# Min-Max Theorems for Packing and Covering Odd $(u, v)$-trails*

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**Abstract.** We investigate the problem of packing and covering odd $(u, v)$-trails in a graph. A $(u, v)$-trail is a $(u, v)$-walk that is allowed to have repeated vertices but no repeated edges. We call a trail *odd* if the number of edges in the trail is odd. Let $\nu(u, v)$ denote the maximum number of edge-disjoint odd $(u, v)$-trails, and $\tau(u, v)$ denote the minimum size of an edge-set that intersects every odd $(u, v)$-trail.

We prove that $\tau(u, v) \leq 2\nu(u, v) + 1$. Our result is *tight*—there are examples showing that $\tau(u, v) = 2\nu(u, v) + 1$—and substantially improves upon the bound of 8 obtained in [5] for $\tau(u, v)/\nu(u, v)$. Our proof also yields a polynomial-time algorithm for finding a cover and a collection of trails satisfying the above bounds.

Our proof is simple and has two main ingredients. We show that (loosely speaking) the problem can be reduced to the problem of packing and covering odd $(\{u, v\}, \{u, v\})$-trails losing a factor of 2 (either in the number of trails found, or the size of the cover). Complementing this, we show that the odd-$(\{u, v\}, \{u, v\})$-trail packing and covering problems can be tackled by exploiting a powerful min-max result of [2] for packing vertex-disjoint nonzero $A$-paths in group-labeled graphs.

## 1 Introduction

Min-max theorems are a classical and central theme in combinatorics and combinatorial optimization, with many such results arising from the study of packing and covering problems. For instance, *Menger's theorem* [10] gives a *tight* min-max relationship for packing and covering edge-disjoint (or internally vertex-disjoint) $(u, v)$-paths: the maximum number of edge-disjoint (or internally vertex-disjoint) $(u, v)$-paths (i.e., *packing number*) is equal to the minimum number of edges (or vertices) needed to cover all $u$-$v$ paths (i.e., *covering number*); the celebrated *max-flow min-cut theorem* generalizes this result to arbitrary edge-capacitated graphs. Another well-known example is the *Lucchesi-Younger theorem* [8], which shows that the maximum number of edge-disjoint directed cuts equals the minimum-size of an arc-set that intersects every directed cut.

Motivated by Menger's theorem, it is natural to ask whether similar (tight or approximate) min-max theorems hold for other variants of path-packing and path-covering problems. Questions of this flavor have attracted a great deal of

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attention. Perhaps the most prominent results known of this type are Mader's min-max theorems for packing vertex-disjoint $\mathcal{S}$-paths [9, 12], which generalize both the *Tutte-Berge formula* and Menger's theorem, and a further far-reaching generalization of this due to Chudnovsky et al. [2] regarding packing vertex-disjoint non-zero $A$-paths in group-labeled graphs.

We consider a different variant of the $(u, v)$-path packing and covering problems, wherein we impose *parity constraints* on the paths. Such constraints naturally arise in the study of multicommodity-flow problem, which can be phrased in terms of packing odd circuits in a signed graph, and consequently, such odd-circuit packing and covering problems have been widely investigated (see, e.g., [13], Chapter 75). Focusing on $(u, v)$-paths, a natural variant that arises involves packing and covering *odd* $(u, v)$-paths, where a $(u, v)$-path is odd if it contains an odd number of edges. However, there are simple examples [5] showing an unbounded gap between the packing and covering numbers in this setting.

In light of this, following [5], we investigate the min-max relationship for packing and covering odd $(u, v)$-trails. An *odd* $(u, v)$*-trail* is a $(u, v)$-walk with *no repeated edges* and an odd number of edges. Churchley et al. [5] seem to have been the first to consider this problem. They showed that the (worst-case) ratio between the covering and packing numbers for odd $(u, v)$-trails is at most 8—which is in stark contrast with the setting of odd $(u, v)$ paths, where the ratio is unbounded—and at least 2, so there is no tight min-max theorem like Menger's theorem. They also motivate the study of odd $(u, v)$-trails from the perspective of studying *totally-odd immersions*. In particular, determining if a graph $G$ has $k$ edge-disjoint odd $(u, v)$-trails is equivalent to deciding if the 2-vertex graph with $k$ parallel edges has a totally-odd immersion into $G$.

*Our results.* We prove a *tight bound* on the ratio of the covering and packing numbers for odd $(u, v)$-trails, which also substantially improves the bound of 8 shown in [5] for this covering-vs-packing ratio.[1] Let $\nu(u, v)$ and $\tau(u, v)$ denote respectively the packing and covering numbers for odd $(u, v)$-trails. Our main result (Theorem 3.1) establishes that $\tau(u, v) \leq 2\nu(u, v) + 1$. Furthermore, we obtain in polynomial time a certificate establishing that $\tau(u, v) \leq 2\nu(u, v) + 1$. This is because we show that, for any integer $k \geq 0$, we can compute in polynomial time, a collection of $k$ edge-disjoint odd $(u, v)$-trails, or an odd-$(u, v)$-trail cover of size at most $2k - 1$. As mentioned earlier, there are examples showing $\tau(u, v) = 2\nu(u, v) + 1$ (see Fig. 1), so our result *settles* the question of obtaining worst-case bounds for the $\tau(u, v)/\nu(u, v)$ ratio.

Notably, our proof is also simple, and noticeably simpler than (and different from) the one in [5]. We remark that the proof in [5] constructs covers of a certain form; in the full version, we prove a *lower bound* showing that such covers cannot yield a bound better than 3 on the covering-vs-packing ratio.

*Our techniques.* We focus on showing that for any $k$, we can obtain either $k$ edge-disjoint odd $(u, v)$-trails or a cover of size at most $2k - 1$. This follows from two other auxiliary results which are potentially of independent interest.

