CO759: Approximation and Randomized Algorithms, Spring 2013

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Assignment 2

Due: By July 23, 2013

You may use anything proved in class directly. I will maintain a FAQ about the assignment on the course webpage. *Acknowledge all collaborators and any external sources of help or reference.* To get full credit for the bonus problems, you should not refer to non-course literature (i.e., papers, books, external sources on the Internet etc.), but you may consult the reference books (Williamson-Shmoys, Vazirani, Motwani-Raghavan) listed on the course webpage. All questions carry equal weightage.

Let $f : 2^V \mapsto \mathbb{Z}$ be a (not-necessarily $\{0,1\}$) cut-requirement function with $f(\emptyset) = f(V) = 0$.

(i) f is symmetric if $f(A) = f(V \setminus A)$ for all $A \subseteq V$.

(ii) f satisfies *maximality* if $f(A \cup B) \leq \max\{f(A), f(B)\}$ for all disjoint $A, B \subseteq V$

(iii) f is downwards monotone if $f(A) \geq f(B)$ for all $\emptyset \neq A \subseteq B$.

(iv) f is proper if f is symmetric and satisfies maximality.

A function $f : 2^V \mapsto \mathbb{R}$ is supermodular if $f(A) + f(B) \leq f(A \cap B) + f(A \cup B)$ for all $A, B \subseteq V$; f is weakly supermodular if $f(A) + f(B) \leq \max\{f(A \cap B) + f(A \cup B), f(A \setminus B) + f(B \setminus A)\}$ for all $A, B \subseteq V$.

The cut-covering LP defined by a cut-requirement function f is as follows.

$$\min \quad \sum_e c_e x_e \qquad \text{s.t.} \qquad x\big(\delta(S)\big) \geq f(S) \quad \forall S \subseteq V, \qquad x \geq 0. \tag{P}$$

Q1: Primal-dual schema for network design

(a) **(Do not hand this in)** Let f be a $\{0,1\}$ cut-requirement function satisfying maximality. Show that a set F of edges is feasible for the corresponding cut-covering problem iff we have $f(C) = 0$ for every connected component C of (V, A).

(b) **(Do not hand this in)** Recall that in the generalized Steiner tree problem, we are given terminal sets $T_1, \ldots, T_p \subseteq V$, and we seek a min-cost set F of edges such that each T_i is contained in a component of (V, F). This can be modeled by the following $\{0,1\}$ cut-requirement function: $f^{\mathsf{GST}}(S) = 1$ if there is some T_i such that $S \cap T_i \neq \emptyset, T_i \setminus S \neq \emptyset$. Prove that f^{GST} is proper.

Show that the following $\{0,1\}$ cut-requirement function is proper. We are given sets $C, D \subseteq V$ with $|C| = |D|$, and we define $f(S) = 1$ if $|S \cap C| \neq |S \cap D|$.

(c) Prove the following generalization of a claim stated in class. Let f be a proper function. Then, $f(C) = 0$ implies that $f(A) = f(C \setminus A)$ for all $A \subseteq C$.

(d) Let f be a $\{0,1\}$ downwards-monotone cut-requirement function. Prove that the primal-dual algorithm described in class yields a $\big(2 - \frac{1}{|\{v: f(\{v\})=1\}|}\big)$-approximation algorithm for the corresponding cut-covering problem. Let F be the solution returned, y be the dual solution constructed, and τ be the "time" when the dual-ascent phase of the algorithm stops. Prove that $c(F) \leq 2\sum_{S:f(S)=1} y_S - \tau$.

(e) **(Bonus part)** Does the $2 - \frac{1}{|\{v:f(\{v\})=1\}|}$ guarantee proved in part (d) match the integrality gap of the cut-covering LP for $\{0,1\}$ downwards monotone functions?

Remark 1. The guarantee of the primal-dual algorithm for a $\{0,1\}$ proper function f can be improved to $2 - \frac{2}{\{v:f(\{v\})=1\}|}$, and this does match the integrality gap of the LP for $\{0,1\}$ proper functions; the simplest example is the spanning-tree problem on a cycle.

(f) Given a subset $T \subseteq V$ with $|T|$ even, the *T-join problem* seeks to find a min-cost set F of edges F such that $|\delta_F(v)|$ is odd if $v \in T$, and even if $v \notin T$. Such a set F is called a T-join. For example, when $T = \{s,t\}$, a T-join is is simply an s-t path.

Give a 2-approximation algorithm for this problem.

Q2: LP rounding for network design

(a) **(Do not hand this in)** Prove that the cut-requirement function $f^{\mathsf{SNDP}}(S) = \max_{i \in S, j \notin S} r_{ij}$, where $r_{ij} = r_{ji} \in \mathbb{Z}_+$ for all i,j, modeling the survivable network design problem (SNDP) is weakly supermodular.

