STAT 334 Fall 2017 Tutorial 1

Names: SOLUTIONS ____________ ____________

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1. Texas Hold'em Poker: Each player receives 2 cards from a standard deck (4 suits x 13 ranks=52 cards). Then 5 cards are dealt which are shared by all the players. Each player tries to make the best 5-card hand they can from the 7 available cards.
A "straight" is 5 cards in a row, not necessarily of the same suit. E.g. 6, 7, 8, 9, 10.
   a. Suppose you have 7 and 9, and the first three shared cards are 2, 6, 10. What is the probability you get a straight once the remaining 2 shared cards are dealt?

We just ~~need~~ need an 8 on either of the next 2 cards (8, J or 5, 8 is contained in that event)

So P(8 on next) + P(not 8 on next, 8 on last)

$$= \frac{4}{47} + \frac{43}{47} \times \frac{4}{46} = 0.165$$

Alternatively, $1 - P(\text{no 8's in next 2}) = 1 - \frac{\binom{43}{2}}{\binom{47}{2}} = 0.165$

   b. Suppose you have 7 and 9, and the first three shared cards are 2, 6, 8. What is the probability you get a straight once the remaining 2 shared cards are dealt?

We need either a 5 or 10 (similarly 4,5 or 10, J are included)

So P(5 or 10 next) + P(not 5 or 10 next, 5 or 10 last)

$$= \frac{8}{47} + \frac{39}{47} \times \frac{8}{46} = 0.315$$

Alternatively, $1 - P(\text{no 5's or 10's}) = 1 - \frac{\binom{39}{2}}{\binom{47}{2}} = 0.315$

   c. Suppose you have 7 and 9, and the first three shared cards are 2, 5, K. What is the probability you get a straight once the remaining 2 shared cards are dealt?

We need a 6 and an 8, in either order

$$P(6,8) + P(8,6) = \frac{4}{47} \times \frac{4}{46} + \frac{4}{47} \times \frac{4}{46} = 0.015$$

Note: it actually doesn't matter how many players there are! Each of the 52 − 5 = 47 cards you can't see is equally likely to come up next.

2. Dependence of Events: Suppose an event E makes another event F more likely to occur, i.e. P(F|E) > P(F). (For example, if the weather forecast predicts rain, you are more likely to bring an umbrella to school.) Is it always the case that F occurring makes E more likely to occur, i.e. is P(E|F) > P(E)? Does dependence act in both directions?

   a. What is your intuition about the statement: **if P(F|E)>P(F), then P(E|F)>P(E)**

There's no wrong answer to this, but I always found that confusing. Why would you bringing an umbrella make it more likely to rain?

Now let's investigate with some examples. For each experiment below, write down the state space S, the sets E and F, calculate P(E), P(F), P(EF), P(E|F), and P(F|E), and determine if the statement holds.

   b. When rolling a fair 6-sided die, let E = "roll is even" and F = "roll is >3"

$E = \{2,4,6\}$ $F = \{4,5,6\}$ so $EF = \{4,6\}$

$S = \{1,2,3,4,5,6\}$ so $P(E) = \frac{3}{6}$ $P(F) = \frac{3}{6}$ $P(EF) = \frac{2}{6}$

$P(E|F) = \frac{2/6}{3/6} = \frac{2}{3} > P(E)$

$P(F|E) = \frac{2/6}{3/6} = \frac{2}{3} > P(F)$

so both events make each other more likely

   c. When flipping 4 fair coins, let E = "at least 3 Heads" and F = "all flips the same"

$E = \{HHHH, HHHT, HHTH, HTHH, THHH\}$ $F = \{HHHH, TTTT\}$

S has 16 elements. $EF = \{HHHH\}$ So $P(E) = \frac{5}{16}$ $P(F) = \frac{2}{16}$ $P(EF) = \frac{1}{16}$

$P(E|F) = \frac{1/16}{2/16} = \frac{1}{2} > P(E)$

$P(F|E) = \frac{1/16}{5/16} = \frac{1}{5} > P(F)$

so both events make each other more likely

   d. Let's return to the umbrella and rain example. Say you check the weather forecast every day and always bring an umbrella if it predicts rain, which it does 20% of the time. The forecast is wrong 10% of the time. If E = "it rains" and F = "you bring an umbrella" then check the statement as above. Does this match your intuition in part a?

$P(E) = P(\text{rain forecast} \cap \text{correct}) + P(\text{no rain forecast} \cap \text{incorrect})$

$= 0.2 \times 0.9 + 0.8 \times 0.1 = 0.26$

$P(F) = 0.2$ since you bring when forecast $P(EF) = 0.2 \times 0.9 = 0.18$

$P(E|F) = \frac{0.18}{0.2} = 0.9 > P(E)$

$P(F|E) = \frac{0.18}{0.26} = 0.69 > P(F)$

The events make each other more likely here too which doesn't match!

   e. Can you prove the statement?

If $P(F|E) > P(F)$ then $\frac{P(EF)}{P(E)} > P(F)$ by def'n of $P(F|E)$

so $P(EF) > P(F)P(E)$ mult by $P(E)$

so $\frac{P(EF)}{P(F)} > P(E)$ provided $P(F) \neq 0$

i.e. $P(E|F) > P(E)$ ■

Dependent events do always influence each other in the same way (making more or less likely.)

STAT 334 Fall 2017 Tutorial 2

Names: SOLUTIONS ____________ ____________

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1. On each turn, you roll a fair 6-sided die. You collect the puzzle piece matching the number you rolled, if it has not been collected yet.
   a. What is the expected number of turns it takes to get all 6 pieces (in any order)?

   Let X_i represent the time to get the i^{th} piece

   $X_i \sim Geo\left(\frac{6-i+1}{6}\right)$ so $E[X_i] = \frac{6}{6-i+1}$. $X = \sum_{i=1}^{6} X_i$

   $E[X] = E[X_1] + E[X_2] + \cdots + E[X_6]$

   $= 1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + 6 = 14.7$

   b. What is the variance of the number of turns?

   Since X_i's are independent

   $Var(X) = Var(X_1) + \cdots + Var(X_6)$

   $= 0 + \frac{1/6}{(5/6)^2} + \frac{2/6}{(4/6)^2} + \frac{3/6}{(3/6)^2} + \frac{4/6}{(5/6)^2} + \frac{5/6}{(1/6)^2}$

   $= 38.99$

   c. If you have to get the pieces in order (1 - 6), what is the expected value and variance of the number of turns?