[1] This bound was later improved to 5 [4, 3, 7]. We build upon some of the ideas in [7].

Our key insight is that one can *decouple* the requirements of parity and $u$-$v$ connectivity when constructing odd $(u,v)$-trails. More precisely, we show that if we have a collection of $k$ edge-disjoint odd $(\{u,v\},\{u,v\})$-trails, that is, odd trails that start and end at a vertex of $\{u,v\}$, and the $u$-$v$ edge connectivity, denoted $\lambda(u,v)$, is at least $2k$, then we can obtain $k$ edge-disjoint odd $(u,v)$-trails (Theorem 3.3). Notice that if $\lambda(u,v) < 2k$, then a min $u$-$v$ cut yields a cover of the desired size. So the upshot of Theorem 3.3 is that it reduces our task to the *relaxed* problem of finding $k$ edge-disjoint odd $(\{u,v\},\{u,v\})$-trails. The proof of Theorem 3.3 relies on elementary arguments (see Section 4). We show that given a fixed collection of $2k$ edge-disjoint $(u,v)$-paths, we can always modify our collection of edge-disjoint trails so as to make progress by decreasing the number of contacts that the paths make with the trails and/or by increasing the number of odd $(u,v)$ trails in the collection. Repeating this process a small number of times thus yields the $k$ edge-disjoint odd $(u,v)$-trails.

Complementing Theorem 3.3 we prove that we can either obtain $k$ edge-disjoint $(\{u,v\},\{u,v\})$-trails, or find an odd-$(\{u,v\},\{u,v\})$-trail cover (which is also an odd-$(u,v)$-trail cover) of size at most $2k-2$ (Theorem 3.2). This proof relies on a powerful result of [2] about *packing and covering nonzero $A$-paths in group-labeled graphs* (see Section 5, which defines these concepts precisely). The idea here is that [2] show that one can obtain either $k$ vertex-disjoint nonzero $A$-paths or a set of at most $2k-2$ vertices intersecting all nonzero $A$-paths, and this can be done in polytime [1, 6]. This is the same type of result that we seek, except that we care about edge-disjoint trails, as opposed to vertex-disjoint paths. However, by moving to a suitable gadget graph where we replace each vertex by a clique, we can encode trails as paths, and edge-disjointness is captured by vertex-disjointness. Applying the result in [2] then yields Theorem 3.2.

*Related work.* Churchley et al. [5] initiated the study of min-max theorems for packing and covering odd $(u,v)$-trails. They cite the question of totally-odd immersions as motivation for their work. We say that a graph $H$ has an *immersion* [11] into another graph $G$, if one can map $V_H$ bijectively to some $U \subseteq V(G)$, and $E_H$ to edge-disjoint trails connecting the corresponding vertices in $U$. (As noted by [5], trails are more natural objects than paths in the context of reversing an edge-splitting-off operation, as this, in general, creates trails.) An immersion is *strong* if the trails do not internally meet $U$, and *weak* otherwise. An immersion is called *totally odd* if all trails are of odd length.

In an interesting contrast to the unbounded gap between the covering and packing numbers for odd $(u,v)$-paths, [14] showed that the covering number is at most twice the *fractional packing number* (which is the optimal value of the natural odd-$(u,v)$-path-packing LP).

The notions of odd paths and trails can be generalized and abstracted in two ways. The first involves *signed graphs* [15], and there are various results on packing odd *circuits* in signed graphs, which are closely related to multicommodity flows (see [13], Chapter 75). The second involves *group-labeled graphs*, for which [2, 1] present strong min-max theorems for packing and covering vertex-disjoint nonzero $A$-paths.

## 2 Preliminaries and notation

Let $G = (V, E)$ be an undirected graph. For $X \subseteq V$, we use $E(X)$ to denote the set of edges having both endpoints in $X$ and $\delta(X)$ to denote set of edges with exactly one endpoint in $X$. For disjoint $X, Y \subseteq V$, we use $E(X, Y)$ to denote the set of edges with one end in $X$ and one end in $Y$.

A $(p, q)$-*walk* is a sequence $(x_0, e_1, x_1, e_2, x_2, \ldots, e_r, x_r)$, where $x_0, \ldots, x_r \in V$ with $x_0 = p$, $x_r = q$, and $e_i$ is an edge with ends $x_{i-1}$, $x_i$ for all $i = 1, \ldots, r$. The vertices $x_1, \ldots, x_{r-1}$ are called the *internal vertices* of this walk. We say that such a $(p, q)$-walk is a:

– $(p, q)$-*path*, if either $r > 0$ and all the $x_i$s are distinct (so $p \neq q$), or $r = 0$, which we call a *trivial path*;
– $(p, q)$-*trail* if all the $e_i$s are distinct (we could have $p = q$).

Thus, a $(p, q)$-trail is a $(p, q)$-walk that is allowed to have repeated vertices but *no repeated edges*. Given vertex-sets $A, B \subseteq V$, we say that a trail is an $(A, B)$-trail to denote that it is a $(p, q)$-trail for some $p \in A, q \in B$. A $(p, q)$-trail is called *odd* (respectively, *even*) if it has an odd (respectively, even) number of edges.

**Definition 2.1.** Let $G = (V, E)$ be a graph, and $u, v \in V$ (we could have $u = v$).
(a) The *packing number for odd* $(u, v)$*-trails*, denoted $\nu(u, v; G)$, is the maximum number of edge-disjoint odd $(u, v)$-trails in $G$.
(b) We call a subset of edges $C$ an *odd* $(u, v)$*-trail cover* of $G$ if it intersects every odd $(u, v)$-trail in $G$. The *covering number for odd* $(u, v)$*-trails*, denoted $\tau(u, v; G)$, is the minimum size of an odd $(u, v)$-trail cover of $G$.

We drop the argument $G$ when it is clear from the context.

