Welcome to

Quantum Morphisms

Some math basics (& some notation)

bra-ket notation

$|\psi\rangle$ – a column vector (a "ket")

$\langle\psi|$ – a row vector, the conjugate transpose of $|\psi\rangle$ (a "bra")

$|i\rangle \in \mathbb{C}^n$ – i^{th} standard basis vector (usually $i \in \{1,\dots,n\}$ or $\{0,\dots,n-1\}$)

$|x\rangle \in \mathbb{C}^{Y}$ – standard basis vector indexed by $x \in Y$

$\langle\psi|\varphi\rangle$ – inner product of $|\psi\rangle$ & $|\varphi\rangle$ (a scalar) $\langle\psi||\varphi\rangle$

$|\psi\rangle\langle\varphi|$ – outer product of $|\psi\rangle$ & $|\varphi\rangle$ (a matrix)

$\mathbb{F}^{m\times n}$ – $m \times n$ matrices over $\mathbb{F}$

Important types of matrices (in $\mathbb{C}^{n\times n}$)

- Hermitian: $M^* = M$ i.e. $M_{ij} = \overline{M_{ji}}$
- Positive semidefinite (psd): Hermitian & $\langle\psi|M|\psi\rangle \geq 0$ $\forall\, |\psi\rangle \in \mathbb{C}^n$
  $M \succeq 0$
- Unitary: $M^*M = MM^* = I$

These are all normal i.e. $M^*M = MM^*$

Spectral Decomposition

$\langle \psi_i | \psi_j \rangle = \delta_{ij} := \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

There exists an orthonormal basis $|\psi_1\rangle, \dots, |\psi_n\rangle \in \mathbb{C}^n$ s.t.

$$M = \sum_{i=1}^{n} \lambda_i |\psi_i\rangle\langle\psi_i| \qquad \lambda_i \in \mathbb{C} \text{ are the eigenvalues of } M$$

Often write $\quad M = \sum_\lambda \lambda E_\lambda \quad$ where $\quad E_\lambda = \sum_{i:\lambda_i=\lambda} |\psi_i\rangle\langle\psi_i|$

Then $\underbrace{E_\lambda^2 = E_\lambda^* = E_\lambda}_{\text{projection}}$, $\quad E_\lambda E_\mu = \delta_{\lambda\mu} E_\lambda \quad \& \quad \sum_\lambda E_\lambda = I$

$\sum_\lambda \sqrt{\lambda}\, E_\lambda$

M Hermitian $\Rightarrow$ real eigenvalues

M psd $\Rightarrow$ non-negative eigenvalues $(M^{1/2}$ well-defined$)$

M unitary $\Rightarrow$ eigenvalues of unit modulus, $\lambda = e^{i\theta} \quad 0 \le \theta < 2\pi$

Quantum States

A quantum state is a description of a quantum system

A pure quantum state is a unit vector $|\psi\rangle \in \mathbb{C}^n$. "state space"

Some PHYSICS: an electron can have spin up $|\uparrow\rangle$ or down $|\downarrow\rangle$ or any superposition of the two: $\alpha|\uparrow\rangle + \beta|\downarrow\rangle \quad |\alpha|^2 + |\beta|^2 = 1$

What (linear) operations map states to states?

$M: \mathbb{C}^n \to \mathbb{C}^n$ must satisfy $\langle\psi| M^* M |\psi\rangle = 1 \quad \forall |\psi\rangle \in \mathbb{C}^n$

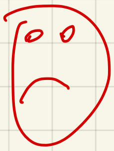
Exercise: Show this implies $M^* M = I$ & thus M is unitary.

General quantum states

Suppose we know that a given system is in state $|0\rangle \in \mathbb{C}^2$ w/ probability $1/2$ & in state $|1\rangle \in \mathbb{C}^2$ w/ prob $1/2$.

How could we describe this?

$$\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle = \frac{1}{2}\big(|0\rangle + |1\rangle\big) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$



not a unit vector!