   Now each piece takes a $Geo(\frac{1}{6})$ amount of time

   $E[X] = 6 \times 6 = 36$

   $Var(X) = 6 \times \frac{5/6}{(1/6)^2} = 180$

   d. Repeat a, b, and c if it is an n-piece puzzle and you roll a fair n-sided die.

   a. $1 + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{2} + n = n\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + 1\right)$

   b. $\frac{1/n}{(\frac{n-1}{n})^2} + \frac{2/n}{(\frac{n-2}{n})^2} + \cdots + \frac{n-2/n}{(2/n)^2} + \frac{n-1/n}{(1/n)^2}$

   $= n\left(\frac{1}{(n-1)^2} + \frac{2}{(n-2)^2} + \cdots + \frac{n-2}{2^2} + \frac{n-1}{1^2}\right)$

   c. n^2 for mean

   $(n-1)n^2 = n^3 - n^2$ for variance

2. Suppose bacteria are distributed randomly and uniformly through a body of water, with an average concentration of 0.2 bacteria per cubic centimetre (cc) of water.
   a. What is the probability that a water sample of 10 cc has at least 2 bacteria?

Let X = # bacteria in 10cc $X \sim Poi(0.2 \times 10) = Poi(2)$

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$
$$= 1 - \frac{e^{-2}2^0}{0!} - \frac{e^{-2}2^1}{1!}$$
$$= 0.594$$

   b. Suppose 10 independent samples of 10 cc are taken. What is the probability that 6 of the samples have at least 2 bacteria?

Let Y = # of samples with ≥ 2 bacteria $Y \sim Bin(10, 0.594)$

$$P(Y=6) = \binom{10}{6} 0.594^6 \, 0.406^4$$
$$= 0.251$$

   c. Samples of 10 cc are tested until 6 with at least 2 bacteria are found. What is the probability that exactly 10 samples in total are required?

Let Z = # of samples to get 6 with ≥ 2 bactria $Z \sim NB(6, 0.594)$

$$P(Z=10) = \binom{9}{5} 0.594^6 \, 0.406^4$$
$$= 0.150$$

   d. Why is the answer for c) greater than/less than/equal to the answer for b)? Explain logically in one sentence.

The probability in c) is smaller because here the 10^{th} sample must have ≥ 2 bacteria, whereas in b) it could be anything. Since there are fewer valid orderings, the probability is lower.

   e. For a different body of water, you know that the probability that a 10 cc sample has no bacteria is 0.3. What is λ, the average concentration of bacteria per cc, for this body of water?

Let W = # of bacteria in 10 cc $W \sim Poi(10\lambda)$

We know $P(W=0) = e^{-10\lambda}$

and we are told $P(W=0) = 0.3$

Equating and solving, $\lambda = \frac{-\ln(0.3)}{10} = 0.1204$

STAT 334 Fall 2017 Tutorial 3

Names: SOLUTIONS ______ ______

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1. This will give you some practice with the Multinomial Distribution.
   Roll the (biased) die 10 times and write the numbers you get in the boxes below:

   $\boxed{2}$ $\boxed{3}$ $\boxed{3}$ $\boxed{2}$ $\boxed{2}$ $\boxed{1}$ $\boxed{1}$ $\boxed{1}$ $\boxed{2}$ $\boxed{2}$

   How many 1's: 3 How many 2's: 5 How many 3's: 2

   a. What is the probability of observing **your** number of 1's, 2's, and 3's in 10 rolls?

   we know $p_1 = \frac{2}{6} = \frac{1}{3}$ $p_2 = \frac{3}{6} = \frac{1}{2}$ and $p_3 = \frac{1}{6}$

   $$p(3,5,2) = \frac{10!}{3!5!2!}\left(\frac{1}{3}\right)^3\left(\frac{1}{2}\right)^5\left(\frac{1}{6}\right)^2 = 0.0810$$

   b. Write down a general expression for the probability of observing x 1's, y 2's, and $10-x-y$ 3's in 10 rolls. Remember to include the ranges of x and y.

   $$p(x,y,10-x-y) = \frac{10!}{x!\,y!\,(10-x-y)!}\left(\frac{1}{3}\right)^x\left(\frac{1}{2}\right)^y\left(\frac{1}{6}\right)^{10-x-y}$$

   for $0 \le x \le 10$, $0 \le y \le 10$, $x+y \le 10$

   c. What is the probability of observing **your** number of 1's in 10 rolls?

   $$p_x(3) = \binom{10}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^7 = 0.2601$$

   since we don't care the exact # of 2's & 3's, just 7 total

   d. Take your general expression in (b) and sum it over all possible values of y (note: possible values of y depend on the value of x) to obtain the probability of observing x 1's in 10 rolls. Remember to include the range of x.

   $$p_x(x) = \sum_{\text{all } y} p(x,y,10-x-y) = \sum_{y=0}^{10-x} \frac{10!}{x!\,y!\,(10-x-y)!}\left(\frac{1}{3}\right)^x\left(\frac{1}{2}\right)^y\left(\frac{1}{6}\right)^{10-x-y}$$

   $$= \frac{10!}{x!}\left(\frac{1}{3}\right)^x \sum_{y=0}^{10-x} \frac{1}{y!\,(10-x-y)!}\left(\frac{1}{2}\right)^y\left(\frac{1}{6}\right)^{10-x-y}$$ (taking out stuff that doesn't depend any)

   $$= \binom{10}{x}\left(\frac{1}{3}\right)^x \sum_{y=0}^{10-x}\binom{10-x}{y}\left(\frac{1}{2}\right)^y\left(\frac{1}{6}\right)^{10-x-y}$$ (multiplying inside and dividing outside by $(10-x)!$)

   $$= \binom{10}{x}\left(\frac{1}{3}\right)^x\left(\frac{1}{2}+\frac{1}{6}\right)^{10-x}$$ (binomial theorem

   $$= \binom{10}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{10-x}$$ for $0 \le x \le 10$

e. What is the probability of observing **your** number of (combined) 1's and 2's in 10 rolls?

$$p_{X+Y}(8) = \binom{10}{8}\left(\frac{1}{3}+\frac{1}{2}\right)^{8}\left(\frac{1}{6}\right)^{2} = 0.2907$$

since we just want 8 1's + 2's total, don't care how many of each

f. Take your general expression in (b) and use it to obtain the probability of observing *z* (combined) 1's and 2's in 10 rolls. Include the range of *z*.
Hint: Use the result $P(Z=z) = P(X=0, Y=z) + P(X=1, Y=z-1) + \ldots + P(X=z, Y=0)$, or equivalently in summation form, $P(Z=z) = \sum P(X=x, Y=z-x)$ for $x = 0$ to z.