For any two distinct vertices $x, y$ of $G$, we denote the size of a minimum $(x, y)$-cut in $G$ by $\lambda(x, y; G)$, and drop $G$ when it is clear from the context. By the max-flow min-cut (or Menger's) theorem, $\lambda(x, y; G)$ is also the maximum number of edge-disjoint $(x, y)$-paths in $G$.

## 3 Main results and proof overview

Our main result is the following *tight* approximate min-max theorem relating the packing and covering numbers for odd $(u, v)$ trails.

**Theorem 3.1.** *Let $G = (V, E)$ be an undirected graph, and $u, v \in V$. For any nonnegative integer $k$, we can obtain in polynomial time, either:*

1. *$k$ edge-disjoint odd $(u, v)$-trails in $G$, or*
2. *an odd $(u, v)$-trail cover of $G$ of size at most $2k - 1$.*

*Hence, we have $\tau(u, v; G) \leq 2 \cdot \nu(u, v; G) + 1$.*

Theorem 3.1 is tight (this was communicated to us by [3]), as can be seen from Fig. 1. The theorem follows readily from the following two results.

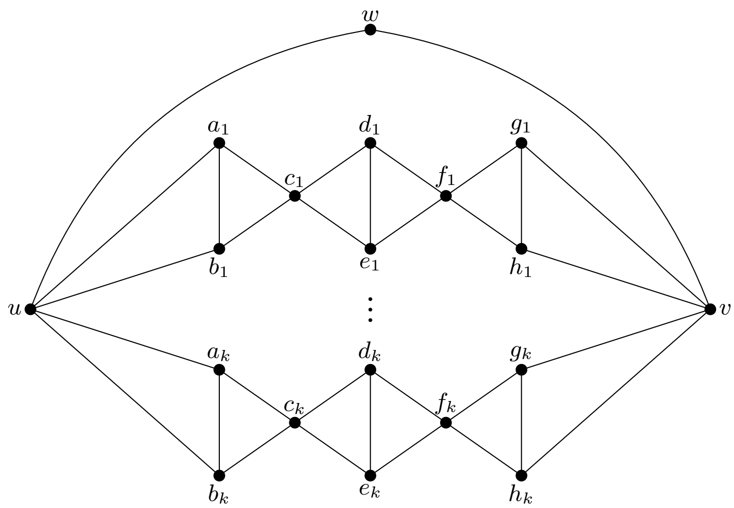



**Fig. 1.** Graph with $\nu(u, v) = k$, $\tau(u, v) = 2k + 1$.

**Theorem 3.2.** *Let $G = (V, E)$ be an undirected graph and $s \in V$. For any nonnegative integer $k$, we can obtain in polynomial time:*
1. *$k$ edge-disjoint odd $(s, s)$-trails in $G$, or*
2. *an odd $(s, s)$-trail cover of $G$ of size at most $2k - 2$.*

**Theorem 3.3.** *Let $G = (V, E)$ be an undirected graph, and $u, v \in V$ with $u \neq v$. Let $\widehat{\mathcal{T}}$ be a collection of edge-disjoint odd $(\{u, v\}, \{u, v\})$-trails in $G$. If $\lambda(u, v) \geq 2 \cdot |\widehat{\mathcal{T}}|$, then we can obtain in polytime $|\widehat{\mathcal{T}}|$ edge-disjoint odd $(u, v)$-trails in $G$.*

*Proof of Theorem 3.1.* If $u = v$, then Theorem 3.2 yields the desired statement. So suppose $u \neq v$. We may assume that $\lambda(u, v) \geq 2k$, since otherwise a minimum $(u, v)$-cut in $G$ is an odd $(u, v)$-trail cover of the required size. Let $E_{uv}$ be the $uv$ edge(s) in $G$ (which could be $\emptyset$). Let $\widehat{G}$ be obtained from $G - E_{uv}$ by identifying $u$ and $v$ into a new vertex $s$. (Note that $\widehat{G}$ has no loops.) Any odd $(u, v)$-trail in $G - E_{uv}$ maps to an odd $(s, s)$-trail in $\widehat{G}$. We apply Theorem 3.2 to $\widehat{G}, s, k' = k - |E_{uv}|$. If this returns an odd-$(s, s)$-trail cover $C$ of size at most $2k' - 2$, then $C \cup E_{uv}$ is an odd-$(u, v)$-trail cover for $G$ of size at most $2k - 2$. If we obtain a collection of $k'$ edge-disjoint odd $(s, s)$-trails in $\widehat{G}$, then these together with $E_{uv}$ yield $k$ edge-disjoint odd $(\{u, v\}, \{u, v\})$-trails in $G$. Theorem 3.3 then yields the required $k$ edge-disjoint odd $(u, v)$-trails. Polytime computability follows from the polytime computability in Theorems 3.2 and 3.3. $\square$

Theorem 3.3 is our chief technical insight, which facilitates the decoupling of the parity and $u$-$v$ connectivity requirements of odd $(u, v)$-trails, thereby driving the entire proof. (It can be seen as a refinement of Theorem 5.1 in [7].) While Theorem 3.2 returns $(\{u, v\}, \{u, v\})$-trails with the right *parity*, Theorem 3.3

supplies the missing ingredient needed to convert these into $(u, v)$-trails (of the same parity). We give an overview of the proofs of Theorems 3.2 and 3.3 below before delving into the details in the subsequent sections. We remark that both Theorem 3.2 and Theorem 3.3 are *tight* as well; we show this in the full version.

