Commutative algebras

Theorem Let $\mathcal{A}$ be a commutative algebra over an algebraically closed[(1)] field. Assume that if $N \in \mathcal{A}$ and $N^2 = 0$[(2)], then $N = 0$. Then $\mathcal{A}$ has a basis of pairwise orthogonal idempotents.

(1) We can make do with less - e.g. if the matrices in $\mathcal{A}$ are symmetric, $\mathbb{R}$ will work.

(2) If N is real and symmetric & $N^2 = 0$, then $N = 0$. (The jargon is that $\mathcal{A}$ has no nonzero nilpotent elements.)

Proof of theorem (Basically an easy exercise with primary decomposition)

(1) If $A \in \mathcal{A}$ then all eigenvalues of A lie in $\mathbb{F}$ and so we have the spectral decomposition

$$A = \sum_r \theta_r E_r$$

(2) If $B \in \mathcal{A}$ and $B = \sum_s \tau_s F_s$, then the distinct non-zero products $E_r F_s$ are pairwise orthogonal idempotents and their span contains A & B.

(3) Induct...

Example: strongly regular graphs

Some strongly regular graphs:

(a) mK_n ; $m, n \geq 2$

(b) $L(K_n)$

(c) Latin square graphs

Algebras & automorphisms

Let X be a graph with diameter d. If $0 \le r \le d$, the r-th distance graph X_r has vertex set $V(X)$ and vertices u, v are adjacent in X_r if $\mathrm{dist}_X(u,v) = r$. Set $A_r = A(X_r)$.

So $A_1 = A$ & $A_0 = I$. If $\mathrm{diam}(X) = 2$, then $A_2 = A(\bar{X})$.

The distance algebra $\mathcal{D}$ is $\langle A_0, A_1, \ldots, A_d \rangle$.

If the permutation matrix P represents an automorphism of X (i.e., $PA = AP$), then $P \in Aut(X_r)$ and $PA_r = A_r P$.

(automorphisms preserve distance)

Therefore P lies in the commutant of $\mathcal{D}$. So:

$$\mathcal{D} \text{ big} \Rightarrow Aut(X) \text{ small}$$

$$Aut(X) \text{ large} \Rightarrow \mathcal{D} \text{ small}$$

We choose to measure size of $\mathcal{D}$ by dimension. We have

$$d+1 \leq \dim(\mathcal{D}) \leq n^2$$

(a) If $\dim(\mathcal{D}) = n^2$, then $\mathrm{Aut}(X) = \langle 1 \rangle$

(b) If $\dim(\mathcal{D}) = d+1$?

If $\dim(\mathcal{D}) = d+1$, then

$$A_i A_j \in \operatorname{span}\{A_0, \ldots, A_d\}$$

and so $A_i A_j$ is symmetric & $A_j A_i = A_i A_j$.

As $\sum_r A_r = J$ we have $J \in \mathcal{D}$ and $JA_r = A_r J$

Therefore $X_1, \ldots, X_d$ are regular.

Formally, $\{X_1, \ldots, X_d\}$ form a symmetric association scheme with d classes.

(and $\mathcal{D}$ is its Bose-Mesner algebra)

If $d=1$, we have $X = K_n$

If $d=2$, then X_1 & X_2 are strongly regular graphs ($X_2 = \bar{X}$).

(If $d>2$ .... we don't care.)

Strongly regular graphs

A graph X is strongly regular if there are constants k, a, c such that:

$$A^2 = kI + aA + c\overbrace{(J-I-A)}^{\bar{A}}.$$

So X is k-regular &

$$\boxed{A^2 - (a-c)A - (k-c)I = cJ}$$

(n, k, a, c) are the parameters of X

Exercises

1) Compute the parameters for $L(K_n)$ and the Latin square graphs.

2) Determine the eigenvalues & their multiplicities.

Eigenthings

(a) $\underset{\sim}{1}$ is an eigenvector with eigenvalue k.