$$p_Z(z) = \sum_{x=0}^{z} p(x, z-x, \underbrace{10-x-(z-x)}_{\text{simplifies to } 10-z})$$

$$= \sum_{x=0}^{z} \frac{10!}{x!(z-x)!(10-z)!}\left(\frac{1}{3}\right)^{x}\left(\frac{1}{2}\right)^{z-x}\left(\frac{1}{6}\right)^{10-z}$$

$$= \frac{10!}{(10-z)!}\left(\frac{1}{6}\right)^{10-z}\sum_{x=0}^{z}\frac{1}{x!(z-x)!}\left(\frac{1}{3}\right)^{x}\left(\frac{1}{2}\right)^{z-x}$$ (taking out stuff that doesn't depend on x)

$$= \binom{10}{z}\left(\frac{1}{6}\right)^{10-z}\sum_{x=0}^{z}\binom{z}{x}\left(\frac{1}{3}\right)^{x}\left(\frac{1}{2}\right)^{z-x}$$ (multiplying inside and dividing outside by z!)

$$= \binom{10}{z}\left(\frac{1}{6}\right)^{10-z}\left(\frac{5}{6}\right)^{z} \quad \text{for } 0 \le z \le 10$$ (binomial theorem)

g. Suppose you are told that the 10 rolls will have **your** number of 1's. What is the probability (given this information) of observing **your** number of 2's in those 10 rolls? Hint: this is a conditional probability and you already have both the numerator and denominator in previous parts.

$$P(5 \text{ 2's} \mid 3 \text{ 1's}) = \frac{p(3,5,2)}{p_X(3)} = \frac{0.0810}{0.2601} = 0.3114$$

since if we want 3 1's and 5 2's, we must also have 2 3's

h. Determine a general expression (and simplify it) for the probability of observing *y* 2's in 10 rolls, given that there are *x* 1's in those 10 rolls. Include the range of *y*.

$$\frac{p(x,y,10-x-y)}{p_X(x)} = \frac{\frac{\cancel{10!}}{\cancel{x!}\,y!(10-x-y)!}\cancel{\left(\frac{1}{3}\right)^{x}}\left(\frac{1}{2}\right)^{y}\left(\frac{1}{6}\right)^{10-x-y}}{\frac{\cancel{10!}}{\cancel{x!}(10-x)!}\cancel{\left(\frac{1}{3}\right)^{x}}\left(\frac{2}{3}\right)^{10-x}}$$

$$= \frac{(10-x)!}{y!(10-x-y)!}\,\frac{\left(\frac{1}{2}\right)^{y}\left(\frac{1}{6}\right)^{10-x-y}}{\left(\frac{2}{3}\right)^{10-x}}$$ split up into $\left(\frac{2}{3}\right)^{y}\left(\frac{2}{3}\right)^{10-x-y}$

$$= \binom{10-x}{y}\left(\frac{1/2}{2/3}\right)^{y}\left(\frac{1/6}{2/3}\right)^{10-x-y}$$

$$= \binom{10-x}{y}\left(\frac{3}{4}\right)^{y}\left(\frac{1}{4}\right)^{10-x-y} \quad \text{for } 0 \le y \le 10-x$$

i. What do you notice about the resulting distributions in d, f, and h?

They are all Binomial rvs! (d) is $\text{Bin}(10, p_1)$
(f) is $\text{Bin}(10, p_1+p_2)$
(h) is $\text{Bin}\left(10-x, \frac{p_2}{1-p_1}\right)$

This makes logical sense because in (d) we can think of S = "1" and F = "not 1", in (f) we can think of S = "1 or 2" and F = "neither", and in (h) we can think of S = "2 given not 1" and F = "3 given not 1" on the $10-x$ remaining rolls.

STAT 334 Fall 2017 Tutorial 4

Names: SOLUTIONS ____________ ____________

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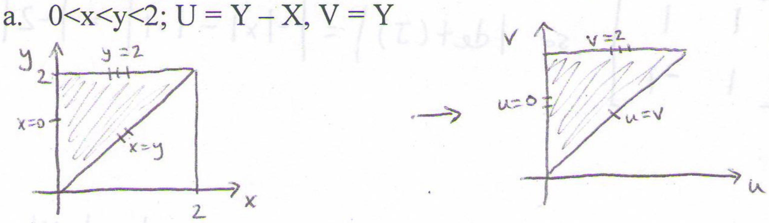
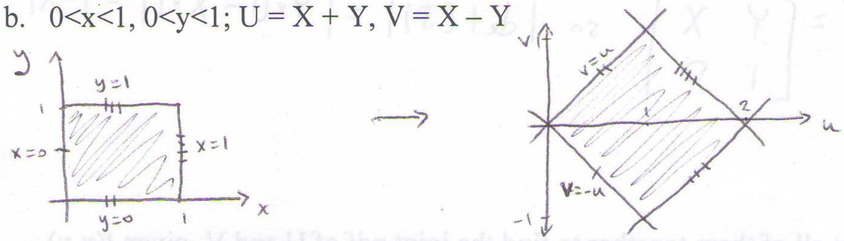
This tutorial will give you some practice with joint transformations, step by step.

1. Step 1: Find (graph) the support of U and V, for the joint transformation of X and Y
   a. $0<x<y<2$; $U = Y - X$, $V = Y$

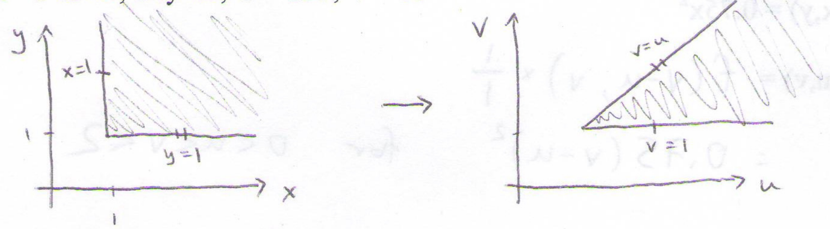


$0 < u < v < 2$

   b. $0<x<1$, $0<y<1$; $U = X + Y$, $V = X - Y$



   c. $1<x<\infty$, $1<y<\infty$; $U = XY$, $V = X$



$1 < v < u < \infty$

2. Step 2: Determine the inverse transformation to get X and Y as functions of U and V
   a.
   $X = V - U$ (since $V - U = Y - (Y - X) = X$)

   $Y = V$ (given)

   b.
   $X = \frac{U+V}{2}$ (since $U + V = X + Y + X - Y = 2X$)

