# Quantum Morphisms HW

Summer 2021

1. Review Chapter 5 here if necessary.

2. A matrix $M \in \mathbb{C}^{n \times n}$ which maps quantum states to quantum states must satisfy $\langle \psi | M^* M | \psi \rangle = 1$ for all unit vectors $|\psi\rangle \in \mathbb{C}^n$. Show that this implies that $M^* M = I$.

3. Suppose that $|\psi_0\rangle$ and $|\psi_1\rangle$ form an orthonormal basis of $\mathbb{C}^2$. Show that if a quantum system is in state $|\psi_i\rangle$ with probability $1/2$ for each $i = 0, 1$, then the density matrix describing this system is $\frac{1}{2} I \in \mathbb{C}^{2 \times 2}$. Prove the analogous result with 2 replaced with $n$.

4. Show that
$$\frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right) := \frac{1}{\sqrt{2}} \left( |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \right) \in \mathbb{C}^2 \otimes \mathbb{C}^2$$
is entangled, i.e., cannot be written as $|\psi_1\rangle \otimes |\psi_2\rangle$ for any $|\psi_1\rangle, |\psi_2\rangle \in \mathbb{C}^2$.

5. Suppose that the density matrix $\rho \in \mathbb{C}^{d_A \times d_A} \otimes \mathbb{C}^{d_B \times d_B}$ is the state of a composite system consisting of system $A$ (corresponding to $\mathbb{C}^{d_A \times d_A}$) and system $B$ (corresponding to $\mathbb{C}^{d_B \times d_B}$). Show that if we measure system $A$ with measurement $\mathcal{A} = (E_1, \ldots, E_k)$ and then measure system $B$ of the post-measurement state with measurement $\mathcal{B} = (F_1, \ldots, F_r)$, the probability of obtaining outcome $i$ from the first measurement and $j$ from the second measurement is the same of the probability of obtaining outcome $(i, j)$ from the global measurement $(E_1 \otimes F_1, E_1 \otimes F_2, \ldots, E_k \otimes F_r)$. Also show that the probability of obtaining outcome $j$ from the measurement on system $B$ does not depend on the choice of measurement $\mathcal{A}$ performed on system $A$.

6. Let $|\psi\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right) \in \mathbb{C}^2 \otimes \mathbb{C}^2$. Show that performing a full basis measurement on the first qubit using basis $|\phi_0\rangle, |\phi_1\rangle \in \mathbb{C}^2$ results in outcome $i \in \{0, 1\}$ with probability $1/2$ and the post-measurement state in this case is $|\phi_i\rangle \otimes \overline{|\phi_i\rangle}$. Conclude that performing a full basis measurement on the second qubit using basis $\overline{|\phi_0\rangle}, \overline{|\phi_1\rangle}$ will always result in the same outcome as the first measurement described above.

7. Let vec denote the linear function that takes $|a\rangle\langle b|$ to $|a\rangle \otimes \overline{|b\rangle}$, i.e., creates a column vector from a matrix by stacking (the transpose of) its rows on top of each other. Use this to show that the quantum state
$$|\Psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{0}^{d-1} |\psi_i\rangle \otimes \overline{|\psi_i\rangle} \in \mathbb{C}^d \otimes \mathbb{C}^d$$
is the same for any choice of orthonormal basis $|\psi_0\rangle, \ldots, |\psi_{d-1}\rangle \in \mathbb{C}^d$. This state is often called the *maximally entangled state* in local dimension $d$.

8. Show that
$$(M \otimes N) \operatorname{vec}(X) = \operatorname{vec}(M X N^T) \tag{1}$$

for all $M, N, X$ such that the product $MXN^T$ is defined. Use this and the fact that $\mathrm{vec}(A)^*\mathrm{vec}(B) = \mathrm{Tr}(A^*B)$ for $A, B \in \mathbb{C}^{m\times n}$ to show that

$$\langle \Psi_d | M \otimes N | \Psi_d \rangle = \frac{1}{d}\,\mathrm{Tr}(MN^T). \tag{2}$$

for $M, N \in \mathbb{C}^{d\times d}$.

9. Recall that in the CHSH game both Alice and Bob's input and output sets are $\{0,1\}$ and the verifier sends each possible pair of inputs with uniform probability, and they win if their inputs $x, y$ and outputs $a, b$ satisfy $x \wedge y = a \oplus b$ where $\wedge$ denotes the AND function and $\oplus$ denotes xor or the sum modulo 2.

