CO452/652: Integer Programming — Winter 2009

Instructor: Chaitanya Swamy

Assignment 1

Due: January 30, 2009 before class

You may use anything proved in class directly. I will maintain a FAQ about the assignment on the course webpage. Acknowledge all collaborators and sources of external help.

Q5, marked (*), is a regular question for *graduate (i.e., CO652) students*, and a bonus question for *undergraduate (i.e., CO452) students*.

Q0: (Do Not Hand This In) Prove the following version of Farkas' Lemma. Exactly one of the following hold:
(1) the system $Ax \leq b$ has a feasible solution, where $A \in \mathbb{R}^{m \times n}$.
(2) there exists $y \in \mathbb{R}^m$ such that $y^T A = 0,\ y \geq 0$ and $y^T b < 0$.

Q1: Let $\{x : Ax \leq b\}$ be non-empty. Show that the following set is convex.

$$S := \{c \in \mathbb{R}^d : (\max\ c^T x \text{ s.t. } Ax \leq b) \leq K\}.$$

(10 marks)

Q2: This question seeks to prove the duality theorem stated in class. Consider the following primal linear program

$$\max\ c^T x \quad \text{s.t.} \quad Ax \leq b, \quad x \in \mathbb{R}^n, \tag{P}$$

where $a_1^T, \ldots, a_m^T$ are the rows of A, and its dual

$$\min\ y^T b \quad \text{s.t.} \quad y^T A = c^T, \quad y \geq 0, \quad y \in \mathbb{R}^m. \tag{D}$$

(a) Prove *weak duality*: if x is a feasible solution to (P) and y is a feasible solution to (D), then $c^T x \leq y^T b$. **(2 marks)**

(b) Suppose (P) and (D) have optimal solutions x^* and y^* respectively. Use part (a) and Farkas' Lemma (in any form) to prove that $c^T x^* = y^{*T} b$. Hence, argue that x^* and y^* are optimal primal and dual solutions iff they are feasible and $y_i^* > 0 \implies a_i^T x^* = b_i$. These latter conditions are called the *complementary slackness* conditions. **(8 marks)**

(c) Finally, prove that (P) has an optimal solution iff (D) has an optimal solution. The statement of part (b) along with this fact is often referred to as *strong duality*. **(5 marks)**

You may use the so called *fundamental theorem of linear programming*, which states that every linear program is either infeasible, or has an optimal solution, or is unbounded.

Q3: Let $P := \{x \in \mathbb{R}^d : Ax \leq b\}$ be a non-empty polyhedron. Prove that the following are equivalent.

(a) A has full column-rank, i.e., $\text{rank}(A) = d$.

(b) There exists $x_0 \in P$ such that for every non-zero $z \in \mathbb{R}^d$, $x_0 + z \notin P$ or $x_0 - z \notin P$ (or both).

(c) There exists $x_0 \in P$ and some vector $c \in \mathbb{R}^d$ such that x_0 is the *unique* optimal solution to the LP: max $c^T x$ s.t. $x \in P$.

(15 marks)

Q4:

(a) Consider the system of inequalities $Ax = b,\ x \geq 0$, where $x \in \mathbb{R}^d$, and assume that it has a feasible solution. Show that there exists a feasible solution x_0 such that the columns of A corresponding to the non-zero components of x_0 are linearly independent (and thus, x_0 has at most rank(A) non-zero components). You may use the result of Q3 if needed. **(8 marks)**

(b) Consider a system of inequalities $Ax \leq b$, where $x \in \mathbb{R}^d$. Show that if this system is infeasible, then there exists a subsystem of at most rank$(A) + 1 \leq d + 1$ inequalities that is infeasible.

(7 marks)

Q5: **(*)** Let $S \subseteq \mathbb{R}^d$ be a set of n points. Prove that there exists a point $z \in \mathbb{R}^d$ (possibly from the set S) such that for every hyperplane H passing through z, there are at least $\frac{n}{d+1}$ points of S contained in both the halfspaces defined by H. That is, for every $\pi \in \mathbb{R}^d$ defining the hyperplane $H := \{x : \pi^T x = \pi^T z\}$ passing through z, we have $|S \cap \{x : \pi^T x \leq \pi^T z\}| \geq \frac{n}{d+1}$ and $|S \cap \{x : \pi^T x \geq \pi^T z\}| \geq \frac{n}{d+1}$. Such a point may be viewed as a "center point" of S. **(15 marks)**

(**Hint:** Argue that for z to be a center point, it is sufficient if z lies in the convex hull of every set $T \subseteq S$ with $|T| > n - \frac{n}{d+1}$. (In fact, this condition is also necessary.) Represent these convex hulls by polyhedra and use the result of Q4(b).)