The proof of Theorem 3.3 relies on elementary arguments and proceeds as follows (see Section 4). Let $\mathcal{P}$ be a collection of $2 \cdot |\widehat{\mathcal{T}}|$ edge-disjoint $(u, v)$-paths. We provide a simple, efficient procedure to iteratively modify $\widehat{\mathcal{T}}$ (whilst maintaining $|\widehat{\mathcal{T}}|$ edge-disjoint odd $(\{u, v\}, \{u, v\})$-trails) and eventually obtain $|\widehat{\mathcal{T}}|$ odd $(u, v)$-trails. Let $\mathcal{P}_0 \subseteq \mathcal{P}$ be the collection of paths of $\mathcal{P}$ that are edge-disjoint from trails in $\widehat{\mathcal{T}}$. First, we identify the trivial case where $|\mathcal{P}_0|$ is sufficiently large. If so, these paths and $\widehat{\mathcal{T}}$ directly yield odd $(u, v)$-trails as follows: odd-length paths in $\mathcal{P}_0$ are already odd $(u, v)$-trails, and even-length paths in $\mathcal{P}_0$ can be combined with odd $(u, u)$- and odd $(v, v)$- trails to obtain odd $(u, v)$-trails.

The paths in $\mathcal{P} \setminus \mathcal{P}_0$, all share at least one edge with some trail in $\widehat{\mathcal{T}}$. Each path is a sequence of edges from $u$ to $v$. If the first edge that a path $P \in \mathcal{P}$ shares with a trail in $\widehat{\mathcal{T}}$ lies on a $(v, v)$-trail $T$, then it is easy to use parts of $P$ and $T$ to obtain an odd $(u, v)$-trail that is edge-disjoint from all other trails in $\widehat{\mathcal{T}}$, and thereby make progress by increasing the number of odd $(u, v)$-trails in the collection. A similar conclusion holds if the last edge that a path shares with a trail in $\widehat{\mathcal{T}}$ lies on a $(u, u)$-trail. If neither of the above cases apply, then the paths in $\mathcal{P} \setminus \mathcal{P}_0$ are in a sense *highly tangled* (which we formalize later) with trails in $\widehat{\mathcal{T}}$. We then infer that $\mathcal{P} \setminus \mathcal{P}_0$ and $\widehat{\mathcal{T}}$ must satisfy some simple structural properties, and leverage this to carefully modify the collection $\widehat{\mathcal{T}}$ (while preserving edge-disjointness) so that the new set of trails are "less tangled" with $\mathcal{P}$ than $\widehat{\mathcal{T}}$, and thereby make progress. Continuing this procedure a polynomial number of times yields the desired collection of $|\widehat{\mathcal{T}}|$ edge-disjoint odd $(u, v)$-trails.

The proof of Theorem 3.2 relies on the key observation that we can cast our problem as the problem of packing and covering nonzero $A$-paths in a group-labeled graph $(H, \Gamma)$ [2] for a suitable choice of $A, H$, and $\Gamma$ (see Section 5). In the latter problem, (1) $H$ denotes an oriented graph whose arcs are labeled with elements of a group $\Gamma$, and (2) a non-zero $A$-path is a path in the undirected version of $H$ whose ends lie in $A$, whose $\Gamma$-length, which is the sum of $\pm\gamma_e$s (suitably defined) for arcs in $P$, is non-zero. Chudnovsky et al. [2] show that either there are $k$ *vertex-disjoint* non-zero $A$-paths, or there is a vertex-set of size at most $2k - 2$ intersecting every non-zero $A$-path (Theorem 1.1 in [2]). We show that applying their result to a suitable "gadget graph" $H$ (essentially the line graph of $G$), yields Theorem 3.2 (see Section 5). Polytime computability follows because a subsequent paper [1] gave a polytime algorithm for finding a maximum-size collection of vertex-disjoint non-zero $A$-paths, and it is implicit in their proof that this also yields a suitable vertex-covering of non-zero $A$-paths [6].

We remark that while the use of the packing-covering result in [2] yields quite a compact proof of Theorem 3.2, it also makes the resulting proof somewhat opaque since we apply the result in [2] to the gadget graph. However, it is possible to translate the min-max theorem for packing vertex-disjoint nonzero

$A$-paths proved in [2] to our setting and obtain the following more-accessible min-max theorem for packing edge-disjoint odd $(s,s)$-trails (stated in terms of $G$ and not the gadget graph). In the full version, we prove that

$$\nu(s,s;G) = \min\left(|E(S) \setminus F| + \sum_{H \in \mathrm{comp}(G-S)} \left\lfloor \frac{|E(S,H)|}{2} \right\rfloor\right)$$

where the minimum is taken over all bipartite subgraphs $(S,F)$ of $G$ such that $s \in S$. (Notice that Theorem 3.2 follows easily from this min-max formula.)

## 4 Proof of Theorem 3.3: converting edge-disjoint odd $(\{u,v\},\{u,v\})$-trails to edge-disjoint odd $(u,v)$-trails

Recall that $\widehat{\mathcal{T}}$ is a collection of edge-disjoint odd $(\{u,v\},\{u,v\})$-trails in $G$. We denote the subset of odd $(u,u)$-trails, odd $(v,v)$-trails, and odd $(u,v)$-trails in $\widehat{\mathcal{T}}$ by $\widehat{\mathcal{T}}_{uu}$, $\widehat{\mathcal{T}}_{vv}$, and $\widehat{\mathcal{T}}_{uv}$, respectively. Let $k_{uu}(\widehat{\mathcal{T}}) = |\widehat{\mathcal{T}}_{uu}|$, $k_{vv}(\widehat{\mathcal{T}}) = |\widehat{\mathcal{T}}_{vv}|$, and $k_{uv}(\widehat{\mathcal{T}}) = |\widehat{\mathcal{T}}_{uv}|$. To keep notation simple, we will drop the argument $\widehat{\mathcal{T}}$ when its clear from the context. Since we are given that $\lambda(u,v) \geq 2 \cdot |\widehat{\mathcal{T}}|$, we can obtain a collection $\mathcal{P}$ of $2 \cdot |\widehat{\mathcal{T}}|$ edge-disjoint $(u,v)$-paths in $G$. In the sequel, while we will modify our collection of odd $(\{u,v\},\{u,v\})$-trails, $\mathcal{P}$ stays fixed.