   Suppose that Alice and Bob play the CHSH game with shared state $|\psi\rangle$ and projective measurements (i.e. measurements whose operators are all projections) $\mathcal{E}_0 = (E_{00}, E_{01})$, $\mathcal{E}_1 = (E_{10}, E_{11})$ for Alice and $\mathcal{F}_0 = (F_{00}, F_{01})$, $\mathcal{F}_1 = (F_{10}, F_{11})$ for Bob. Define $A_0 = E_{00} - E_{01}$, $A_1 = E_{10} - E_{11}$, and analogously for $B_i$ (these are called *observables*). Note that $A_i^2 = B_i^2 = I$ for all $i$. Show that the expression

$$\frac{1}{4}\langle\psi|\Big(A_0\otimes B_0 + A_0\otimes B_1 + A_1\otimes B_0 - A_1\otimes B_1\Big)|\psi\rangle \tag{3}$$

   is equal to the *bias* of their strategy, i.e., their probability of winning minus their probability of losing. By bounding the maximum eigenvalue of the square of the operator in parentheses in (3), show that the bias of any quantum strategy using projective measurements for the CHSH game is at most $\frac{\sqrt{2}}{2}$. In fact it is known that any quantum correlation can be produced by a strategy that uses projective measurements, thus we have shown that the optimal quantum bias for CHSH is at most $\frac{\sqrt{2}}{2}$.

   Now show that this bias can be obtained by letting $|\psi\rangle = |\Psi_2\rangle$, and $A_0 = Z$, $A_1 = X$, $B_0 = \frac{1}{\sqrt{2}}(Z+X)$, $B_1 = \frac{1}{\sqrt{2}}(Z-X)$, where $Z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ respectively (these are the *Pauli* $X$ and $Z$ matrices). These are the observables corresponding to the strategy presented in class.

10. Recall that a correlation $p : O_A \times O_B \times I_A \times I_B$ is *non-signalling* if

$$\begin{aligned}&\sum_{b\in O_B} p(a,b|x,y) \text{ is independent of } y \in I_B, \text{ and}\\ &\sum_{a\in O_A} p(a,b|x,y) \text{ is independent of } x \in I_A.\end{aligned} \tag{4}$$

   Find a non-signalling correlation that wins the CHSH game with probability 1.

11. Show that if $P_1, \ldots, P_k$ are projections such that $\sum_{i=1}^k P_i = I$, then $P_iP_j = \delta_{ij}P_i$ where $\delta_{ij}$ is the Kronecker delta.

12. Suppose that Alice and Bob play the $(G,H)$-homomorphism game with shared state $|\psi'\rangle \in \mathbb{C}^{d_A\times d_B}$ and POVMs $\mathcal{E}'_g = \{E'_{gh} \in \mathbb{C}^{d_A\times d_A} : h \in V(H)\}$ for $g \in V(G)$ for Alice and $\mathcal{F}'_g = \{F'_{gh} \in \mathbb{C}^{d_B\times d_B} : h \in V(H)\}$ for $g \in V(G)$ for Bob. By considering the singular value decomposition of the $d_A \times d_B$ matrix $X$ such that $\mathrm{vec}(X) = |\psi'\rangle$, we can write

$$|\psi'\rangle = \sum_{i=1}^{d} \lambda_i |\alpha_i\rangle \otimes |\beta_i\rangle$$

where $|\alpha_1\rangle, \ldots, |\alpha_d\rangle \in \mathbb{C}^{d_A}$ and $|\beta_1\rangle, \ldots, |\beta_d\rangle \in \mathbb{C}^{d_B}$ are orthonormal sets of vectors and $\lambda_i > 0$ for all $i$ (this is called the *Schmidt decomposition* of $|\psi'\rangle$. Define

$$\begin{aligned}
P_A &:= \sum_{i=1}^{d} |i\rangle\langle\alpha_i| \in \mathbb{C}^{d \times d_A}; \\
P_B &:= \sum_{i=1}^{d} |i\rangle\langle\beta_i| \in \mathbb{C}^{d \times d_B}; \\
|\psi\rangle &:= P_A \otimes P_B |\psi'\rangle = \sum_{i=1}^{d} \lambda_i |ii\rangle; \\
E_{gh} &:= P_A E'_{gh} P_A^*; \\
F_{gh} &:= P_B F'_{gh} P_B^*.
\end{aligned}$$

