CO759: Approximation and Randomized Algorithms, Spring 2013

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Assignment 3

Due: By August 12, 2013

You may use anything proved in class directly. I will maintain a FAQ about the assignment on the course webpage. *Acknowledge all collaborators and any external sources of help or reference.* To get full credit for the bonus problems, you should not refer to non-course literature (i.e., papers, books, external sources on the Internet etc.), but you may consult the reference books (Williamson-Shmoys, Vazirani, Motwani-Raghavan) listed on the course webpage. All questions carry equal weightage.

Given two metrics (V, d) and (V', ℓ), where $V' \supseteq V$, we say that ℓ dominates d, or that ℓ is a d-dominating metric, if $d(u, v) \leq \ell(u, v)$ for all $u, v \in V$.

Let $G = (V, E)$ be a complete graph, and $\{c_{uv}\}_{u,v \in V}$ be nonnegative edge costs/capacities. A cut tree $\big(T, \{P_T(x, y)\}_{(x,y)\in T}\big)$ is a spanning tree of V along with an x-y path $P_T(x, y)$ in G for every edge (x, y) of T. For an edge (x, y) of T, define $C_T(x, y) := c\big(\delta(S_T(x, y))\big)$, where $\big(S_T(x, y), V \setminus S_T(x, y)\big)$ is the partition of V induced by deleting (x, y) from T. A cut-tree packing is a convex combination of $\big(\lambda_1; T^1, \{P_{T^1}(x, y)\}_{(x,y)\in T^1}\big), \ldots, \big(\lambda_k; T^k, \{P_{T^k}(x, y)\}_{(x,y)\in T^k}\big)$ of cut-trees, where $\lambda_1, \ldots, \lambda_k \geq 0$ and $\sum_{i=1}^k \lambda_i = 1$. The distortion of such a cut-tree packing is the smallest α such that $\sum_i \lambda_i \big(\sum_{(x,y)\in T^i:(u,v)\in P_{T^i}(x,y)} C_{T^i}(x, y)\big) \leq \alpha c_{uv}$ for all $u, v \in V$.

Q1: Applications of cut-tree packings

(a) **(Do not hand this in)** Use cut-tree packings to give alternate $O(\log n)$-approximation algorithms for the sparsest-cut, and multicut problems.

(b) Give an $O(\log n)$-approximation algorithm for finding a b-balanced cut of minimum value. Note that we seek a "true" approximation algorithm here, that is, the algorithm should return a b-balanced cut whose value is at most $O(\log n)$ times the value of optimal b-balanced cut.

Q2: Metric embeddings

Let $C_n = (V, E)$ denote the simple cycle on n nodes.

(a) Show that any C_n-metric can be isometrically embedded into ℓ_1, that is, any C_n-metric is an ℓ_1-metric, or equivalently, a weighted cut-metric.

(b) **(Williamson-Shmoys, Ex. 8.7)** Let d be the shortest path metric of C_n induced by unit edge lengths. Show that for any d-dominating tree metric (V, T) there must be an edge $(u, v) \in E$ (so $d(u, v) = 1$) such that $T(u, v) \geq n-1$. Do this by showing that among all d-dominating tree metrics with optimal distortion, if (V, T) is the tree metric that has minimum total edge-length $\sum_{(x,y)\in E(T)} T(x, y)$, then T must be a path of vertices of degree 2. Hence, deduce the above statement.

Q3: Tree embeddings

Let (V, d) be metric, and let $(s_1, t_1), \ldots, (s_k, t_k)$ be pairs of distinct elements of V. We showed in class that one can obtain a random tree metric (V', T), where $V' \supseteq V$ such that T dominates d with probability 1, and $\mathrm{E}\big[T(u, v)\big] \le O(\log n) d(u, v)$ for all $u, v \in V$. In this question, we investigate if better distortion is possible if we relax the d-dominating condition to only apply to the (s_i, t_i) pairs, that is, we only require that $d(s_i, t_i) \le T(s_i, t_i)$ for all $i = 1, \ldots, k$. (Note that for ℓ_1-embeddings, this allowed us to improve the distortion from $O(\log n)$ to $O(\log k)$.)

(a) Show that one can efficiently obtain a random tree metric (V', T), where $V' \supseteq V$ such that $d(s_i, t_i) \le T(s_i, t_i)$ for all $i = 1, \ldots, k$ with probability 1, and $\mathrm{E}\big[T(u, v)\big] \le O(\log k) d(u, v)$ for all $u, v \in V$.

(b) Let $\{c_{uv}\}_{u,v \in V}$ be nonnegative costs. Show that one can efficiently obtain a (deterministic) tree metric (V', T), where $V' \supseteq V$, such that $d(s_i, t_i) \le T(s_i, t_i)$ for all $i = 1, \ldots, k$, and $\sum_{u,v} c_{uv} T(u, v) \le O(\log k) \sum_{u,v} c_{uv} d(u, v)$.

Q4: Tree embeddings and cut-tree packings

Let (V, d) be a metric.

(a) **(Williamson-Shmoys, Ex. 15.7)** Let $\{c_{uv}\}_{u,v \in V}$ be nonnegative costs. Let G be the complete graph on V. Suppose we have a polytime algorithm that returns a cut-tree packing with distortion α. Show that one can obtain in polytime a d-dominating tree metric (V, T) such that $\sum_{u,v} c_{uv} T(u, v) \le \alpha \sum_{u,v} c_{uv} d(u, v)$.

(b) **(Williamson-Shmoys, Ex 15.9)** Suppose we have a polytime (deterministic) algorithm that for any nonnegative costs $\{c_{uv}\}_{u,v \in V}$, returns a d-dominating tree metric (V', T), where $V' \supseteq V$, such that $\sum_{u,v} c_{uv} T(u, v) \le O(\log n) \sum_{u,v} c_{uv} d(u, v)$. Show that one can efficiently obtain a random tree metric (V'', T''), where $V'' \supseteq V$, such that T'' dominates d with probability 1, and $\mathrm{E}\big[T''(u, v)\big] \le O(\log n) d(u, v)$ for all $u, v \in V$.