Instead we use density matrices

$$\rho \in \mathbb{C}^{n\times n} \text{ s.t. } \rho \geq 0 \ \& \ Tr(\rho) = 1$$

pure state $\longrightarrow$ density matrix

$$|\psi\rangle \in \mathbb{C}^n \longrightarrow |\psi\rangle\langle\psi| \in \mathbb{C}^{n\times n}$$

$$\alpha|\psi\rangle \longrightarrow \alpha\bar{\alpha}|\psi\rangle\langle\psi|$$

mixed state/ensemble $\longrightarrow$ density matrix

$$|\psi_i\rangle \text{ w/ prob } p_i \longrightarrow \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

$$\rho_i \text{ w/ prob } p_i \longrightarrow \sum_i p_i \rho_i$$

Above example: $\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Exercise: Show that if $|\psi_0\rangle$ & $|\psi_1\rangle$ are an orthonormal basis for $\mathbb{C}^2$, then the density matrix for the ensemble

$$|\psi_i\rangle \text{ w/ prob } 1/2 \text{ for } i = 0, 1$$

is $\frac{1}{2}I$. Do for $\mathbb{C}^n$.

Measurements

A quantum measurement is a tuple $\mathcal{M} = (M_1, \dots, M_k)$ with $M_i \in \mathbb{C}^{m \times n}$ s.t. $\sum_{i=1}^{k} M_i^* M_i = I$.

When a state $\rho \in \mathbb{C}^{n \times n}$ is measured with $\mathcal{M}$ it results in

① a classical outcome $i \in [k]$ w/ prob

$$\mathrm{Tr}(M_i \rho M_i^*) = \underbrace{\mathrm{Tr}(\rho M_i^* M_i)}_{\geq 0}$$

$$\sum_i \mathrm{Tr}(\rho M_i^* M_i) = \mathrm{Tr}(\rho) = 1$$

pure case:

$$\mathrm{Tr}(M_i |\psi\rangle\langle\psi| M_i^*) = \langle\psi| M_i^* M_i |\psi\rangle = \| M_i |\psi\rangle \|^2$$

② and the corresponding post-measurement state

$$\frac{1}{\mathrm{Tr}(M_i \rho M_i^*)} M_i \rho M_i^* = M_i \rho^{1/2} (M_i \rho^{1/2})^*$$

$\rho = \frac{1}{n} I \quad M_i = |i\rangle\langle i| \quad = \frac{1}{n}\mathrm{Tr}(|i\rangle\underline{\langle i|i\rangle}\langle i|) = \frac{1}{n}\mathrm{Tr}(|i\rangle\langle i|) = \frac{1}{n}$

pure case: $\frac{1}{\| M_i |\psi\rangle \|} M_i |\psi\rangle$

Note: The outcome probabilities only depend on the operators $M_i^* M_i$.

The $M_i^* M_i$ form a positive operator valued measure (POVM), i.e. a tuple $(P_1, \dots, P_k)$ with $P_i \in \mathbb{C}^{n \times n}$

s.t. $P_i \succeq 0 \ \forall i$ & $\sum_i P_i = I$.

$(P_1^{1/2}, \dots, P_k^{1/2})$

Example: Full basis measurement of a pure state

$|\psi\rangle \in \mathbb{C}^n$, $|\varphi_1\rangle, \dots |\varphi_n\rangle \in \mathbb{C}^n$ ONB $M_i = |\varphi_i\rangle\langle\varphi_i|$

Note $M_i^* M_i = |\varphi_i\rangle\langle\varphi_i|\varphi_i\rangle\langle\varphi_i| = |\varphi_i\rangle\langle\varphi_i| = M_i = M_i^*$

special case of projective measurement: $M_i^* = M_i = M_i^2$.

Obtain outcome i w/ prob

$$\langle\psi|M_i^* M_i|\psi\rangle = \langle\psi|\varphi_i\rangle\langle\varphi_i|\psi\rangle = |\langle\varphi_i|\psi\rangle|^2$$

Post-measurement state:

$$\frac{1}{\| M_i|\psi\rangle \|} M_i|\psi\rangle = \frac{1}{\| |\varphi_i\rangle\langle\varphi_i|\psi\rangle \|} |\varphi_i\rangle\langle\varphi_i|\psi\rangle$$

$= |\varphi_i\rangle$ up to a phase.