Verify that $\mathcal{E}_g = \{E_{gh} \in \mathbb{C}^{d \times d} : h \in V(H)\}$ and $\mathcal{F}_g = \{F_{gh} \in \mathbb{C}^{d \times d} : h \in V(H)\}$ are valid POVMs for $g \in V(G)$ and that using these with the state $|\psi\rangle$ produces the same correlation as the original strategy, i.e.,

$$\langle\psi| E_{gh} \otimes F_{g'h'} |\psi\rangle = \langle\psi'| E'_{gh} \otimes F'_{g'h'} |\psi'\rangle \text{ for all } g \in V(G), h \in V(H).$$

Conclude that we can always assume that the state used in a quantum strategy has the form $\sum_{i=1}^{d} \lambda_i |ii\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ with $\lambda_i > 0$ for all $i$.

13. Suppose that $A, B, X \in \mathbb{C}^{d \times d}$ are Hermitian matrices satisfying $AX = AXB = XB$ and $X$ is positive definite (i.e., positive semidefinite and invertible). Show that $A = B$ and this is a projection that commutes with $X$ (not necessarily in that order).

14. Suppose that the quantum strategy using shared state $|\psi\rangle = \sum_{i=1}^{d} \lambda_i |ii\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ with $\lambda_i > 0$ for all $i$, and POVMs $\mathcal{E}_g$ and $\mathcal{F}_g$ for $g \in V(G)$ for Alice and Bob respectively wins the $(G, H)$-homomorphism game with probability 1. Note that this implies that

$$\langle\psi| E_{gh} \otimes F_{gh'} |\psi\rangle = 0 \text{ if } h \neq h'.$$

Use this and the identities in Exercise 8 to show that $E_{gh} X F_{gh'}^T = 0$ if $h \neq h'$ where $X = \sum_{i=1}^{d} \lambda_i |i\rangle\langle i|$ is the matrix in $\mathbb{C}^{d \times d}$ such that $\mathrm{vec}(X) = |\psi\rangle$. Combine this with the result of Exercise 13 (and properties of POVMs) to show that $E_{gh} = F_{gh}^T$ and these are projections that commute with $X$ for all $g \in V(G)$ and $h \in V(H)$. Use this to show that

$$\langle\psi| E_{gh} \otimes F_{g'h'} |\psi\rangle = 0 \Leftrightarrow E_{gh} F_{g'h'}^T = 0.$$

From this conclude that replacing $|\psi\rangle$ with $\mathrm{vec}\left(\frac{1}{\sqrt{d}} I\right) = |\Psi_d\rangle$ gives a quantum strategy whose correlation is zero precisely when the correlation of the original strategy was zero, and therefore the latter is a perfect quantum strategy for the $(G, H)$-homomorphism game. Finally, combine all of the above results from this exercise with those of Exercise 12 to obtain our "Main Theorem" from class, i.e., that if there is a perfect quantum strategy for the $(G, H)$-homomorphism game then there is one using shared state $|\Psi_d\rangle$ for some $d$ and $E_{gh} = F_{gh}^T$ are projections.

15. Use the "Main Theorem" from class (see also Exercise 14) to prove the "Main Corollary", i.e., that $G \xrightarrow{q} H$ if and only if there exist projections $E_{gh} \in \mathbb{C}^{d \times d}$ for $g \in V(G)$ and $h \in V(H)$ satisfying

$$\begin{aligned}
&E_{gh} E_{g'h'} = 0 \text{ if } (g = g' \;\&\; h \neq h') \text{ or } (g \sim g' \;\&\; h \not\sim h'); \\
&\sum_{h \in V(H)} E_{gh} = I \text{ for all } g \in V(G).
\end{aligned} \tag{5}$$

16. Show that the relation $\xrightarrow{q}$ is transitive, i.e., if $G \xrightarrow{q} H$ and $H \xrightarrow{q} K$, then $G \xrightarrow{q} K$.

17. Suppose that there are projections $E_{gh}$ for $g \in V(G)$, $h \in V(H)$ that give a quantum homomorphism from $G$ to $H$ and these projections pairwise commute. Show that there exists a homomorphism from $G$ to $H$.

18. Show that for any graph $G$, there is a non-signalling correlation that wins the $(G, K_2)$-homomorphism game perfectly. Show that there is no correlation that wins the $(G, K_1)$-homomorphism game unless $G$ has no edges.

