Missing bit from the proof of Nash's theorem using Brouwer's fixed-point theorem

Recall the following definitions from lecture.

$$D = \Delta_{S_1} \times \Delta_{S_2} \times \ldots \times \Delta_{S_n}$$
$$\delta_{it}(x) = \mathrm{E}_{s_{-i} \sim x^{-(i)}}\left[u_i(t, s_{-i})\right] - \mathrm{E}_{s \sim x}\left[u_i(s)\right] \qquad \text{for all } i, t \in S_i \text{ and } x \in D$$
$$f\big(x^{(1)}, \ldots, x^{(n)}\big) = \big(y^{(1)}, \ldots, y^{(n)}\big), \quad \text{where} \quad y_t^{(i)} = \frac{x_t^{(i)} + \max\big\{\delta_{it}(x), 0\big\}}{\sum_{t' \in S_i}\Big[x_{t'}^{(i)} + \max\big\{\delta_{it'}(x), 0\big\}\Big]}.$$

Claim 1 *If $\overline{x} = f(\overline{x})$, then $\delta_{it}(\overline{x}) \le 0$ for all $i, t \in S_i$, which implies that $\overline{x}$ is a mixed NE.*

Proof : We observed in lecture that if $\delta_{it}(\overline{x}) > 0$, then by the definition of f, we have $\overline{x}_t^{(i)} = f(\overline{x})_t^{(i)} > 0$. We also observed that $\sum_{t \in S_i} \delta_{it}(\overline{x}) \cdot \overline{x}_t^{(i)} = 0$. To complete the proof, the preceding two observations imply that we cannot have $\delta_{it}(\overline{x}) > 0$ for *all* $t \in S_i$. So there is some $t \in S_i$ with $\delta_{it}(\overline{x}) \le 0$. Now since $\overline{x}_t^{(i)} = f(\overline{x})_t^{(i)}$, by definition of f, this means that we must have

$$1 = \sum_{t' \in S_i}\Big[\overline{x}_{t'}^{(i)} + \max\big\{\delta_{it'}(\overline{x}), 0\big\}\Big] = 1 + \sum_{t' \in S_i} \max\big\{\delta_{it'}(\overline{x}), 0\big\}.$$

This implies that $\delta_{it'}(\overline{x}) \le 0$ for all $t' \in S_i$. ∎