Linear Algebra Exercises

1. Prove that any real square matrix is a product of two symmetric matrices.

2. Let W be the cyclic A-module generated by the vector w. If B is a matrix and $Bw \in W$, prove that there is a polynomial p such that $Bw = p(A)w$.

3. If the sets of vectors $u_1, \dots, u_m$ and $v_1, \dots, v_m$ are linearly independent, show that the vectors

$$u_i \otimes v_j, \quad 1 \le i, j \le m$$

   are linearly independent.

4. Assume A is a real matrix and let P be the linear map that sends $u \otimes v$ to $v \otimes u$. Show that the matrix $P(A \otimes A^T)$ is symmetric.

5. Let V be $\mathrm{Mat}_{n \times n}(\mathbb{F})$ and let A be a fixed matrix. If $X \in V$, define the map Ad_A in $\mathrm{End}(V)$ by

$$\mathrm{Ad}_A(X) := AX - XA.$$

   If $A^n = 0$, prove that $\mathrm{Ad}_A^{2n} = 0$.

6. Compute the singular values of a companion matrix. (Remark: this is asking for the eigenvalues of CC^T or C^TC; one of these is much simpler to work with than the other.)

7. Show that the sum of the singular values of a matrix is a norm.

8. Let A be an $n \times n$ complex matrix with all eigenvalues simple. Prove that

$$\mathrm{rk}(A \otimes I - I \otimes A) = n^2 - n,$$

   and deduce from this that any matrix that commutes with A is a polynomial in A.

9. Let C_f denote the companion matrix of the polynomial f. Let $p \vee q$ and $p \wedge q$ denote respectively the lcm and gcd of the polynomials p and q. Prove that the matrices

$$\begin{pmatrix} C_p & 0 \\ 0 & C_q \end{pmatrix}, \quad \begin{pmatrix} C_{p \vee q} & 0 \\ 0 & C_{p \wedge q} \end{pmatrix}$$

   are similar.

10. Assume $A \in \mathrm{End}(V)$ and let x be a vector such that $\psi_{A,x} = \psi_A$. Prove that any submodule of $\langle z \rangle_A$ is cyclic.

11. The proof in the lectures for the singular value decomposition works over the reals. Write out a proof of the complex version.

12. Prove that matrix A is normal if and only if $\langle Ax, Ax \rangle = \langle A^* x, A^* x \rangle$ for all x. Use this to prove that if A is normal and x is an eigenvector for A, it is also an eigenvector for A^*.

13. If S is a completely regular subset of $V(X)$ with covering radius r, prove that the set of vertex at distance r from X is completely regular.

14. Apply the inertia bound to a weighted Cartesian product of the Petersen graph with K_2 to derive an upper bound on the maximum size of an induced bipartite subgraph of the Petersen graph. Show your bound is tight. [If you're ambitious, repeat the exercise for the Kneser graph $K_{7:3}$ (4-regular on 35 vertices, look up the eigenvalues) and again derive a tight bound. For $K_{9:4}$, the bound is good but not tight.]

15. If X is perfect, prove that the weighted inertia bound gives the exact value for $\alpha(X)$.

16. Using the machinery employed in the lectures to prove the Hoffman bound on $\chi(X)$, prove that if equality holds, the multiplicity of τ is at least $\chi(X)-1$.

17. Let W and Z. be type-II matrices of the same order. Let A and B be matrices such that $\circ B = 0$ and $A + tB$ is type-II for all non-zero t. Prove that $W \otimes A = tZ \otimes B$ is type-II for all non-zero t.

18. Assume W is type-II of order $n \times n$ and let P be the $n^2 \times n^2$ permutation matrix such that $P(x \otimes y) = y \otimes x$ for all x and y in $\mathbb{C}^n$. Prove that $\widehat{W} = (W \otimes W^{(-)T})P$ is a type-II matrix with constant diagonal and that $\widehat{W}^2 = I$.

19. Assume P and Q are quantum permutations of order $n \times n$. Define the $n \times n$ matrix $P \star Q$ by setting
$$(P \star Q_{i,j}) = \sum_{r=1}^{n} P_{i,r} \otimes Q_{r,j}.$$
(Remark: the entries of P are $d \times d$, those of Q are $e \times e$. We are not assuimng that $d = e$.) Show that $P \star Q$ is a quantum permutation. Show that this $\star$-operation is associative.