   $Y = \frac{U-V}{2}$ (since $U - V = X + Y - (X - Y) = 2Y$)

   c.
   $X = V$ (given)

   $Y = \frac{U}{V}$ (since $\frac{U}{V} = \frac{XY}{X} = Y$)

3. Step 3: Find the determinant of the Jacobian matrix to divide by for the joint pdf

recall
$U = Y - X$
$V = Y$

a. $J = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ so $|\det(J)| = |-1 \times 1 - 1 \times 0| = |-1| = 1$

$U = X + Y$
$V = X - Y$

b. $J = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ so $|\det(J)| = |-1 \times 1 - 1 \times 1| = |-2| = 2$

$U = XY$
$V = X$

c. $J = \begin{bmatrix} Y & X \\ 1 & 0 \end{bmatrix}$ so $|\det(J)| = |Y \times 0 - X \times 1| = |-X| = X = V$

4. Step 4: put all of them together to find the joint pdf of U and V, given f(x,y)

a. $f(x,y) = 0.75x^2$

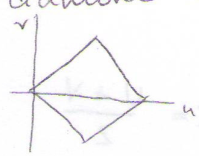
$g(u,v) = f(v-u, v) \times \frac{1}{1}$

$= 0.75(v-u)^2$ for $0 < u < v < 2$

b. $f(x,y) = 1$

$g(u,v) = f\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \times \frac{1}{2}$

$= \frac{1}{2}$ (uniform on the diamond shape)



c. $f(x,y) = \lambda^2(xy)^{-\lambda+1}$

$g(u,v) = f\left(v, \frac{u}{v}\right) \times \frac{1}{v}$

$= \lambda^2 \left(v \frac{u}{v}\right)^{-\lambda+1} \times \frac{1}{v}$

$= \lambda^2 u^{-\lambda+1} v^{-1}$ for $1 < v < u < \infty$

STAT 334 Fall 2017 Tutorial 5

Names: SOLUTIONS ______________ ______________

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This tutorial will give you some practice with conditional expectation and double averaging.

1. Let X = your group number divided by 2 (round up). Let Y = your shoe size in US sizes.
   a. Calculate E[Y|X=x] where x is the value for your group. Note: you will need to combine your information with another group to obtain this number.

      depends on size of groups : $\frac{y_1 + y_2 + \cdots + y_k}{k}$ where k is # people

   b. Calculate P(X=x) where x is the value for your group. Note: you will need the total number of people in the class for this so wait until everyone arrives. Send a representative to report these numbers (a) and (b) to Diana.

      $\frac{\#\text{people with } X=x}{48}$

   c. Given the information on the board, which is the distribution of the random variable E[Y|X], calculate E[Y].

the distribution of E[Y|X] was:

E[Y|X=x]	8.42	8.2	9	6.67	6.9	7.7	8	8.2	8.5
P(X=x)	0.125	0.125	0.104167	0.125	0.104168	0.104167	0.125	0.125	0.0625

$$\text{so } E[Y] = E[E[Y|X]] = \sum_{x=1}^{9} E[Y|X=x]P(X=x)$$

by double averaging

$$= 7.9$$ ← this is also the number you would get if you just averaged all the shoe sizes in the class

2. Let X = your age (integer part only). Let Y = your heart rate.
   a. Find the other people who are the same age as you and calculate E[Y|X=x]

      depends on how many your age

   b. Calculate P(X=x) for your age x. Send a representative to report these numbers (a) and (b) to Diana.

      $\frac{\#\text{people your age}}{48}$

   c. Go back to your original groups and find the distribution of the random variable E[Y|X] by filling in the chart below:

x	19	20	21	22	23	24	25
value E[Y|X=x]	76	75.88	82.5	76.875	60		
probability the rv E[Y|X] takes the value E[Y|X=x]	0.041667	0.5	0.270833	0.166667	0.020833		

   d. Calcuate E[Y]

$$E[Y] = E[E[Y|X]]$$

$$= \sum_{x=19}^{23} E[Y|X=x]\,P(X=x)$$ ← which is the prob. that E[Y|X] equals E[Y|X=x]

$$= 77.5$$ ← this is also the number you would get if you averaged all the heart rates in the class

3. Suppose you roll a 6-sided die until you obtain a 4, and let X = number of rolls it took. Then you flip a fair coin X times, and let Y = number of heads you get.

   a. What is the distribution of X?

      $X \sim Geo(\frac{1}{6})$

   b. What is the conditional distribution of Y|X=x?

      $Y|X=x \sim Bin(x, \frac{1}{2})$

   c. What is its conditional range?

      $\{0, 1, \ldots, x\}$

   d. Find its conditional expectation E[Y|X=x]

      $E[Y|X=x] = x \cdot \frac{1}{2} = \frac{x}{2}$ since mean of $Bin(n,p)$ is np and we have $n = x$ $p = \frac{1}{2}$

   e. What is E[Y|X]?

      $\frac{X}{2}$ just replace x with X. It's a r.v. since it's a function of X.

   f. Find the unconditional expectation of Y, E[Y].

      $E[Y] = E[E[Y|X]]$ using double averaging

      $= E[\frac{X}{2}]$

      $= \frac{1}{2} E[X] = \frac{1}{2} \times 6 = 3$ ← since mean of $Geo(\frac{1}{6})$ is $\frac{1}{1/6} = 6$

   g. Find P(Y ≥ 1|X=x) (in other words, the probability that the variable Y|X=x is ≥ 1)

      $P(Y \geq 1 | X=x) = 1 - P(Y=0 | X=x)$

      $= 1 - \binom{x}{0}(\frac{1}{2})^0(\frac{1}{2})^x$ ← since $Y|X=x \sim Bin(x, \frac{1}{2})$

      $= 1 - (\frac{1}{2})^x$

   h. What is your guess for how we would define P(Y ≥ 1|X)? (Hint: it's similar to e)

      again we would replace x with X to allow x to vary.

