is controllable iff $\phi(X, t)$ & $\phi(X \setminus a, t)$ are coprime.

Lemma If X is controllable, $\mathrm{Aut}(X) = \langle 1 \rangle$.

If (X, e_a) is controllable, then $\mathrm{Aut}(X)_a = \langle 1 \rangle$.

Proof. Every permutation matrix commutes with J, and a permutation matrix P commutes with A if & only if $P \in \mathrm{Aut}(X)$.

So if $P \in \mathrm{Aut}(X)$, then $P \in \mathrm{comm}(\langle A, J \rangle)$ and so $P = I$. $\square$

An isomorphism invariant for controllable graphs.

Assume $|V(X)|=n$ and define

$$W_X = [\underset{\sim}{1} \ A\underset{\sim}{1} \cdots A^{n-1}\underset{\sim}{1}]$$ walk matrix

We can order the rows lexicographically, in this case we say W_X is ordered.

(Note that if W_X is invertible, no two rows of W_X are equal.)

Theorem Let X and Y be controllable graphs with ordered walk matrices W_X and W_Y respectively. Then $X \cong Y$ if and only if $\phi(X,t) = \phi(Y,t)$ and $W_X = W_Y$

Proof. We have

$$A_X W_X = W_X C_{\phi(X)}, \quad A_Y W_Y = W_X C_{\phi(Y)}$$

If $W_X = W_Y$, then

$$(A_X - A_Y) W_X = W_X (C_{\phi(X)} - C_{\phi(Y)})$$

If W_X is invertible, $A_X = A_Y$. $\square$

Reminder: if $W = [h\ Ah \cdots A^{n-1}h]$, then

$$AW = [Ah\ A^2h \cdots A^n h]$$

$$= [h\ Ah \cdots A^{n-1}h] \begin{bmatrix} 0 \cdots & 0 & * \\ I_{n-1} & & \vdots \\ & & * \end{bmatrix}$$

companion matrix of $\phi(X,t)$

$$= W\, C_{\phi(X,t)}$$

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Theorem If X & Y are controllable and

$$W(X,t) = W(Y,t),$$

$A^k \underset{\sim}{1}$

then $\phi(X,t) = \phi(Y,t)$.

Proof. Set $A = A(X)$, $B = A(Y)$. Define

$$\hat{W}_X = [\underset{\sim}{1}\ A\underset{\sim}{1} \cdots A^n \underset{\sim}{1}], \quad \hat{W}_Y = [\underset{\sim}{1}, B\underset{\sim}{1} \ldots B^n \underset{\sim}{1}]$$

$n \times (n+1)$

Then

$$(\hat{W}_X^T \hat{W}_X)_{ij} = \underset{\sim}{1}^T A^{i+j-2} \underset{\sim}{1}$$

and therefore

$$\hat{W}_X^T \hat{W}_X = \hat{W}_Y^T \hat{W}_Y$$

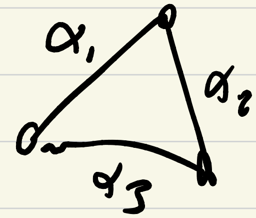
Gram matrices of columns

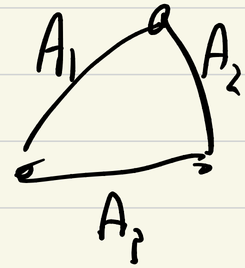
Two sets of vectors $\{b_1, \dots, b_k\}$ & $\{c_1, \dots, c_k\}$ are congruent if $\langle b_i, b_j \rangle = \langle c_i, c_j \rangle \ \forall i, j$

Theorem. The vectors $\{b_1, \dots, b_k\}$ & $\{c_1, \dots, c_k\}$ are congruent if & only if there is an orthogonal matrix Q such that $Qb_i = c_i$ for all i.

So if $B = [b_1, \dots, b_k]$, $C = [c_1, \dots, c_k]$ and $B^T B = C^T C$, then $C = QB$, for some orthogonal Q.

Euclidean geometry





Sketch of proof:

(a) reduce to case where the columns of B and C are linearly independent.

(b) If $b_1, \dots, b_j$ are linearly independent and $\langle x, x\rangle = \langle y, y\rangle$ and $\langle x, b_i\rangle = \langle y, b_i\rangle$ then there exists orthogonal Q s.t. $Qb_i = b_i$ $(1 \le i \le j)$ and $Qx = y$.

To prove (b), show that there is a reflection swapping x & y and fixing $b_1, \dots, b_j$

- $\tau_a(u) := u - \dfrac{2\langle a,u\rangle}{\langle a,a\rangle} a$
- $\tau_a(a) = -a$, $\quad \tau_a(u) = u$ if $u \in a^\perp$
- τ_a is orthogonal

$a = x - y$

We had $\hat{W}_x^T \hat{W}_x = \hat{W}_y^T \hat{W}_y$.

It follows that there is an orthogonal matrix Q such that $Q\hat{W}_x = \hat{W}_y$. Therefore

$$QA^k \underset{\sim}{1} = B^k \underset{\sim}{1} \qquad (0 \le k \le n)$$

and $Q\underset{\sim}{1} = \underset{\sim}{1}$. Hence

$$QA^nQ^T \cdot \underset{\sim}{1} = B^n \underset{\sim}{1}$$

and so QAQ^T & B have the same characteristic polynomial. $\square$

Let $N_i(t)$ be the generating function for walks in X that start at i and never return. Then

$$W_i(X,t) = W_{i,i}(X,t)\, N_i(t)$$

$$W(X,t) - W(X \setminus i, t) = W_{i,i}(X,t)\, N_i(t)^2$$

$$\Rightarrow W(X,t) - W(X \setminus i, t) = \frac{W_i(X,t)^2}{W_{i,i}(X,t)} \qquad \frac{\phi(X \setminus i, t)}{\phi(X,t)}$$

Corollary $W_i(X,t)$ is determined by $W(X,t)$, $W(X \setminus i; t)$ and $\phi(X \setminus i, t)$; $W(X \setminus i; t)$ is determined by $\phi(X \setminus i, t)$ & $\phi(\overline{X} \setminus i, t)$

Theorem If X is controllable, it is vertex reconstructible.

Lemma [Hagos] The polynomials $\phi(X\setminus i, t)$ and $\phi(\overline{X\setminus i}, t)$ $(i \in V(X))$ determine $\phi(X,t)$ and $\phi(\overline{X}, t)$.

Proof of theorem. Suppose $V(Y) = V(X)$ and $X \setminus i \cong Y \setminus i$ for all i in $V(X)$. Then X & Y have the same walk matrix W. Now

$$A(X)W = WC_{\phi(X,t)}, \quad A(Y)W = WC_{\phi(Y,t)}$$

$$\Rightarrow A(X) = A(Y)$$