We now introduce the key notion of a *contact* between a trail $T$ and a $(u,v)$-path $P$. Suppose that $P = (x_0, e_1, x_1, \ldots, e_r, x_r)$ for some $r \geq 1$.

**Definition 4.1.** A *contact* between $P$ and $T$ is a *maximal subpath* $S$ of $P$ containing at least one edge such that $S$ is also a subtrail of $T$ i.e., for $0 \leq i < j \leq r$, we say that $(x_i, e_{i+1}, x_{i+1}, \ldots, e_j, x_j)$ is a contact between $P$ and $T$ if $(x_i, e_{i+1}, x_{i+1}, \ldots, e_j, x_j)$ is a subtrail of $T$, but neither $(x_{i-1}, e_i, x_i, \ldots e_j, x_j)$ (if $i > 0$) nor $(x_i, e_{i+1}, x_{i+1}, \ldots, x_j, e_{j+1}, x_{j+1})$ (if $j < r$) is a subtrail of $T$.

$$\text{Define} \quad \mathcal{C}(P,T) = \Big|\big\{(i,j) : 0 \leq i < j \leq r, \quad (x_i, e_{i+1}, x_{i+1}, \ldots, e_j, x_j) \text{ is a contact between } P \text{ and } T\big\}\Big|$$

By definition, contacts between $P$ and $T$ are edge disjoint. For an edge-disjoint collection $\mathcal{T}$ of trails, we use $\mathcal{C}(P,\mathcal{T})$ to denote $\sum_{T\in\mathcal{T}} \mathcal{C}(P,T)$. So if $\mathcal{C}(P,\mathcal{T}) = 0$, then $P$ is edge-disjoint from every trail in $\mathcal{T}$. Otherwise, we use the term *first contact* of $P$ to refer to the contact arising from the first edge that $P$ shares with some trail in $\mathcal{T}$ (note that $P$ is a $(u,v)$-walk so is a sequence from $u$ to $v$). Similarly, the *last contact* of $P$ is the contact arising from the last edge that $P$ shares with some trail in $\mathcal{T}$. If $\mathcal{C}(P,\mathcal{T}) = 1$, then the first and last contacts of $P$ are the same. We further overload notation and use $\mathcal{C}(\mathcal{P},\mathcal{T})$ to denote $\sum_{P\in\mathcal{P}} \mathcal{C}(P,\mathcal{T}) = \sum_{P\in\mathcal{P},T\in\mathcal{T}} \mathcal{C}(P,T)$. We use $\mathcal{C}(\mathcal{P},\mathcal{T})$ as a measure of how "tangled" $\mathcal{T}$ is with $\mathcal{P}$. The following lemma classifies five different cases that arise for any pair of edge-disjoint collections of odd $(\{u,v\},\{u,v\})$-trails and $(u,v)$-paths.

**Lemma 4.2.** *Let $\mathcal{T}$ be a collection of edge-disjoint odd $(\{u,v\},\{u,v\})$-trails in $G$. If $|\mathcal{P}| \geq 2 \cdot |\mathcal{T}|$, then one of the following conditions holds.*

*(a) There are at least $k_{uu}(\mathcal{T}) + k_{vv}(\mathcal{T})$ paths in $\mathcal{P}$ that make no contact with any trail in $\mathcal{T}$.*
*(b) There exists a path $P \in \mathcal{P}$ that makes its first contact with a trail $T \in \mathcal{T}_{vv}$.*
*(c) There exists a path $P \in \mathcal{P}$ that makes its last contact with a trail $T \in \mathcal{T}_{uu}$.*
*(d) There exist three distinct paths $P_1, P_2, P_3 \in \mathcal{P}$ which make their first contact with a trail $T \in \mathcal{T}_{uu} \cup \mathcal{T}_{uv}$.*
*(e) There exist three distinct paths $P_1, P_2, P_3 \in \mathcal{P}$ which make their last contact with a trail $T \in \mathcal{T}_{uv} \cup \mathcal{T}_{vv}$.*

*Proof.* To keep notation simple, we drop the argument $\mathcal{T}$ in the proof. Suppose that conclusion (a) does not hold. Then there are at at least $2 \cdot |\mathcal{T}| - (k_{uu} + k_{vv} - 1) = 2k_{uv} + k_{uu} + k_{vv} + 1$ paths in $\mathcal{P}$ that make at least one contact with some trail in $\mathcal{T}$. Let $\mathcal{P}' \subseteq \mathcal{P}$ be this collection of paths. If either conclusions (b) or (c) hold (for some $P \in \mathcal{P}'$), then we are done, so assume that this is not the case. Then, every path $P \in \mathcal{P}'$ makes its first contact with a trail in $\mathcal{T}_{uu} \cup \mathcal{T}_{uv}$ and its last contact with a trail in $\mathcal{T}_{uv} \cup \mathcal{T}_{vv}$. Note that the number of first and last contacts are both at least $2k_{uv} + k_{uu} + k_{vv} + 1 > 2 \cdot \min(k_{uv} + k_{uu}, k_{uv} + k_{vv})$. So if $k_{uu} \leq k_{vv}$, then by the Pigeonhole principle, there are at least 3 paths that make their first contact with some $T \in \mathcal{T}_{uu} \cup \mathcal{T}_{uv}$, i.e., conclusion (d) holds. Similarly, if $k_{vv} \leq k_{uu}$, then conclusion (e) holds. $\square$

We now leverage the above classification and show that in each of the above five cases, we can make progress by "untangling" the trails (i.e., decreasing $C(\mathcal{P}, \mathcal{T})$) and/or increasing the number of odd $(u,v)$-trails in our collection.

