CO452/652: Integer Programming — Winter 2009

Instructor: Chaitanya Swamy

Assignment 4

Due: April 3, 2009 before class

You may use anything proved in class directly. I will maintain a FAQ about the assignment on the course webpage. Acknowledge all collaborators and sources of external help.

The *undergraduate (i.e., CO452) students* may omit *one* of the following: Q1(c) or Q7(a), or attempt these as bonus questions.

Q1:

(a) Consider the knapsack polytope $P_{\text{Knap}} = P_I$, where $P = \{x \in [0,1]^n : a^T x \le B\}$ with $a \ge 0$ and $B \ge 0$. Recall that we showed that if $C \subseteq \{1, \dots, n\}$ is a dependent set, i.e., $\sum_{i \in C} a_i > B$, then defining $E(C) := C \cup \{j : a_j \ge a_i \text{ for all } i \in C\}$, the inequality $\sum_{i \in E(C)} x_i \le |C| - 1$ is valid for P_{Knap}. Show that this inequality has Chvátal rank at most 1 (relative to P). **(5 marks)**

(b) Let $V = \{1, \dots, n\}$. Consider the TSP polytope for the n-node complete graph:

$$P_{\text{TSP}} = P_I \text{ where } P = \{x \in \mathbb{R}^n : \ x(\delta(v)) = 2 \ \forall v \in V, \quad x(E(S)) \le |S| - 1 \ \forall S \ne \emptyset, V, \quad 0 \le x \le e\}.$$

A *comb* is a subgraph induced by a node-set $(H, W_1, W_2, \dots, W_k)$ satisfying the following conditions: (i) $H \cap W_i \ne \emptyset$, $W_i \setminus H \ne \emptyset$ for all i; (ii) $W_i \cap W_j = \emptyset$ for all $i \ne j$; and (iii) $k \ge 3$ and is odd. The set H is called the handle of the comb and the W_is are called the teeth of the comb. Given a comb $(H, W_1, \dots, W_k)$, prove that the following *comb inequality* is valid for P_{TSP} by showing that it has Chvátal rank at most 1.

$$x(E(H)) + \sum_{i=1}^{k} x(E(W_i)) \le |H| + \sum_{i=1}^{k} (|W_i| - 1) - \left\lceil \frac{k}{2} \right\rceil$$

(5 marks)

(c) Given a graph $G = (V, E)$, consider its stable-set polytope $\text{STAB}(G) := \text{conv}\big(\{\chi^S : S \subseteq V \text{ is a stable set}\}\big) = P_I$, where $P := \{x \in \mathbb{R}^n : x_u + x_v \le 1 \ \ \forall (u,v) \in E, \quad 0 \le x \le e\}$. Let C be a clique of G. Give a tight bound on the Chvátal rank of the clique-inequality $x(C) \le 1$ (which is valid for $\text{STAB}(G)$). That is, you should establish lower and upper bounds on the Chvátal rank that are within a constant factor of each other. (One can in fact compute the Chvátal rank exactly.)

You may use the fact that the *Chvátal closure* of P, which is defined as $P' := P \cap \{x \in \mathbb{R}^n : x \text{ satisfies all rank-1 Chvátal-Gomory inequalities for } P\}$, is a polyhedron, if necessary. Thus, defining $P^{(0)} := P$ and $P^{(i+1)} := \big(P^{(i)}\big)'$, we obtain that each $P^{(k)}$ is a polyhedron. **(5 marks)**

Q2: The purpose of this question is to introduce a notion of duality for integer programs based on (nondecreasing) superadditive functions. (Throughout, when we say that F is superadditive, we also require that $F(0) = 0$.) Let (IP): $\max \ c^T x \ \text{ s.t. } x \in P_I$ be an integer program, where $P = \{x : Ax \le b, \ x \ge 0\} \subseteq \mathbb{R}^n$. Let A have m rows, and $A_j \in \mathbb{R}^m$ denote the j-th column of A.

(a) Prove the following *weak-duality* statement. If $F : \mathbb{R}^m \mapsto \mathbb{R}$ is a nondecreasing superadditive function such that $F(A_j) \geq c_j$ for all $j = 1, \ldots, n$, and x is a feasible solution to (IP), then $c^T x \leq F(b)$. **(2 marks)**

(b) Show that for every inequality $\alpha^T x \leq \beta$ that is valid for P_I, there exists a nondecreasing superadditive function $F : \mathbb{R}^m \mapsto \mathbb{R}$ such that $F(A_j) \geq \alpha_j$ for all $j = 1, \ldots, n$ and $F(b) \leq \beta$.