19. Prove that $\chi_q(K_n) = n$.

20. Show that if $H$ is vertex transitive (i.e. for any $h, h' \in V(H)$ there is an automorphism $\sigma$ of $H$ such that $\sigma(h) = h'$), and there is a quantum homomorphism from $G$ to $H$, then there is a quantum homomorphism from $G$ to $H$ using projections of all the same rank.

21. Show that if $G$ has a rank-1 quantum 3-coloring then it is 3-colorable.

22. Show that if $D \in \mathbb{C}^{d \times d}$ is a diagonal matrix and $F \in \mathbb{C}^{d \times d}$ is a flat unitary, then $F^* D F$ has constant diagonal.

23. Can you find projections that give a quantum 4-coloring of $K_4$ that do not all pairwise commute? What about a quantum 3-coloring of $K_3$? What about a quantum 3-coloring of $K_2$? Neither the '3' nor the '2' in the previous sentence is a typo. This question may be challenging.

24. If projections $E_{gh}$ for $g \in V(G)$, $h \in V(H)$ give a quantum homomorphism from $G$ to $H$, we say that it is a *locally commuting* quantum homomorphism if $E_{gh}E_{g'h'} = E_{g'h'}E_{gh}$ whenever $g \sim g'$. We then define the *locally commuting quantum chromatic number* of a graph $G$, denoted $\chi_q^{lc}(G)$, as the minimum $n$ such that $G$ has a locally commuting quantum homomorphism to $K_n$. Show that if $G'$ is the graph obtained from $G$ by adding an apex vertex (i.e., a new vertex adjacent to all vertices of $G$), then

$$\chi_q^{lc}(G') = \chi_q^{lc}(G) + 1.$$

    Recall that this is not the case for $\chi_q$.

25. Show that if $G \xrightarrow{q} H$, then $\omega_p(G) \leq \omega_p(H)$ and $\xi_f(G) \leq \xi_f(H)$. It's fine to just prove one of the statements.

26. Show that if $\alpha_q(G) = k$, then $G$ has a projective packing of value $k$ and therefore $\alpha_q(G) \leq \alpha_p(G)$.

27. Show that if $E_1, \ldots, E_k \in \mathbb{C}^{d \times d}$ are projections, then their sum is a projection if and only if they are mutually orthogonal. In the case where they are mutually orthogonal, show that their sum is the identity if and only if the sum of their ranks is $d$.

28. The *lexicographic* and *disjunctive* products of graphs $G$ and $H$, denoted $G[H]$ and $G * H$ respectively, have vertex sets $V(G) \times V(H)$, and adjacencies

    (a) $(g, h) \sim (g', h')$ in $G[H]$ if $g \sim g'$ or ($g = g'$ and $h \sim h'$);
    (b) $(g, h) \sim (g', h')$ in $G * H$ if $g \sim g'$ or $h \sim h'$.

    Thus $G[H]$ is a subgraph of $G * H$. Show that

$$\alpha_p(G)\alpha_p(H) \leq \alpha_p(G * H) \leq \alpha_p(G[H]) \leq \alpha_p(G)\alpha_p(H)$$

and thus there is equality throughout. This means that $\alpha_p$ is *multiplicative* with respective to the lexicographic and disjunctive products. It may be helpful to think of assigning subspaces to vertices rather than projections. The projective rank $\xi_f$ is also multiplicative with respect to these products, and the proof is very similar.

29. Let $G \cup H$ denote the disjoint union of the graphs $G$ and $H$. Show that $\alpha_p(G \cup H) = \alpha_p(G) + \alpha_p(H)$.

30. In class we saw that $\alpha_q(G) \leq \alpha_p(G) \leq \chi_f(\overline{G}) \leq \chi(\overline{G})$, where $\chi_f$ is the fractional chromatic number. We further saw (but did not prove) that if there is a projective packing of value $\chi(\overline{G})$ (i.e., $\alpha_p(G) = \chi(\overline{G})$ and this value is attained by some projective packing), then $\alpha_q(G) = \chi(\overline{G})$. Prove the following strengthening: there exists a projective packing of $G$ of value $\chi_f(\overline{G})$ if and only if there exists an $r \in \mathbb{N}$ such that $\alpha_q(G[\overline{K_r}]) = r\chi_f(\overline{G})$ if and only if there exists an $r \in \mathbb{N}$ such that $\alpha_q(rG) = r\chi_f(\overline{G})$ (where $rG$ denotes the disjoint union of $r$ copies of $G$). Exercises (28) and (29) may be useful for part of this. You may freely use the fact that $\chi_f\left(\overline{G[\overline{K_r}]}\right) = r\chi_f(\overline{G})$. There are many formulations of fractional chromatic number, one that is useful for this problem is that $\chi_f(G)$ is equal to the minimum $n/r$ such that there exist $n$ (not necessarily distinct) independent sets in $G$ such that every vertex appears in exactly $r$ of them.