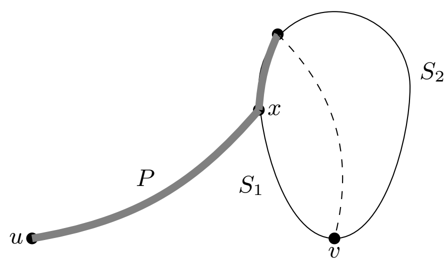
**Lemma 4.3.** *Let $\mathcal{T}$ be a collection of edge-disjoint odd $(\{u,v\},\{u,v\})$-trails. If $|\mathcal{P}| \geq 2 \cdot |\mathcal{T}|$, we can obtain another collection $\mathcal{T}'$ of edge-disjoint odd $(\{u,v\},\{u,v\})$-trails such that at least one of the following holds.*

*(i) $k_{uv}(\mathcal{T}') = |\mathcal{T}|$.*
*(ii) $\mathcal{C}(\mathcal{P}, \mathcal{T}') \leq \mathcal{C}(\mathcal{P}, \mathcal{T})$ and $k_{uv}(\mathcal{T}') = k_{uv}(\mathcal{T}) + 1$.*
*(iii) $\mathcal{C}(\mathcal{P}, \mathcal{T}') \leq \mathcal{C}(\mathcal{P}, \mathcal{T}) - 1$ and $k_{uv}(\mathcal{T}') \geq k_{uv}(\mathcal{T}) - 1$.*

*Proof.* If $k_{uv}(\mathcal{T}) = |\mathcal{T}|$, then (i) holds trivially by taking $\mathcal{T}' = \mathcal{T}$. So we may assume that $\mathcal{T}$ contains some odd $(u,u)$- or odd $(v,v)$-trail. Observe that $\mathcal{T}$ and $\mathcal{P}$ satisfy the conditions of Lemma 4.2, so at least one of the five conclusions of Lemma 4.2 applies. We handle each case separately.

(a) At least $k_{uu}(\mathcal{T}) + k_{vv}(\mathcal{T})$ paths in $\mathcal{P}$ have zero contacts with $\mathcal{T}$. Let $\mathcal{P}_0 = \{P \in \mathcal{P} : \mathcal{C}(P, \mathcal{T}) = 0\}$. Consider some $P \in \mathcal{P}_0$. If $P$ is odd, we can replace an odd $(u,u)$- or odd $(v,v)$- trail in $\mathcal{T}$ with $P$. If $P$ is even, then $P$ can be combined with an odd $(u,u)$- or odd $(v,v)$- trail to obtain an odd $(u,v)$-trail. Since $|\mathcal{P}_0| \geq k_{uu}(\mathcal{T}) + k_{vv}(\mathcal{T})$, we create $k_{uu}(\mathcal{T}) + k_{uv}(\mathcal{T})$ odd $(u,v)$-trails this way, and this new collection $\mathcal{T}'$ satisfies (i).
(b) Some $P \in \mathcal{P}$ makes its first contact with an odd $(v,v)$-trail $T \in \mathcal{T}$. Let the first vertex in the first contact between $P$ and $T$ be $x$. Observe that $x$

partitions the trail $T$ into two subtrails $S_1$ and $S_2$. Since $T$ is an odd trail, exactly one of $S_1$ and $S_2$ is odd. We can now obtain an odd $(u, v)$-trail $T'$ by traversing $P$ from $u$ to $x$, and then traversing $S_1$ or $S_2$, whichever yields odd parity (see Fig. 2). Since $P$ already made a contact with $T$, we have $\mathcal{C}(P, T') \leq \mathcal{C}(P, T)$, and $\mathcal{C}(Q, T') \leq \mathcal{C}(Q, T)$ for any other path $Q \in \mathcal{P}$. Thus, taking $\mathcal{T}' = (\mathcal{T} \cup \{T'\}) \setminus \{T\}$, we have $\mathcal{C}(\mathcal{P}, \mathcal{T}') \leq \mathcal{C}(\mathcal{P}, \mathcal{T})$, and (ii) holds.




$$T_1 = u \xrightarrow[P]{} x \xrightarrow[S_1]{} v$$

$$T_2 = u \xrightarrow[P]{} x \xrightarrow[S_2]{} v$$

**Fig. 2.** Path $P$ makes its first contact with an odd $(v, v)$-trail.

(c) Some $P \in \mathcal{P}$ makes its last contact with an odd $(u, u)$-trail $T \in \mathcal{T}$. This is completely symmetric to (b), so a similar strategy works and we satisfy (ii).

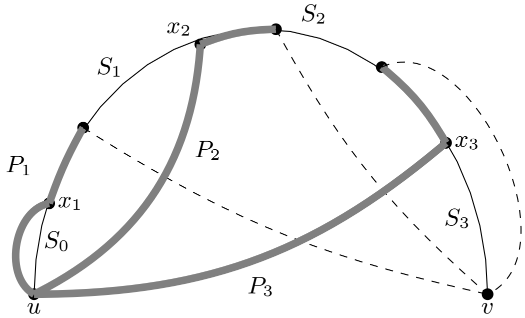
(d) Paths $P_1, P_2, P_3 \in \mathcal{P}$ that make their first contact with an odd $(u, \{u, v\})$-trail $T \in \mathcal{T}$. Note that all contacts between paths in $\mathcal{P}$ and trails in $\mathcal{T}$ are edge disjoint, since the paths in $\mathcal{P}$ are edge disjoint and the trails in $\mathcal{T}$ are edge disjoint. For $i = 1, 2, 3$, let the first vertex in the first contact of $P_i$ (with $T$) be $x_i$. Let $Q_i$ denote the subpath of $P_i$ between $u$ and $x_i$. Note that $T$ is a sequence of edges from $u$ to some vertex in $\{u, v\}$. Without loss of generality, assume that in $T$, the first contact of $P_1$ appears before the first contact of $P_2$, which appears before the first contact of $P_3$. The vertices $x_1, x_2, x_3$ partition the trail $T$ into four subtrails $S_0, S_1, S_2, S_3$ (see Fig. 3). For a trail $X$, we denote the reverse sequence of $X$ by $\overline{X}$. Now consider the following trails (where + denotes concatenation):

$$T_1 = S_0 + \overline{Q}_1, \quad T_2 = Q_1 + S_1 + \overline{Q}_2, \quad T_3 = Q_2 + S_2 + \overline{Q}_3, \quad T_4 = Q_3 + S_3.$$