You may use the fact that every valid inequality for P_I has finite Chvátal rank, and that $P^{(k)}$ (as defined in Q1(c)) is a polyhedron for all $k \geq 0$. **(7 marks)**

(c) Deduce from parts (a) and (b) that if (IP) has an optimal solution, then its optimal value is equal to

$$\min \ F(b) \quad \text{s.t.} \quad F(A_j) \geq c_j \ \forall j = 1, \ldots, n, \quad F(0) = 0, \quad F : \mathbb{R}^m \mapsto \mathbb{R} \text{ is nondecr., superadditive.}$$

(6 marks)

Q3: This question considers a generalization of the lifting procedure (not lift-and-project) described in class for strengthening valid inequalities, where we simultaneously lift more than one variable at a time. Let $P \subseteq [0,1]^n$ be a polyhedron with $P_I \neq \emptyset$. Let $J \subseteq \{1, \ldots, n\}$, and z be a vector in $\{0,1\}^J$. Define $S := \{x \in P_I : x_j = z_j \text{ for all } j \in J\}$ and suppose $S \neq \emptyset$. Suppose $\sum_{j \notin J} \pi_j x_j \leq \delta$ is a valid inequality for S. Consider the set

$$Q := \Big\{\alpha \in \mathbb{R}^J : \ \sum_{j \in J} \alpha_j (x_j - z_j) + \sum_{j \notin J} \pi_j x_j \leq \delta \ \text{ is valid for } P_I\Big\}.$$

Prove that Q is a non-empty polyhedron, and it is pointed iff $\text{proj}_J(P_I)$ is full-dimensional (i.e., $\dim\big(\text{proj}_J(P_I)\big) = |J|$), where $\text{proj}_J(P_I)$ denotes the projection of P_I onto the $(x_j)_{j \in J}$-space. Argue that if $\hat{\alpha}$ is an extreme point of Q, then the inequality $\sum_{j \in J} \hat{\alpha}_j x_j + \sum_{j \notin J} \pi_j x_j \leq \delta + \sum_{j \in J} \hat{\alpha}_j z_j$ defines a face of P_I of dimension at least $\dim\big(\{x \in S : \sum_{j \notin J} \pi_j x_j = \delta\}\big) + |J|$. **(15 marks)**

Q4: Let K be the set of solutions to

$$\begin{aligned} 2x_1 - 2x_2 &\leq 1 && (1) \\ 2x_1 - 2x_2 &\geq -1 && (2) \\ 0 \leq x_1 &\leq 1 && (3) \\ 0 \leq x_2 &\leq 1. && (4) \end{aligned}$$

(a) Using the Balas-Ceria-Cornuéjols lift-and-project method compute $P_1(K)$. Also compute $N(K)$ using the Lovász-Schrijver lift-and-project procedure.

(b) Verify (geometrically) that $P_1(K)$ is indeed equal to $\text{conv}\big(\{x \in K : x_1 \in \{0,1\}\}\big)$, and that $N(K) \subseteq P_1(K) \cap P_2(K)$.

(15 marks)

Q5: Consider the integer program

$$\min \ x_{n+1} \ \text{ s.t. } \ 2x_1 + 2x_2 + \cdots + 2x_n + x_{n+1} = n, \quad x \in \{0,1\}^{n+1}.$$

Prove that a branch-and-bound algorithm that branches by setting a fractional variable to 0 or 1 will require the enumeration of an exponential (in n) number of subproblems when n is odd.

(10 marks)

Q6:

(a) Consider an undirected graph $G = (V, E)$ with distinct vertices s, t and nonnegative edge costs $\{c_e\}$. Call an s-t path is odd if it contains an odd number of edges. Show that one can find an minimum-cost odd s-t-path in time polynomial in the input length. **(7 marks)**

(**Hint:** Let G_1, G_2 be two disjoint copies of G, and $u_i,\ i = 1, 2$ denote the copy of u in G_i. Let G' be the graph obtained by taking the union of G_1 and $G_2 \setminus \{s_2, t_2\}$ (i.e., the graph obtained by removing s_2, t_2, and their incident edges, from G_2). Add suitable edges connecting the nodes of G_1 and G_2, and give these edges suitable costs so that a minimum-cost perfect matching in G', if one exists, corresponds to a minimum-cost odd s-t path in G. You may use the fact that minimum-cost perfect matchings in arbitrary (i.e., not necessarily bipartite) graphs can be computed in polynomial time.)