(b) If $\underset{\sim}{1}^T z = 0$ and z is an eigenvector, eval λ, then $Jz = 0$ & so

$$0 = A^2 z - (a-c)Az - (k-c)z$$
$$= (\lambda^2 - (a-c)\lambda - (k-c))z$$

$\Rightarrow \lambda$ is a zero of $t^2 - (a-c)t - (k-c)$

$\frac{1}{2}\left(a-c \pm \sqrt{(a-c)^2 + 4(k-c)}\right)$

Denote the zeros of $t^2-(a-c)t-(k-c)$ by θ & τ. (As $\theta\tau = c-k<0$, we assume $\theta>0$ and $\tau<0$.)

A strongly regular graph is primitive if X and $\bar{X}$ are connected.

If $\ell := n-1-k$ (valency of $\bar{X}$) and X is primitive,

$$\theta < k, \quad -\tau-1<\ell$$

X k-regular, X is connected iff k is a simple eigenvalue.

Spectral idempotents of A are polynomials in A. So

$$E_\theta = \alpha I + \beta A + \gamma \bar{A}$$

Hence

(a) $m_\theta = tr(E_\theta) = n\alpha \Rightarrow \alpha = \frac{m_\theta}{n}$

(b)
$$\left.\begin{aligned} sum(E_\theta \circ A) &= nk\beta \\ sum(E_\theta \circ A) &= tr(E_\theta A) = tr(\theta E_\theta) = \theta m_\theta \end{aligned}\right\} \beta = \frac{m_\theta \theta}{n k}$$

(c) $\bar{A} E_\theta = (-\theta - 1) E_\theta$

$$\left. n(n-1-k)\gamma = sum(\bar{A} \circ E_\theta) = tr(E_\theta \bar{A}) = -(\theta+1) m_\theta \right\} \gamma = -\frac{m_\theta(\theta+1)}{n(n-1-k)}$$

$\ell := n-1-k$

$$E_\theta = \frac{m_\theta}{n}\left(I + \frac{\theta}{k}A - \frac{\theta+1}{\ell}\bar{A}\right)$$

Now $E_k = \frac{1}{n}J = \frac{1}{n}(I + A + \bar{A})$ and

$E_k + E_\theta + E_\tau = I$. So:

$$\begin{array}{llcllll}
E_0 & E_k & = & \frac{1}{n}(I & + & A & + & \bar{A}) \\
E_1 & E_\theta & = & \frac{m_\theta}{n}(I & + & \frac{\theta}{k}A & - & \frac{\theta+1}{\ell}\bar{A}) \\
E_2 & E_\tau & = & ?\, I & & ?\, A & & ?\, \bar{A})
\end{array}$$

$1 + m_\theta + m_\tau = n$; $k + \theta m_\theta + \tau m_\tau = 0$; $\ell - (\theta+1)m_\theta - (\tau+1)m_\tau = 0$

Lemma. Any two vertices in a strongly regular graph are cospectral.

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2×2 submatrices of idempotents

$$E_\theta[u,v] = \begin{cases} \dfrac{m_\theta}{n}\begin{bmatrix} 1 & \theta/k \\ \theta/k & 1 \end{bmatrix}, & u \sim v; \\[2ex] \dfrac{m_\theta}{n}\begin{bmatrix} 1 & -\frac{\theta+1}{\ell} \\ -\frac{\theta+1}{\ell} & 1 \end{bmatrix}, & u \not\sim v,\ u \neq v. \end{cases}$$

$$E_\tau[u,v] = \begin{cases} \dfrac{m_\tau}{n}\begin{bmatrix} 1 & \tau/k \\ \tau/k & 1 \end{bmatrix}, & u \sim v \\[2ex] \dfrac{m_\tau}{n}\begin{bmatrix} 1 & -\frac{\tau+1}{\ell} \\ -\frac{\tau+1}{\ell} & 1 \end{bmatrix} & u \not\sim v,\ u \neq v \end{cases}$$

all invertible if X is primitive