Composite Systems

System A: $|\psi_A\rangle \in \mathbb{C}^{d_A}$ $\quad (\rho_A \in \mathbb{C}^{d_A \times d_A})$

System B: $|\psi_B\rangle \in \mathbb{C}^{d_B}$ $\quad (\rho_B \in \mathbb{C}^{d_B \times d_B})$

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \cdots \\ A_{21}B & \ddots & \\ \vdots & & A_{mn}B \end{pmatrix}$$

Combined system:

$$|\psi_A\rangle \otimes |\psi_B\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \cong \mathbb{C}^{d_A d_B}$$

$$\rho_A \otimes \rho_B \in \mathbb{C}^{d_A \times d_A} \otimes \mathbb{C}^{d_B \times d_B} \cong \mathbb{C}^{d_A d_B \times d_A d_B}$$

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

whenever AC + BD exist

A <u>pure</u> state $|\psi\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ is entangled if it cannot be written as

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

for states $|\psi_A\rangle \in \mathbb{C}^{d_A}$, $|\psi_B\rangle \in \mathbb{C}^{d_B}$.

A state $\rho \in \mathbb{C}^{d_A \times d_A} \otimes \mathbb{C}^{d_B \times d_B}$ is entangled if it cannot be written as

$$\rho = \sum_{i=1}^{k} p_i \, \rho_A^i \otimes \rho_B^i$$

for states $\rho_A^i \in \mathbb{C}^{d_A \times d_A}$, $\rho_B^i \in \mathbb{C}^{d_B \times d_B}$ & prob dist $p_1, \ldots, p_k$.

Exercise: Show that $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is entangled.

$= |0\rangle \otimes |0\rangle$ $\quad |\psi\varphi\rangle = |\psi\rangle \otimes |\varphi\rangle$

$|\psi\rangle|\varphi\rangle$

Joint Measurements

$\rho \in \mathbb{C}^{d_A \times d_A} \otimes \mathbb{C}^{d_B \times d_B}$

Alice "has" $\mathbb{C}^{d_A \times d_A}$ $\quad$ Bob "has" $\mathbb{C}^{d_B \times d_B}$

Alice performs measurement $\mathcal{A} = (E_1, \ldots, E_k)$ on her system

Equivalent to "global" measurement $(E_1 \otimes I, \ldots, E_k \otimes I)$

Outcome i w/ prob $Tr((E_i \otimes I)\rho(E_i^* \otimes I))$

post-measurement state $(E_i \otimes I)\rho(E_i^* \otimes I)$ normalized

If Bob also measures with $\mathcal{B} = (F_1, \ldots, F_r)$:

outcome i,j w/ prob

$$Tr((E_i \otimes F_j)\rho(E_i^* \otimes F_j^*))$$

post meas state

$$(E_i \otimes F_j)\rho(E_i^* \otimes F_j^*) \text{ normalized}$$

Exercise: Show that doing both measurements "at the same time" is equivalent to doing one & then the other.

Example: $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$

Measure first qubit in standard basis.

Outcome 0: $\frac{1}{2}\mathrm{Tr}\big((|0\rangle\langle 0|\otimes I)(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)\big)$

$$= \frac{1}{2}\mathrm{Tr}(|0\rangle\langle 0|) = \frac{1}{2}$$

Outcome 1: $\frac{1}{2}$

post measurement state: $M_i|\Psi\rangle = |0\rangle\langle 0|\otimes I \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$= |00\rangle$$

" : $|11\rangle$

Alice measures her qubit w/ full basis meas $|\varphi_0\rangle, |\varphi_1\rangle$

Bob " his " " " " " $\overline{|\varphi_0\rangle}, \overline{|\varphi_1\rangle}$