31. Below is a pair of dual semidefinite programs whose optimal values are equal to Lovasz' theta function of the complement of a graph $G$, denoted $\bar{\vartheta}(G)$. Note that $u \not\simeq v$ means that the vertices $u, v$ are neither adjacent nor equal, and $\text{sum}(B)$ is the sum of all of the entries of the matrix $B$. Suppose that $B$ is a feasible solution of objective value $d$ to the maximization program and $M$ is a feasible solution of objective value $p$ to the minimization program. Show that $\text{Tr}(MB) = p - d$. Conclude that $M$ and $B$ are both optimal if and only if $MB = 0$. Further show that any feasible solution $B$ to the maximization program is optimal if and only if $\vartheta(G)B_{uu} = \sum_v B_{uv}$ for all $u \in V(G)$.

$$\begin{array}{rllcll} & & \text{PRIMAL} & & & \text{DUAL} \\ \bar{\vartheta}(G) = & \min & t & = & \max & \text{sum}(B) \\ & \text{s.t.} & M_{uu} = t - 1 \text{ for } u \in V(G) & & \text{s.t.} & B_{uv} = 0 \text{ for } u \not\simeq v \\ & & M_{uv} = -1 \text{ for } u \sim v & & & \text{tr}(B) = 1 \\ & & M \succeq 0 & & & B \succeq 0 \end{array} \tag{6}$$

32. Use whatever formulation(s) you like to prove that $\bar{\vartheta}(K_n) = n$.

33. Prove that $\vartheta(G)\bar{\vartheta}(G) \geq |V(G)|$ with equality if $G$ is vertex transitive. The formulations in Equation 6 can be used to do this.

34. Recall from class that we saw the following two formulations of $\vartheta$:

$$\vartheta(G) = \max \lambda_{\max}\Big(\sum_{u \in V(G)} E_u\Big) \text{ s.t. } u \mapsto E_u \text{ is a projective packing of } G$$

$$\vartheta(G) = \max \sum_{u \in V(G)} |\langle \varphi | \psi_u \rangle|^2 \text{ s.t. } u \mapsto |\psi_u\rangle \text{ is an orthogonal representation of } G,\ |\varphi\rangle \text{ a unit vector}$$

Prove that these two formulations are in fact equivalent.

35. Let $M \in \mathbb{C}^{n \times n}$. Recall that a subspace $U \subseteq \mathbb{C}^n$ is $M$-isotropic if $\langle \psi | M | \phi \rangle = 0$ for all $|\psi\rangle, |\phi\rangle \in U$. Show that if $U$ is $M$-isotropic, then

$$U \cap \text{span}\{|\psi\rangle \in \mathbb{C}^n : M|\psi\rangle = \lambda|\psi\rangle \text{ for some } \lambda > 0\} = \{0\}.$$

36. Let $G$ be a graph. Show that if $U \subseteq \mathbb{C}^{V(G)}$ is an $M$-isotropic subspace for all weighted adjacency matrices $M$, then $\mathrm{span}\{v \in V(G) : |v\rangle \notin U^\perp\}$ is an independent set of $G$. Here $U^\perp = \{|\psi\rangle \in \mathbb{C}^{V(G)} : \langle\phi|\psi\rangle = 0 \text{ for all } |\phi\rangle \in U\}$.

37. Let $G$ be a graph. We say that a matrix $M \in \mathbb{C}^{V(G)\times V(G)}$ *fits* $G$ if $M_{uv} = 0$ whenever $u$ and $v$ are distinct non-adjacent vertices of $G$. This differs from a weighted adjacency matrix of $G$ in that the diagonal entries are allowed to be nonzero for a matrix that fits $G$. Show that $\alpha(G)$ is equal to the maximum number $r \in \mathbb{N}$ such that there exist orthonormal vectors $|\psi_1\rangle, \ldots, |\psi_r\rangle \in \mathbb{C}^{V(G)}$ such that $\langle\psi_i|M|\psi_j\rangle = 0$ if $i \neq j$ for any matrix $M$ that fits $G$.