      $P(Y \geq 1 | X) = 1 - (\frac{1}{2})^X$

   i. Give an expression for how to find the unconditional probability P(Y ≥ 1). (no calculations needed)

      to remove the information about X, take the expected value like in part (f)

      $P(Y \geq 1) = E[P(Y \geq 1 | X)] = E[1 - (\frac{1}{2})^X] = 1 - E[(\frac{1}{2})^X]$

      ↑ not linear so not equal to $1 - \frac{1}{2^{E[X]}}$

   j. Circle which of the following things are random variables:

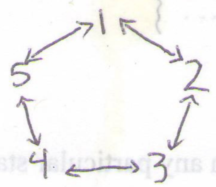
Y|X=x	E[Y|X=x]	E[Y|X]	E[E[Y|X]]	P(Y ≥ 1|X=x)	P(Y ≥ 1|X)
you found in (a) that it has a $Bin(x, \frac{1}{2})$ dist'n	it's a value, $\frac{x}{2}$	it's $\frac{X}{2}$, a function of X.	it's just E[Y], a number	it's a value, $1-(\frac{1}{2})^x$	it's $1-(\frac{1}{2})^X$, a function of X

STAT334 Fall 2017 Tutorial 6

Names: Solutions

This tutorial will give you some practice with Markov chains and transition matrices, and how we can classify the states.

1. Markov Chain 1
   a. Write down the transition probability matrix **P** for this chain.



$$\mathbb{P} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

   b. Can you ever get from state 1 to state 3? Can you ever get from state 3 to state 1?

   Yes (through 2 or 5,4) Yes (through 2 or 4,5)

   c. Imagine you are in state 4. How many steps can it take to return to state 4?

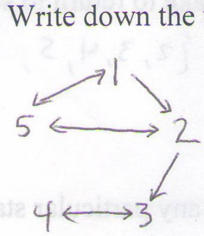
   any even number or any multiple of 5

   so $\{2, 4, 5, 6, \ldots\}$

   d. Do you think it is more likely for the chain to be in any particular state at a random future time? Why or why not?

   No. The chain has all its states behaving the same way, so in the long term no state will be more likely than any other.

2. Markov Chain 2
   a. Write down the transition probability matrix **P** for this chain.



$$\mathbb{P} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$$

   b. Can you ever get from state 1 to state 3? Can you ever get from state 3 to state 1?

   Yes (through 2) No!

c. In what way(s) do states 3 and 4 behave differently from states 1, 2, and 5? (There are at least 3 ways, get as many as you can think of)

- once in 3 or 4, stuck there forever (can get out of 1,2,5)
- no randomness in 3 or 4, go to other state for sure
- 3 and 4 can only be returned to in an even number of steps whereas 1,2,5 can have any number ≥ 2 steps to return
- possible to never return to 1,2,5 but guaranteed to return to 3,4.

d. Imagine you are in state 4. How many steps can it take to return to state 4?

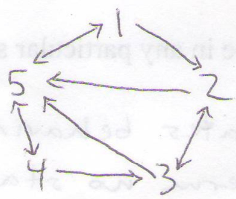
any even number $\{2, 4, 6, \ldots\}$

e. Do you think it is more likely for the chain to be in any particular state at a random future time? Why or why not?

Yes. Once the chain enters state 3 it will go back and forth between 3 and 4 forever, so more likely to be in 3 and 4 vs 1, 2, 5.

3. Markov Chain 3

a. Write down the transition probability matrix **P** for this chain.



$$\mathbb{P} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

b. Can you ever get from state 1 to state 3? Can you ever get from state 3 to state 1?

Yes (through 2) Yes (through 5)

c. Imagine you are in state 4. How many steps can it take to return to state 4?

any number other than 1 $\{2, 3, 4, 5, \ldots\}$

d. Do you think it is more likely for the chain to be in any particular state at a random future time? Why or why not?

Yes. All states go to 5 and there are no other states the chain can get "stuck" in, so 5 is more likely than other states.

Group: 1

STAT 334 Fall 2017 Tutorial 7

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This tutorial will give you some concept review and practice questions for Test 1 on Thursday.

On this side of the page, summarize the most important results from Lectures 1-3 (Sept 7, 12, 14)

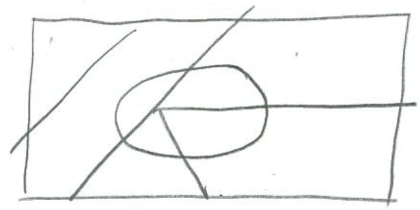
On the other side of the page is a question relating to that material. **Everyone** in the group must be comfortable explaining both sides of the sheet.

Independence: $P(A \cap B) = P(A) \cdot P(B)$

Mutually exclusive: $P(A \cap B) = \emptyset$ $\quad P(A \cup B) = P(A) + P(B)$

Conditional prob: $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

Baye's rule: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A) \cdot P(A)}{P(B)}$



$P(E) = P(EF) + P(E\bar{F})$.

$P(E) = \sum_{i=1}^{k} P(EA_i)$

$= \sum_{i=1}^{k} P(A_i) \cdot P(E|A_i)$.

Discrete distributions:

1. Bernoulli distribution
2. Binomial distribution
3. Poisson distribution
4. Geometric / Negative Binomial distribution

Continuous distributions:

1. uniform distribution
2. exponential distribution
3. normal distribution
4. Gamma distribution

$f(x) = \frac{x^{\alpha-1} e^{-\lambda x} \lambda^{\alpha}}{\Gamma(\alpha)}$

trick: $\int_0^{\infty} \frac{x^{\alpha-1} e^{-\lambda x} \lambda^{\alpha}}{\Gamma(\alpha)} dx = 1$

$\Rightarrow \frac{\Gamma(\alpha)}{\lambda^{\alpha}} = \int_0^{\infty} x^{\alpha-1} e^{-\lambda x} dx$.

Transformation of a random variable: $Y = h(x)$

M1: (cdf)

1. find the range of Y
2. find cummulative density fn for Y and sub X to Y
3. use the cdf of x to evaluate
4. differentiate cdf & get the pdf

M2: (pdf - only for 1-1 transformations

$f(y) = f(g(y)) \cdot |g'(y)|$

where $g(y)$ is the inverse of $h(x)$

1. A call center has 5 phone lines which are independent of each other. For each line, calls arrive at the center following a Poisson process at a rate of 10 calls per hour. There is a red light for each line, and if there have been more than 2 calls in the past 10 minutes on this line, the red light will flash.

   a. For any phone line, find the probability that the red light is flashing.

$\mu = \lambda t = 10 \cdot \frac{10}{60} = \frac{5}{3}.$

$$P(X>2) = 1 - P(X=1) - P(X=0) - P(X=2)$$

$$= 1 - \frac{e^{-\frac{5}{3}} \frac{5}{3}^1}{1!} - \frac{e^{-\frac{5}{3}}(\frac{5}{3})^2}{2!} - \frac{e^{-\frac{5}{3}}(\frac{5}{3})^0}{0!}$$

$$= 0.23.$$

   b. What is the probability that at least two phone lines have the red light flashing?