Observe that the disjoint union of edges in $T_1, T_2, T_3$, and $T_4$ has the same parity as that of $T$, and hence at least one of the $T_i$s is an odd trail; call this trail $T'$. Let $\mathcal{T}' = \mathcal{T} \cup \{T'\} \setminus \{T\}$. By construction, *every* $T_i$ avoids at least one of the (first) contacts made by $P_1$, $P_2$, or $P_3$ (with $T$). Also, for any other path $Q \in \mathcal{P} \setminus \{P_1, P_2, P_3\}$, we have $\mathcal{C}(Q, T') \leq \mathcal{C}(Q, T)$. Therefore, $\mathcal{C}(\mathcal{P}, \mathcal{T}') \leq \mathcal{C}(\mathcal{P}, \mathcal{T}) - 1$. It could be that $T$ was an odd $(u, v)$-trail, which is now replaced by an odd $(u, u)$-trail, so $k_{uv}(\mathcal{T}') \geq k_{uv}(\mathcal{T}) - 1$. So we satisfy (iii).

(e) Paths $P_1, P_2, P_3 \in \mathcal{P}$ make their last contact with an odd $(\{u, v\}, v)$-trail in $\mathcal{T}$. This is symmetric to (d); the same approach works, so (iii) holds. $\square$

Theorem 3.3 now follows by simply applying Lemma 4.3 starting with the initial collection $\mathcal{T}^0 := \widehat{\mathcal{T}}$ until conclusion (i) of Lemma 4.3 applies. The $\mathcal{T}'$ returned by this final application of Lemma 4.3 then satisfies the theorem statement.




$$T_1 = u \xrightarrow[S_0]{} x_1 \xrightarrow[\overline{P}_1]{} u$$

$$T_2 = u \xrightarrow[P_1]{} x_1 \xrightarrow[S_1]{} x_2 \xrightarrow[\overline{P}_2]{} u$$

$$T_3 = u \xrightarrow[P_2]{} x_2 \xrightarrow[S_2]{} x_3 \xrightarrow[\overline{P}_3]{} u$$

$$T_4 = u \xrightarrow[P_3]{} x_3 \xrightarrow[S_3]{} v$$

**Fig. 3.** Paths $P_1, P_2, P_3$ make their first contact with an odd $(u, v)$-trail.

We now argue that this process terminates in at most $2 \cdot |E(G)| + |\widehat{\mathcal{T}}|$ steps, which will conclude the proof. Let $k = |\widehat{\mathcal{T}}|$. Consider the following potential function defined on a collection $\mathcal{T}$ of $k$ edge-disjoint odd $(\{u, v\}, \{u, v\})$-trails: $\phi(\mathcal{T}) := 2 \cdot \mathcal{C}(\mathcal{P}, \mathcal{T}) - k_{uv}(\mathcal{T})$. Consider any iteration where we invoke Lemma 4.3 and move from a collection $\mathcal{T}$ to another collection $\mathcal{T}'$ with $k_{uv}(\mathcal{T}') < k$. Then, either conclusion (ii) or (iii) of Lemma 4.3 applies, and it is easy to see that $\Phi(\mathcal{T}') \leq \phi(\mathcal{T}) - 1$. Finally, we have $-k \leq \Phi(\mathcal{T}) \leq 2 \cdot |E(G)|$ for all $\mathcal{T}$ since $0 \leq \mathcal{C}(\mathcal{P}, \mathcal{T}) \leq |E(G)|$ as the contacts between paths in $\mathcal{P}$ and trails in $\mathcal{T}$ are edge-disjoint, so the process terminates in at most $2|E(G)| + k$ steps.

## 5 Proof of Theorem 3.2

Our proof relies on two reductions both involving non-zero $A$-paths in a group-labeled graph, which we now formally define. A *group-labeled graph* is a pair $(H, \Gamma)$, where $\Gamma$ is a group, and $H = (N, E')$ is an oriented graph (i.e., for any $u, v \in N$, if $(u, v) \in E'$ then $(v, u) \notin E'$) whose arcs are labeled with elements of $\Gamma$. All addition (and subtraction) operations below are always with respect to the group $\Gamma$. A path $P$ in $H$ is a sequence $(x_0, e_1, x_1, \ldots, e_r, x_r)$, where the $x_i$s are distinct, and each $e_i$ has ends $x_i$, $x_{i+1}$ but could be oriented either way (i.e., as $(x_i, x_{i+1})$ or $(x_{i+1}, x_i)$). (So upon removing arc directions, $P$ yields a path in the undirected version of $H$.) We say that $P$ traverses $e_i$ in the direction $(x_i, x_{i+1})$. The *$\Gamma$-length* (or simply length) of $P$, denoted $\gamma(P)$, is the sum of $\pm\gamma_e$s for arcs in $P$, where we count $+\gamma_e$ for $e$ if $P$'s traversal of $e$ matches $e$'s orientation and $-\gamma_e$ otherwise. Given $A \subseteq N$, an $A$-path is a path $(x_0, e_1, \ldots, e_r, x_r)$ where $r \geq 1$, and $x_0, x_r \in A$; finally, call an $A$-path $P$ a nonzero $A$-path if $\gamma(P) \neq 0$ (where 0 denotes the identity element for $\Gamma$).