(b) Prove that if f is a proper function then f is weakly supermodular.

(c) Let $f : 2^V \mapsto \mathbb{R}$ be weakly supermodular, and F be a set of edges. Prove that the function f' defined by $f'(S) = f(S) - |\delta_F(S)|$ is also weakly supermodular.

(d) The *tree augmentation* problem is defined as follows. We are given a complete undirected graph $G = (V,E)$ with edge costs $\{c_e \geq 0\}_{e \in E}$ and a spanning tree $T = (V,E')$ of G. The goal is to augment T with a min-cost set F of edges so that the graph $(V, E' \cup F)$ is 2-edge connected, that is, there are at least 2 edge disjoint paths in $E' \cup F$ between every pair of nodes.

Devise a 2-approximation algorithm for this problem.

Q3: Extreme points

Given a family $\mathcal{S}$ of sets on a ground set V, we say that a function $f : \mathcal{S} \mapsto \mathbb{R}$ is supermodular on $\mathcal{S}$ if for any two sets $A, B \in \mathcal{S}$ where $A \cap B, A \cup B \in \mathcal{S}$, we have $f(A) + f(B) \leq f(A \cap B) + f(A \cup B)$. Similarly, we say that f is submodular on $\mathcal{S}$ if for any two sets $A, B \in \mathcal{S}$ such that $A \cap B, A \cup B \in \mathcal{S}$, we have $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$.

(a) Let $G = (V,E)$ be an undirected graph. We call a collection $\mathcal{S}$ of subsets of V a *strong crossing family* if whenever $A, B \in \mathcal{S}$, we also have that $A \cap B, A \cup B \in \mathcal{S}$. Consider the following variant of the cut-covering LP, where $\mathcal{S}$ is a strong crossing family and $f : \mathcal{S} \mapsto \mathbb{Z}_+$ is supermodular on $\mathcal{S}$.

$$\min \quad \sum_e c_e x_e \qquad \text{s.t.} \qquad x\big(\delta(S)\big) \geq f(S) \quad \forall S \in \mathcal{S}, \qquad x \geq 0. \tag{C-P}$$

Prove that every extreme-point solution to (C-P) is integral.

(b) Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two laminar families on a ground set V. Let $a, b \in \mathbb{Z}^{\mathcal{L}_1}$ and $\ell, u \in \mathbb{Z}^{\mathcal{L}_2}$ be integer (but not necessarily nonnegative) vectors. Show that all extreme points of the polyhedron defined by the following inequalities is integral.

$$a_S \leq \sum_{v \in S} x_v \leq b_S \quad \forall S \in \mathcal{L}_1, \qquad \ell_S \leq \sum_{v \in S} x_v \leq u_S \quad \forall S \in \mathcal{L}_2.$$

(c) Let $D = (N, A)$ be a directed graph. We say that a family $\mathcal{C}$ of subsets of N is a *crossing family* if for every $A, B \in \mathcal{C}$ such that $A \cap B \neq \emptyset$, $A \cup B \neq N$, we have that $A \cap B, A \cup B \in \mathcal{C}$. Let $\mathcal{C} \subseteq 2^N$ be a crossing family, and $f : \mathcal{C} \mapsto \mathbb{Z}$ be submodular on $\mathcal{C}$. Note that $f(S)$ could be negative. Consider the following LP, where a indexes the arcs in A.

$$\min \quad \sum_a c_a x_a \qquad \text{s.t.} \qquad x\big(\delta^{\text{out}}(S)\big) - x\big(\delta^{\text{into}}(S)\big) \leq f(S) \quad \forall S \in \mathcal{C}, \qquad x \geq 0. \qquad \text{(SF-P)}$$

Feasible solutions to (SF-P) are called *submodular flows for* $\mathcal{C}$. Prove that every extreme point of (SF-P) is integral.

(**Hint:** Given a subfamily $\mathcal{F} \subseteq \mathcal{C}$ of non-crossing tight sets, construct an equivalent laminar family $\mathcal{L}$ with $|\mathcal{L}| = |\mathcal{F}|$.)

Remark 2. A variety of results in combinatorial optimization fall out as special cases of the above result, including integrality of the intersection of two matroid polyhedra and the Lucchesi-Younger theorem.