38. For any $d \in \mathbb{N}$ let $\alpha^d(G)$ denote maximum number $r \in \mathbb{N}$ such that there exist orthonormal vectors $|\psi_1\rangle, \ldots, |\psi_r\rangle \in \mathbb{C}^{V(G)} \otimes \mathbb{C}^d$ such that $\langle\psi_i|M \otimes I_d|\psi_j\rangle = 0$ if $i \neq j$ for any matrix $M$ that fits $G$. Thus $\alpha^1(G) = \alpha(G)$ by the previous exercise. Show that

$$\alpha_p(G) = \sup_d \frac{\alpha^d(G)}{d}.$$

The parameter $\alpha^d(G)$ is equal to the one-shot zero-erro *classical* capacity of the *quantum* channel consisting of a noiseless quantum channel of dimension $d$ and a noisy classical channel with confusability graph equal to $G$. See `arXiv:1002.2514` for definitions.

39. Suppose that $E_{gh}$ are projections that give a quantum homomorphism from $G$ to $H$. Let $g, g' \in V(G)$ and suppose there is a walk of length $\ell$ between $g, g'$, i.e., there are vertices $g_0 = g, g_1, \ldots, g_\ell = g'$ such that $g_{i-1} \sim g_i$ for $i \in [\ell]$. Show that if $E_{gh}E_{g'h'} \neq 0$, then $h$ and $h'$ have a walk of length $\ell$ between them.

40. Let $C_n$ be the cycle of length $n$. Show that $C_n \xrightarrow{q} H$ if and only if $C_n \to H$ for any graph $H$.

41. Let $G$ be a connected graph. Show that if $G \xrightarrow{q} H$ then there is a connected component $H'$ of $H$ such that $G \xrightarrow{q} H'$.

42. Let $p : (V(G) \cup V(H))^4 \to [0, 1]$ be a non-signalling correlation that wins the $(G, H)$-isomorphism game. Show that

$$p(h, h|g, g) = p(h, g|g, h) = p(g, h|h, g) = p(g, g|h, h)$$

for all $g \in V(G)$, $h \in V(H)$. Use this to prove that the $V(G) \times V(H)$ matrix $D$ such that $D_{gh} = p(h, h|g, g)$ is doubly stochastic. Finally, show that $A_G D = D A_H$. Thus if there is a non-signalling correlation that wins the $(G, H)$-isomorphism game, then $G \cong_f H$.

43. Recall from class that graphs $G$ and $H$ are fractionally isomorphic if and only if they have a *common equitable partition*. This means that there are partitions $(C_1, \ldots, C_r)$ and $(C'_1, \ldots, C'_r)$ of $V(G)$ and $V(H)$ respectively with $|C_i| = |C'_i|$ for all $i \in [r]$ and there are numbers $d_{ij}$ for $i, j \in [r]$ such that the number of neighbors a vertex in $C_i$ (respectively $C'_i$) has in $C_j$ (respectively $C'_j$) is $d_{ij}$. Use such a common equitable partition of $G$ and $H$ to construct a non-signalling correlation that wins the $(G, H)$-isomorphism game. Together with the previous exercise this shows that there is a winning non-signalling correlation for the $(G, H)$-isomorphism game if and only if $G \cong_f H$.

44. Suppose that $\mathcal{P} = (P_{ij}) \in M_n(\mathbb{C}^{d\times d})$ is such that $P_{ij} = P_{ij}^2 = P_{ij}^*$ for all $i, j \in [n]$. Show that $\mathcal{P}$ is unitary if and only if it is a quantum permutation matrix.

45. Recall that a tracial state on a $C^*$-algebra $\mathcal{A}$ is a linear functional $s : \mathcal{A} \to \mathbb{C}$ such that $s(\mathbf{1}) = 1$, $s(x^*x) \geq 0$ for all $x \in \mathcal{A}$, and $s(xy) = s(yx)$ for all $x, y \in \mathcal{A}$. Show that the only tracial state on the algebra of $d \times d$ complex matrices is

$$\mathrm{tr}(M) = \frac{1}{d} \sum_{i=1}^{d} M_{ii}.$$

46. Let $p_1, \ldots, p_k$ be projections in a $C^*$-algebra, i.e., $p_i = p_i^2 = p_i^*$ for all $i = 1, \ldots, k$. Show that if $\sum_{i=1}^{k} p_i = \mathbf{1}$, where $\mathbf{1}$ is the identity element, then $p_i p_j = 0$ for $i \neq j$. This does not hold for arbitrary algebras. Hint: use the GNS theorem. If you find a short proof that does not use this, please let me know. How does this proof compare to your proof from Exercise 11?