$Y \sim Bin(6, 0.23)$

$$P(Y \geq 2) = 1 - P(Y=0) - P(Y=1)$$

$$= 1 - \binom{5}{0} 0.23^0 \, 0.77^5 - \binom{5}{1} 0.23 \cdot 0.77^4$$

$$= 0.33.$$

   c. Given that at least two phone lines have the red light flashing, what is the probability that all 5 are flashing?

$$P(Y=5 \mid Y \geq 2)$$

$$= \frac{P(Y=5)}{P(Y \geq 2)}$$

$$= \frac{\binom{5}{5} 0.33^5}{0.33}$$

$$= 0.33^4$$

Group: 2

STAT 334 Fall 2017 Tutorial 7

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This tutorial will give you some concept review and practice questions for Test 1 on Thursday.

On this side of the page, summarize the most important results from Lectures 4-5 (Sept 19, 21)

On the other side of the page is a question relating to that material. **Everyone** in the group must be comfortable explaining both sides of the sheet.

$\Rightarrow X \sim Gam(\alpha,\lambda)$ then $f(x) = \dfrac{x^{\alpha-1}e^{-\lambda x}\lambda^\alpha}{\Gamma(\alpha)}$ where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx = (\alpha-1)!$ if $\alpha = 1, 2, 3$

$\Rightarrow$ Gammatrick:- $\int_0^\infty f(x)\,dx = 1 \Rightarrow$ Multiplying both sides. $\dfrac{\Gamma(\alpha)}{\lambda^\alpha} = \int_0^\infty x^{\alpha-1}e^{-\lambda x}\,dx$.

$\Rightarrow X \sim Exp(\lambda) \Rightarrow E[X] = 1/\lambda \Rightarrow Var(X) = \dfrac{1}{\lambda^2}$.

$\Rightarrow$ Relationship between Gam & Exp:- If α is an integer and $X_i \sim Exp(\lambda)$ then $X = \sum_{i=1}^{\alpha} X_i \sim Gam(\alpha,\lambda)$. $\Rightarrow E(X) = \dfrac{\alpha}{\lambda}$. $\Rightarrow Var(X) = \dfrac{\alpha}{\lambda^2}$.

$\Rightarrow$ Transformations of RV:- Interested in the distribution of a function of X, whose distribut we know of.

1) CDF Technique:- Step 0: Find the range (support of Y). Step 1:- Express the cdf of Y, $F_Y(y) = P(Y \le y)$ in terms of prob involving X (sub in $Y = h(X)$) & isolate X. Step 2: Use the cdf of X to evaluate

2) PDF Technique:- Same as CDF Technique but Step 1: Find inverse transforma Step 2: $f_Y(y) = f_X(g(y)) \cdot |g'(y)|$.

$\Rightarrow$ Moment Generating function:-

$$\Rightarrow M_X(t) = E[e^{tX}] = \underbrace{\sum_{\text{all } x} e^{tx}p(x)}_{\text{discrete}} \text{ or } \underbrace{\int_{\text{all } x} e^{tx}f(x)\,dx}_{\text{continuous}}.$$

$\Rightarrow M_X(0) = 1$.

$\Rightarrow M_X'(0) = E[X]$.

$\Rightarrow M_X''(0) = E[X^2]$

Joint distributions

$p(x,y) = P(X=x, Y=y)$ $\quad \sum_x\sum_y p(x,y) = 1$

marginal $p_X(x) = \sum_y p(x,y)$

$p_Y(y) = \sum_x p(x,y)$

X and Y are indep. iff $p(x,y) = p_X(x)p_Y(y)\ \forall x,$

2. Suppose X and Y are discrete random variables with joint pf p(x,y) given by:

X Y	1	2	3		
1	1/12	1/3	1/4	2/3	$P_X(1)$
2	1/4	0	1/12	1/3	$P_X(2)$
	1/3	1/3	1/3	1	
	$P_Y(1)$	$P_Y(2)$	$P_Y(3)$		

a. Fill in the marginal distributions of X and Y in the table above

b. Find the mgf of Y

$$M_Y(t) = E[e^{Yt}] = \sum_{\text{all } y} e^{Yt} P_Y(y) = \tfrac{1}{3}(e^{1t}) + \tfrac{1}{3}e^{2t} + \tfrac{1}{3}e^{3t}.$$

$$= \tfrac{1}{3}(e^{1t} + e^{2t} + e^{3t})$$

c. Find the covariance of X and Y. Does this value make logical sense?

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$= \sum_{\text{all } x}\sum_{\text{all } y} xy\, p(x,y) - \sum_{\text{all } x} x\, p(x) \sum_{\text{all } y} y\, p(y)$$

$$= \left[\left(1\cdot 1\cdot \tfrac{1}{2}\right) + (1\cdot 2)\tfrac{1}{4} + 2\cdot 1\left(\tfrac{1}{3}\right) + 3\cdot 1\left(\tfrac{1}{4}\right) + 3\cdot 2\left(\tfrac{1}{12}\right)\right]$$

$$- \left[1\cdot\tfrac{2}{3} + 2\cdot\tfrac{1}{3}\right]\left[1\cdot\tfrac{1}{3} + 2\cdot\tfrac{1}{3} + 3\left(\tfrac{1}{3}\right)\right].$$

$$= \tfrac{5}{2} - \tfrac{4}{3}(2) = -\tfrac{1}{6}$$

Yes, it makes logical sense because X & Y move in opposite direction. X ↑, Y ↓

& Y ↑, X ↓

Group: 3

STAT 334 Fall 2017 Tutorial 7

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This tutorial will give you some concept review and practice questions for Test 1 on Thursday.

On this side of the page, summarize the most important results from Lectures 6-7 (Sept 26, 28)

On the other side of the page is a question relating to that material. **Everyone** in the group must be comfortable explaining both sides of the sheet.

Expectation in the Joint Case

- Definition: The expected value of $g(X, Y)$ is:

$$E[g(x,y)] = \begin{cases} \sum_{\text{all } x} \sum_{\text{all } y} g(x,y) f(x,y) \\ \int_{\text{all } x} \int_{\text{all } y} g(x,y) f(x,y)\, dy\, dx \end{cases}$$

Properties of expectation.

If $g(X,Y)$ is a linear function of X and Y. $E[g(X,Y)] = E[a+bX+cY] = a + bE[X] + cE[Y]$

If X and Y are independent, we have $g(X,Y) = h(X)j(Y)$. $E[g(X,Y)] = E[h(X)]E[j(Y)]$.