Chudnovsky et al. [2] proved the following theorem as a consequence of a min-max formula they obtain for the maximum number of nonzero vertex-disjoint $A$-paths. Subsequently, [1] devised a polytime algorithm to compute the maximum number of vertex-disjoint $A$-paths. Their algorithm also implicitly computes the quantities needed in (the minimization portion of) their min-max formula to show the optimality of the collection of $A$-paths they return [6]; this in turn easily yields the vertex-set mentioned in Theorem 5.1.

**Theorem 5.1 ([2, 1]).** *Let $\big(H = (N, E'), \Gamma\big)$ be a group-labeled graph, and $A \subseteq V$. Then, for any integer $k$, one can obtain in polynomial time, either:*

1. *$k$ vertex-disjoint nonzero $A$-paths, or*
2. *a set of at most $2k-2$ vertices that intersects every nonzero $A$-path.*

Recall that $G$ is the undirected graph in the theorem statement, and $s \in V$. For a suitable choice of a group-labeled graph $(H, \Gamma)$, and a vertex-set $A$, we show that: (a) vertex-disjoint nonzero $A$-paths in $(H, \Gamma)$ yield edge-disjoint odd $(s, s)$-trails; and (b) a *vertex-set* covering all nonzero $A$-paths in $(H, \Gamma)$ yields an odd $(s, s)$-trail cover of $G$. Combining this with Theorem 5.1 finishes the proof.

Since we are dealing with parity, it is natural to choose $\Gamma = \mathbb{Z}_2$ (so the orientation of edges in $H$ will not matter). To translate vertex-disjointness (and vertex-cover) to edge-disjointness (and edge-cover), we essentially work with the line graph of $G$, but slightly modify it to incorporate edge labels. We replace each vertex $x \in V$ with a clique of size $\deg_G(x)$, with each clique-node corresponding to a distinct edge of $G$ incident to $x$; we use $[x]$ to denote this clique, both its set of nodes and edges; the meaning will be clear from the context. For every edge $e = xy \in E$, we create an edge between the clique nodes of $[x]$ and $[y]$ corresponding to $e$. We arbitrarily orient the edges to obtain $H$. We give each clique edge a label of 0, and give every other edge a label of 1. Finally, we let $A = [s]$. The proof of the following lemma is straightforward.

**Lemma 5.2.** *The following properties hold.*

(a) *Every $A$-path $P$ in $H$ maps to an $(s, s)$-trail $T = \pi(P)$ in $G$ such that $\gamma(P) = 1$ iff $T$ is an odd trail.*

(b) *If two $A$-paths $P, Q$ are vertex disjoint then the $(s, s)$-trails $\pi(P)$ and $\pi(Q)$ are edge disjoint.*

(c) *Every $(s, s)$-trail $T$ in $G$ with at least one edge maps to an $A$-path $P = \sigma(T)$ in $G$ such that: $T$ is an odd trail iff $\gamma(P) = 1$, and $P$ contains a vertex $x$ iff $T$ contains the corresponding edge of $G$.*

To complete the proof of Theorem 3.2, we apply Theorem 5.1 to the nonzero $A$-paths instance constructed above. If we obtain $k$ vertex-disjoint nonzero $A$-paths in $H$, then parts (a) and (b) of Lemma 5.2 imply that we can map these to $k$ edge-disjoint odd $(s, s)$-trails. Alternatively, if we obtain a set $C$ of at most $2k-2$ vertices of $H$ that intersect every nonzero $A$-path, then we obtain a cover $F$ for odd $(s, s)$-trails in $G$ by taking the set of edges in $G$ corresponding to the vertices in $C$. To see why $F$ is a cover, suppose that the graph $G - F$ has an odd $(s, s)$-trail. This then maps to a nonzero $A$-path $P$ in $H$ such that $P \cap C = \emptyset$ by part (c) of Lemma 5.2, which yields a contradiction.

## 6 Extensions

*Odd trails in signed graphs.* A *signed graph* is a tuple $\big(G = (V, E), \Sigma\big)$, where $G$ is undirected and $\Sigma \subseteq E$. A set $F$ of edges is now called odd if $|F \cap \Sigma|$ is odd. Our results extend to the more-general setting of packing and covering odd $(u, v)$-trails in a signed graph. In particular, Theorems 3.1, 3.2 and 3.3 *hold without any changes*. Theorem 3.2 follows simply because it utilizes Theorem 5.1, which applies to the even more-general setting of group-labeled graphs. Theorem 3.3 holds because it uses basic parity arguments: if we simply replace parity with parity with respect to $\Sigma$ (i.e., instead of parity of $F$, we now consider parity of $|F \cap \Sigma|$), then everything goes through. Finally, as before, combining the above two results yields (the extension of) Theorem 3.1.

*Odd $(C, D)$-trails.* This is the generalization of the odd $(u, v)$-trails setting, where we have disjoint sets $C, D \subseteq V$. Our results yield a factor-2 gap between the the minimum number of edges needed to cover all odd $(C, D)$-trails and the maximum number of edge-disjoint odd $(C, D)$-trails. First, we utilize Theorem 5.1 to prove a generalization of Theorem 3.2 showing that for any integer $k \geq 0$, we can either obtain $k$ edge-disjoint odd $(C \cup D, C \cup D)$-trails, or an odd-$(C \cup D, C \cup D)$-trail cover of size at most $2k - 2$. Next, we observe that Theorem 3.3 can still be applied in this more-general setting to show that if we have a collection $\widehat{\mathcal{T}}$ of $k$ edge-disjoint odd $(C \cup D, C \cup D)$-trails, and (at least) $2k$ edge-disjoint $(C, D)$-paths, then we can obtain $k$ edge-disjoint odd $(C, D)$-trails.

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