Q4: Multiway cut and multicut

(a) Given an undirected graph $G = (V, E)$ with edge costs $\{c_e \geq 0\}$, a *Gomory-Hu tree* of G is a tree $H = (V, F)$, where F may contain edges not in E, with weights $\{w_e \geq 0\}_{e \in F}$, with the following property: for any two nodes $s, t \in V$, if e is a minimum-weight edge on the unique s-t path in H, then w_e is the value of the min s-t cut in G and the partition $(S_e, V \setminus S_e)$ of V induced by deleting e from H is a minimum s-t cut. A Gomory-Hu tree can be computed in polynomial time.

Consider the following algorithm for finding a multiway cut in G for the terminal-set T. Find a Gomory-Hu tree $H = (V, F)$ of G. For every pair $s, t \in T$, let $e_{s,t} \in F$ be a minimum-weight edge on the unique s-t path in H, where we break ties by imposing an arbitrary ordering on F. Let C be the set of all such edges, and $(S_1, \ldots, S_p)$ be the partition of V induced by removing C from H. Prove that $\bigcup_{i=1}^p \delta_G(S_i)$ is a multiway cut of cost at most $\big(2 - \frac{2}{|T|}\big) \cdot OPT$, where OPT is the cost of an optimal multiway cut.

(b) The *node multiway cut* problem is as follows. We are again an undirected graph $G = (V, E)$ and a set $T \subseteq V$ of terminals, but now we have costs $\{c_v \geq 0\}_{v \in V}$ on the nodes. The goal is to choose a min-cost set $A \subseteq V \setminus T$ of nodes whose deletion from G disconnects every pair of terminals. The following is a natural LP-relaxation of the problem. Let $\mathcal{P}_{uv}$ denote the collection of all u-v paths in G.

$$\min \quad \sum_{v \notin T} c_v x_v \qquad \text{s.t.} \qquad \sum_{v \in P \setminus T} x_v \geq 1 \quad \forall P \in \mathcal{P}_{s,t}, \forall s, t \in T, \qquad x \geq 0. \qquad \text{(MWC-P)}$$

Show that extreme points of (MWC-P) are half-integral, and use this to obtain a $\big(2 - \frac{2}{|T|}\big)$-approximation algorithm.

Remark 3. Note that the node multiway cut problem is a generalization of the (edge) multiway cut problem since one can create a node in the "middle" of every edge e and set its cost equal to c_e and set the costs of the other nodes, that is, the original nodes of G to be extremely high (essentially infinity). For such an instance, the above LP reduces to the path-covering LP stated in class.

(c) Design a 2-approximation algorithm for the multicut problem on trees.

Remark 4. The previous version asked you to prove the potentially *incorrect* statement that "every extreme point x of the natural path-covering LP described in class has the property that there is some edge e such that $x_e \geq \frac{1}{2}$." I do not know if this is correct, or if there is a counterexample.

Q5: Random edge-contraction algorithm Contract for min-cuts and its variants

(a) **(Do not hand this in)** Give a minimum s-t cut instance on an undirected graph where the number of minimum s-t cuts is exponential in n (the number of nodes).

(b) Give an example of a minimum s-t cut instance on an undirected graph with no (s,t) edges where the probability that Contract returns some minimum s-t cut is exponentially small in n. (Note that we are considering the probability of returning any min s-t cut, not a specific min s-t cut.)

(c) **(Motwani-Raghavan, Exercise 10.9)** Consider running algorithm Contract until the number of nodes is reduced to t, then using an $O(n^3)$-time algorithm to find a min-cut in the reduced graph, and repeating this process as many times as necessary to ensure success probability at least $\frac{1}{2}$. Show that by choosing t suitably, this yields an $O\big(n^{8/3}\big)$ time algorithm, and that no better running time is possible using this approach.

(d) The *minimum k-cut* problem seeks to find a partition of the node-set of a given undirected multigraph G into k non-empty subsets $(S_1, \ldots, S_k)$ so as to minimize $\left|\bigcup_{i=1}^{k} \delta(S_i)\right|$. Modify algorithm Contract suitably to give a devise a randomized algorithm for this problem, and hence obtain an upper bound on the number of distinct minimum k-cuts in an n-node multigraph. Your bound should be polynomial in n for any fixed k.

(e) **(Motwani-Raghavan, Problem 10.13, 10.14(b))** For any $\alpha \geq 1$, an α-approximate min-cut of $G = (V, E)$ is a cut $(S, V \setminus S)$, where $\emptyset \neq S \subsetneq V$, such that $|\delta(S)| \leq \alpha \cdot \min_{\emptyset \neq A \subsetneq V} |\delta(A)|$. Adapt algorithm Contract so that it returns any fixed α-approximate min-cut with non-zero probability. Use this to give an upper bound on the number of α-approximate min-cuts in an n-node multigraph that is polynomial in n for any fixed α.