Covariance and Correlation

$Cov(X,Y) = E[XY] - E[X]E[Y]$. $Corr(X,Y) = \rho = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$ $\quad -1 \le \rho \le 1$

Joint MGFs

$M_{X_1,\dots,X_k}(t_1,\dots,t_k) = E[e^{t_1X_1 + \dots + t_kX_k}] = E[e^{t_1X_1} e^{t_2X_2} \cdots e^{t_kX_k}]$

when independent, $= E[e^{t_1X_1}] \cdots E[e^{t_kX_k}] = M_{X_1}(t_1) \cdots M_{X_k}(t_k)$.

If we want mgf of $X_1 + X_2$, $M_{X_1+X_2}(t) = E[e^{t(X_1+X_2)}] = E[e^{tX_1} e^{tX_2}]$

$= M_{X_1 \dots X_k}(t, t, \dots 0, \dots 0)$

Marginal: $M_{X_1}(t) = M_{X_1,\dots,X_k}(t, 0, 0, \dots, 0)$.

Joint pdf technique

Step 1: Find the support of U, V will depend on the support of X, Y and functions.

Step 2: Find the inverse transformation

Step 3: Find the Jacobian matrix.

$$J = \left| \det \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{bmatrix} \right|$$

$g(u,v) = f(\text{inverse of } x, \text{inverse of } y) \frac{1}{J}$

$$\int x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^{\alpha}}$$

3. Suppose X and Y have joint pdf $f(x,y) = x^2ye^{-xy}$ for $x>0$, $y>2$
   a. Find $f_Y(y)$ and determine if the variables are independent

$$f_Y(y) = \int_{0}^{\infty} x^2 y e^{-xy} dx$$

all x

$$= y \int_0^{\infty} x^2 e^{-xy} dx \qquad \begin{matrix} \lambda = y \\ \alpha = 3 \end{matrix}$$

$$= y \cdot \frac{(3-1)!}{y^3}$$

$$= \frac{2}{y^2}$$

There are no functions which only contain x, that when multiplied with $\frac{2}{y^2}$, gives us $f(x,y)$ so they cannot be independent.

   b. Find E[XY], using a gamma function to simplify your integration

$$E[XY] = \int_2^{\infty}\int_0^{\infty} xy\,(x^2ye^{-xy})\,dx\,dy$$

$$= \int_2^{\infty} y^2 \int_0^{\infty} x^3 e^{-xy}\,dx\,dy \qquad \begin{matrix} \alpha = 4 \\ \lambda = y \end{matrix}$$

$$= \int_2^{\infty} y^2 \frac{\Gamma(4)}{y^4}\,dy$$

$$= \int_2^{\infty} y^2 \frac{3!}{y^4}\,dy$$

$$= \int_2^{\infty} \frac{6}{y^2}\,dy \qquad 6y^{-2}$$

$$= \frac{6y^{-1}}{-1}\Big]_2^{\infty}$$

$$= -\frac{6}{y}\Big]_2^{\infty}$$

$$= \frac{6}{2} = 3$$

Group: 4.

STAT 334 Fall 2017 Tutorial 7

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This tutorial will give you some concept review and practice questions for Test 1 on Thursday.

On this side of the page, summarize the most important results from Lectures 8-9 (oct 3, 5)

On the other side of the page is a question relating to that material. **Everyone** in the group must be comfortable explaining both sides of the sheet.

oct 3rd lec 8

Conditional R.V.

- $Y|X=x$
- fixed x and Y takes on values in its conditional range
- conditional range (possible value of Y, knowing $X=x$)
- random variable which has a conditional range, conditional pf/pdf, conditional mean + variance, conditional mgf

Conditional mean/expectation

$E[Y|X=x] = \sum_{\text{all } y \text{ in conditional range}} y\,P(y|x)$

- if X + Y are independent. $P(y|x)=P_Y(y)$

$E[Y|X=x] = \sum_{\text{all } y \text{ in range}} y(P_Y(y)) = E(Y)$

continuous case

$f(x|y) = \dfrac{f(x,y)}{f_Y(y)}$

cond pdf; joint pdf; marginal pdf

oct 5th lec 9

$E[Y|X]$ is the random variable that takes the value $E[Y|X=x]$ exactly when X takes the value of x.

properties of conditional Expectation

1) Linearity

$E[a+bY+cZ|X=x] = a+bE[Y|X=x] + cE[Z|X=x]$

2) independence

$E[g(Y)|X=x] = E[g(Y)]$

3) Double Averaging

$E[E[Y|X]] = E[Y]$

4. Suppose X is the number of trials (with probability of Success p) to obtain a Success. A fair coin is then flipped until X heads occur, and the number of flips required is called Y.
   a. Consider the conditional random variable Y|X=x
      i. What distribution does it have?

$X \sim Geo(p)$

$Y|X \sim NB(X, \tfrac{1}{2})$

$E[Y|X=x] = \frac{x}{1/2} = 2x$

$\cancel{V[Y|X=x] = 2}$

      ii. What is its conditional range?

$[x, \infty)$ integers

      iii. What is its conditional mean E[Y|X=x]?

$E[Y|X=x] = \frac{x}{1/2} = 2x$

   b. What is E[Y|X]?

$E[Y|X] = 2X$

   c. Find E[Y].

$E[E[Y|X]] = E[2X] = 2E[X] = 2\left[\frac{1}{p}\right] = \frac{2}{p}$

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Group: 5 .

STAT 334 Fall 2017 Tutorial 7

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This tutorial will give you some concept review and practice questions for Test 1 on Thursday.

On this side of the page, summarize the most important results from Lectures 10 - first half of 11 (Oct 17, 19)

On the other side of the page is a question relating to that material. **Everyone** in the group must be comfortable explaining both sides of the sheet.

1) Computing probabilities of Conditioning:

$$P(A) = E[P(A|X)] = \begin{cases} \sum_{\text{all } x} P(A|X=x)P(X=x) & \text{if } X \text{ is discret} \\ \int_{\text{all } x} P(A|X=x)f_X(x)\,dx & \text{if " " continio} \end{cases}$$

partition rule applied

2) compute Var by cond.

$Var(Z) = E[Z^2] - E[Z]^2$.

$Var(Y|X=x) = E[Y^2|X=x] - E[Y|X=x]^2$ use same trick allowing x to vary over all X.

Law of total Var.

$Var(Y) = E[Var(Y|X)] + Var[E[Y|X]]$

3) Compound R.Vs.

↳ combo of two R.Vs

$S = X_1 + X_2 + \cdots + X_N$

$E[S] = E[N]E[X]$

$Var(S) = E[N]Var(X) + E[X]^2 Var(N)$

4) Mgf of compound R.Vs.