47. Let $M \in \mathbb{Z}_2^{m \times n}$ and $b \in \mathbb{Z}_2^m$. Also let $S_\ell = \{i \in [n] : M_{\ell i} = 1\}$ and let $\mathcal{H}$ be a Hilbert space. Show that the existence of projections $P_f^\ell \in B(\mathcal{H})$ for $\ell \in [m]$ and $f : S_\ell \to \mathbb{Z}_2$ satisfying

   (a) $\sum_{f:S_\ell \to \mathbb{Z}_2} P_f^\ell = I$ for all $\ell \in [m]$,
   (b) $P_f^\ell = 0$ if $\sum_{i \in S_\ell} f(i) \neq b_\ell$, and
   (c) $P_f^\ell P_{f'}^k = 0$ if there is $i \in S_\ell \cap S_k$ such that $f(i) \neq f'(i)$,

   is equivalent to the existence of $A_i \in B(\mathcal{H})$ for $i \in [n]$ satisfying

   (a) $A_i = A_i^*$ and $A_i^2 = I$ for all $i \in [n]$,
   (b) $A_i A_j = A_j A_i$ whenever there exists $\ell \in [m]$ such that $i, j \in S_\ell$, and
   (c) $\prod_{i \in S_\ell} A_i = (-1)^{b_\ell} I$ for all $\ell \in [m]$,

   i.e., the $A_i$ form a *quantum solution* for $Mx = b$.

48. Show that the following system of equations over $\mathbb{Z}_2$ has no quantum solution:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + x_4 + x_5 &= 0 \\ x_2 + x_4 + x_6 &= 0 \\ x_3 + x_5 + x_6 &= 0 \end{aligned}$$

   Hint: the coefficient matrix for this system is the incidence matrix of $K_4$, so you can think of the variables as the edges of $K_4$ and the equations as its vertices.

49. Recall that given a binary linear system $Mx = b$ with $M \in \mathbb{Z}_2^{m \times n}$ and $b \in \mathbb{Z}_2^m$ we let $S_\ell = \{i \in [n] : M_{\ell i} = 1\}$, and the graph $G(M, b)$ has vertex set

$$\bigcup_{\ell \in [m]} \{f : S_\ell \to \mathbb{Z}_2 | \sum_{i \in S_\ell} f(i) = b_\ell\}$$

   and vertices $f : S_\ell \to \mathbb{Z}_2$ and $f' : S_k \to \mathbb{Z}_2$ are adjacent if there exists $i \in S_\ell \cap S_k$ such that $f(i) \neq f'(i)$. Prove that the following are equivalent:

   (a) $Mx = b$ has a solution;
   (b) $G(M, b) \cong G(M, 0)$;
   (c) $\alpha(G(M, b)) = m$.

50. Let $\mathcal{A}$ be a $C^*$-algebra with a unit $\mathbf{1}$, and faithful tracial state $s : \mathcal{A} \to \mathbb{C}$. Recall that this means that $s$ is linear, $s(\mathbf{1}) = 1$, $s(xy) = s(yx)$ for all $x, y \in \mathcal{A}$, and $s(x^*x) \geq 0$ for all $x \in \mathcal{A}$ with equality if and only if $x = 0$. Suppose that $p_i \in \mathcal{A}$ for $i \in [k]$ are projections (i.e., $p_i = p_i^* = p_i^2$) such that $p_i p_j = 0$ for $i \neq j$. Show that $\sum_{i=1}^k p_i = \mathbf{1}$ if and only if $\sum_{i=1}^k s(p_i) = 1$.

51. Complete the proof of the reduction of binary linear system games to isomorphism games from class (for the $q$-case) by showing that if $G(M, b)$ has a projective packing of value $m$ (i.e., the number of equations in $Mx = b$), then $Mx = b$ has a finite dimensional quantum solution. It may help to recall from Lecture 9 that having a finite dimensional quantum solution is equivalent to the existence of a particular set of projections satisfying certain constraints.