(b) Given a graph G with nonnegative edge costs $\{c_e\}$, the MAXCUT problem is to find a set $\emptyset \neq S \subsetneq V$ that maximizes $c(\delta(S))$. An odd circuit is a cycle with an odd number of edges and no repeated nodes. Consider the following polyhedron.

$$P \ := \ \{x \in \mathbb{R}^E : \ \ x(C) \leq |C| - 1 \quad \text{for every odd circuit } C; \qquad 0 \leq x \leq e\}.$$

Show that max $c^T x$ s.t. $x \in P_I$ is equal to the optimal value of the MAXCUT problem on G. Show that one can solve max $c^T x$ s.t. $x \in P$ in polynomial time. **(8 marks)**

(c) **(Bonus part)** *There was an error in this question, which has been corrected below.*

A semidefinite program (SDP) is an optimization problem involving a symmetric matrix X that has the following form:

$$\max \quad \sum_{i,j} c_{ij} X_{ij} \quad \text{s.t.} \quad \sum_{i,j} a_{ij}^{(\ell)} X_{ij} \leq b^{(\ell)} \quad \forall \ell = 1, \ldots, k, \qquad X \succeq 0 \qquad \text{(SDP)}$$

where $X \succeq 0$ denotes the constraint that X is required to be positive semidefinite (PSD). Consider the following semidefinite-programming relaxation for the MAXCUT problem.

$$\max \quad \sum_{e=(u,v) \in E} c_e \Big(\frac{1 - z_u^T z_v}{2} \Big) \qquad \text{s.t.} \qquad z_u^T z_u = 1 \quad \text{for all } u \in V. \qquad \text{(MC-SDP)}$$

This is a semidefinite program because if we use X to denote ZZ^T, where Z is an $n \times d$ matrix (for some d) with rows z_u^T for $u = 1, \ldots, n$, then substituting X_{uv} for $z_u^T z_v$, we obtain a problem of the form (SDP). Moreover, if X is a PSD matrix representing a solution to this resulting SDP, then by a well-known result called the *Cholesky decomposition*, we can write $X = ZZ^T$ for some $n \times d$ matrix Z; hence, X encodes a solution to (MC-SDP). (MC-SDP) is a relaxation of the MAXCUT problem, because given any cut $(S, V \setminus S)$ we can set z_u for all $u \in S$ to some common unit vector, and z_v for all $v \notin S$ to the opposite unit vector, so that the objective function of (MC-SDP) evaluates precisely to $c(\delta(S))$.

Now define $K \in \mathbb{R}^{E+V}$ as the set of feasible solutions to the following system.

$$\begin{aligned} & d_e \geq x_u - x_v, \quad d_e \geq x_v - x_u, \qquad d_e \leq x_u + x_v, \quad d_e \leq 2 - x_u - x_v \qquad && \forall e = (u,v) \in E, \\ & \qquad 0 \leq d_e, x_u \leq 1 && \forall e \in E, u \in V. \end{aligned}$$

The integer program max $\sum_e c_e d_e$ s.t. $(d, x) \in \mathbb{Z}(K)$ is a valid formulation for the MAXCUT problem, where x_u indicates which side of the cut (the 0-side or 1-side) u is on, and d_e thus encodes if edge e is cut. Let $M^+(K)$ be the convex set in the higher-dimensional space obtained by applying the semidefinite version of the Lovász-Schrijver procedure to K. Prove that $M^+(K)$ yields a relaxation for MAXCUT that is at least as strong as (MC-SDP) by showing that any point in $M^+(K)$ maps to a solution to (MC-SDP) of no smaller value. (Thus, the maximum value of $c^T d$ over points in $M^+(K)$ is at most the optimal value of (MC-SDP).) **(10 marks)**

Q7: In this question, we compare the Chvátal-Gomory (CG) procedure for generating valid inequalities with the Balas-Ceria-Cournuéjols (BCC) lift-and-project method.

(a) Consider again the sable-set polytope STAB(G) for a graph G, the polyhedron P defined in Q1(c), which we now denote as K, and a clique inequality $x(C) \leq 1$ obtained from a clique C of G. Show that starting with the polyhedron K, one requires at least $|C| - 3$ sequential applications of the BCC lift-and-project method (no matter what sequence of variables is chosen) before we obtain a polyhedron for which this clique inequality is valid. **(5 marks)**

(b) **(Bonus part)** Consider the polyhedron

$$K := \{(x, y) \in \mathbb{R}^2 : \quad x \leq B, \quad x \leq B^2 y, \quad x \geq 0, \quad 0 \leq y \leq 1\},$$

where B is a positive integer. Notice that $x \leq B^2 y$ denotes a big-M constraint that, for integer y, forces $y = 1$ if $x > 0$, and thus, $\mathbb{Z}(P) = \{(0, 0)\} \cup \{(x, 1) : 0 \leq x \leq B, \ x \in \mathbb{Z}\}$. Observe that this big-$M$ constraint can be strengthened to $x \leq By$, that is, $x \leq By$ is valid for K_I. It is easy to see that $P_y(K) = K_I$. (Although we defined the lift-and-project operators in the context pure {0,1}-IPs, one can also apply them to (mixed) IPs where only a subset of the variables are {0,1}-variables. The only difference is that now only the {0,1}-variables x_j are candidates for multiplying our constraint-system by x_j and $(1 - x_j)$; the linearization, and projection steps are unchanged.)

Show however that the Chvátal-rank of $x \leq By$ is at least $\gamma B - \delta$ for some constants $\gamma, \delta, \ \gamma > 0$. **(10 marks)**