52. Now check that essentially the same proof works for the $qc$-case by showing that if $G(M, b)$ has a *tracial packing* of value $m$ then $Mx = b$ has a quantum solution. A tracial packing of a graph $G$ is an assignment $g \to E_g \in \mathcal{A}$ of projections to the vertices of $G$ such that $E_g E_{g'} = 0$ for $g \sim g'$ and where $\mathcal{A}$ is a unital $C^*$-algebra that has a faithful tracial state $s$. The value of a tracial packing is $\sum_g s(E_g) = s\left(\sum_g E_g\right)$. Exercise 50 may help here.

53. In class we proved that $Mx = b$ having a quantum solution implies that $G(M, b) \cong_{qc} G(M, 0)$ purely in terms of the games, i.e., we did not take a quantum solution to $Mx = b$ and use this to construct a quantum permutation matrix certifying that $G(M, b) \cong_{qc} G(M, 0)$. However, our proof does implicitly provide a way of doing this. Do this. Does the quantum permutation matrix you construct have a special form? For instance, is it block diagonal? If so, each of the blocks is themselves is a quantum permutation matrix. Do these blocks have some special form? For instance, within a block do all of the entries commute? How many distinct entries are there in each block?

54. Let $G$ be a graph and let $\mathrm{Aut}(G) = \{P \in \mathbb{C}^{V(G) \times V(G)} : P \text{ is a perm mtx } \& \ A_G P = P A_G\}$ be the automorphism group of $G$ viewed as permutation matrices. Define $u_{ij} : \mathrm{Aut}(G) \to \mathbb{C}$ to be the function defined as $u_{ij}(P) = P_{ij}$. Show that the quantum permutation matrix $\mathcal{U} = (u_{ij})$ commutes with $A_G$.

55. Let $G$ be a graph and define $\mathcal{A}(G)$ to be the universal $C^*$-algebra generated by elements $p_{ij}$ for $i, j \in V(G)$ satisfying the relations

   (a) $p_{ij} = p_{ij}^2 = p_{ij}^*$ for all $i, j \in V(G)$;
   (b) $\sum_k p_{ik} = \mathbf{1} = \sum_\ell p_{\ell j}$ for all $i, j \in V(G)$;
   (c) $A_G \mathcal{P} = \mathcal{P} A_G$ where $\mathcal{P} = (p_{ij})$;
   (d) the $p_{ij}$ pairwise commute.

   Recall that in class we saw a proposition stating that there is a $*$-isomorphism $\phi : \mathcal{A}(G) \to C(\mathrm{Aut}(G))$ such that $\phi(p_{ij}) = u_{ij}$ (where $u_{ij}$ is defined as in the above exercise). We did not give a complete proof, but noted that the universal $C^*$-algebra construction of $\mathcal{A}(G)$ implies the existence of a surjective $*$-*homomorphism* $\phi : \mathcal{A}(G) \to C(\mathrm{Aut}(G))$ such that $\phi(p_{ij}) = u_{ij}$. Complete the proof by showing that this $\phi$ must be injective.

56. Prove that the matrix $M^{1,2} \in \mathbb{C}^{V(G) \times V(G)^2}$ defined as $M^{1,2}(e_i \otimes e_j) = \delta_{ij} e_i$ is a $(1,2)$-intertwiner of $\mathrm{Qut}(G)$ by showing that $\mathcal{P} M^{1,2} = M^{1,2} \mathcal{P}^{\otimes 2}$ for any quantum permutation matrix $\mathcal{P}$.

57. Let $\mathcal{P}$ be an $n \times n$ quantum permutation matrix. Show that the matrix $S \in \mathbb{C}^{n^2 \times n^2}$ defined as $S(e_i \otimes e_j) = e_j \otimes e_i$ commutes with $\mathcal{P}^{\otimes 2}$ if and only if the entries of $\mathcal{P}$ pairwise commute.

58. Let $\mathcal{P}$ be an $n \times n$ quantum permutation matrix and let $\psi = \sum_{i=1}^{n} e_i \otimes e_i \in \mathbb{C}^n \otimes \mathbb{C}^n$. Show that $\mathcal{P}^{\otimes 2}\psi = \psi\mathcal{P}^{\otimes 0}$. Recall that $\mathcal{P}^{\otimes 0}$ is a $1 \times 1$ matrix with entry $\mathbf{1}$.