$M_S(t) = M_N(\ln(M_X(t)))$ from: $E[e^{tS}] =$

5. Suppose M ~ U(10,30). Errors occur according to a Poisson process with rate M errors per hour, and the number of errors observed in an hour is called X.
Thus, X|M=m ~ Poi(m)
   a. Find E[X].

$E[X] = E[E[X|M]]$ since $X|M=m \sim Poi(m)$

$= E[M]$ $\quad X|M \sim Poi(M)$

$= 20.$ $\quad \Rightarrow E(X|M) = M$

$M \sim U(10,30)$

$E[M]$

   b. Find Var(X)

$Var(X) = E[Var(X|M)] + Var(E[X|M])$

$= E[M] + Var(M)$

$= 20 + \frac{1}{12}(20)^2$

$= 20 + \frac{400}{12}$

$= 53.\overline{33}$

   c. Suppose for each error, the cost to fix it has mean 5 and variance 9. Find the mean and variance of the total cost to fix all errors in an hour.

$E[S]$ $\quad C = \text{cost}$

$= E[X]\, E[C]$ $\quad X = \#\text{ of errors}$

$= 20 \times 5 = 100$

$Var(S)$

$= E[X]\, Var(C) + E[C]^2\, Var[X]$

$= 20 \times 9 + 5^2 \times 53.33.$

$= 1513.33.$

Group: 6 .

STAT 334 Fall 2017 Tutorial 7

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This tutorial will give you some concept review and practice questions for Test 1 on Thursday.

On this side of the page, summarize the most important results from Lectures second half of 11 – first half of 13 (Oct 19, 2

On the other side of the page is a question relating to that material. **Everyone** in the group must be comfortable explaining both sides of the sheet.

Stochastic process: sequence . r.v index . time

State space: value X_n's can take on

Markov property. $P(X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \dots, X_0=i_0)$

$$P(X_{n+1}=j \mid X_n=i) = P(X_{n+2}=j \mid X_{n+1}=i) = p_{ij}$$

Transition probability matrix (tpm) $\mathbb{P} = \begin{bmatrix} p_{11} & p_{12} & \cdot & p_{1K} \\ p_{21} & & & \\ \vdots & & & \\ p_{K1} & \cdots & & p_{KN} \end{bmatrix}$ ①

$\mathbb{P}^2$

$\sum p_{1K}$

property of tpm

1. $\forall p_{ij} \geq 0 \quad \forall i,j$
2. $\sum_{j=1} p_{ij} = 1$

1 – K 1

2'

Chapman-Kolmogorov Equations (CK)

$$p_{ij}^{(n)} = P(X_n=j \mid X_0=i)$$

$$\mathbb{P}^{(n)} = \mathbb{P}^n$$

initial probability vector: $\underline{\pi_0} = [\,(P(X_0=1)\ P(X_0=2) \cdots P(X_0=k)]$

$$\underline{\pi_n} = \underline{\pi_0}\,\mathbb{P}^n$$

6. Suppose the weather today depends on the weather yesterday in the following way:
   - If it was Rainy yesterday, it will be Rainy today with probability 0.5.
   - If it was Sunny yesterday, it will be Sunny today with probability 0.4.
   - If it was Cloudy yesterday, it will be Cloudy today with probability 0.2

   all other transitions not mentioned are equally likely.

   a. If the state space is S = {R, S, C}, write down the one-step transition matrix **P**

   Let $R \to 0$
   $S \to 1$
   $C \to 2$

   $$\mathbb{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix}$$

   $$= \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.4 & 0.2 \end{bmatrix}$$

   b. Suppose you can't remember for sure, but you think it either was Rainy or Cloudy yesterday (with equal probability). Find the distribution of the weather today.

   $$\underline{\pi_0} = [\,0.5 \quad 0 \quad 0.5\,]$$

   $$\underline{\pi_1} = [\,0.5 \quad 0 \quad 0.5\,]\,\mathbb{P} = [\,0.45 \quad 0.325 \quad 0.225\,]$$

   c. Same situation as b, now find the probability the weather tomorrow is the same as today.

   $$\pi_{2same} = \underline{\pi_1} * P = \begin{bmatrix} P_{00} & & 0 \\ & P_{11} & \\ 0 & & P_{22} \end{bmatrix}$$

   $$= [\,0.225 \quad 0.13 \quad 0.045\,]$$

   $$P(same) = 0.225 + 0.13 + 0.045 = 0.4$$

STAT 334 Fall 2017 Tutorial 9

Names: Solutions __________ __________

IDs: __________

This tutorial will give you some practice working with the properties of the Poisson process.

1. Suppose calls to 911 follow a Poisson process with an average of 5 calls per minute. Unfortunately, 20% of calls are for non-emergencies.

   a. Discuss briefly whether you think the properties of stationary and independent increments would hold for this situation

   stationary – probably not, there would likely be more calls during the day when people are awake

   independent – probably OK, unless there is a major disaster that many people call about

   b. Interpret in one sentence what these probabilities represent, and calculate them:

   i. $P(N(1) = 3)$ The probability of 3 calls in 1 minute

   $$= \frac{e^{-5}5^3}{3!} = \boxed{0.140}$$

   ii. $P(N(4) = 12 \mid N(2) = 7)$ The probability of 12 calls in 4 minutes given there were 7 in the first 2 min.

   $$= P(N(4) - N(2) = 5)$$

   $$= \frac{e^{-10}10^5}{5!} = \boxed{0.0378}$$

   iii. $P(N(4) = 12 \mid N(2) = 7, N(1) = 2)$ The probability of 12 calls in 4 min given 7 in first 2 min, 2 of which were in first min.

   same as ii.

   can ignore past if you know present. (Markov property!)

   iv. $P(N(1.5) = 0 \mid N(2) = 1)$ The probability of no calls in first 1.5 minutes given 1 call by 2 min.

   $$T_1 \mid N(2) = 1 \sim U(0, 2)$$

   so simply $1 - \frac{1.5}{2} = \boxed{0.25}$

   v. $P(T_1 > 1/3)$ The probability the first call happens after $\frac{1}{3}$ min

   $$T_1 \sim Exp(5)$$

   $$P(T_1 > \tfrac{1}{3}) = 1 - F(\tfrac{1}{3}) = 1 - (1 - e^{-5 \times \frac{1}{3}}) = e^{-\frac{5}{3}} = \boxed{0.1889}$$

   c. What do you think is the distribution of the number of **emergency** calls in the first 10 minutes?

   $$Poi(5 \times 0.80 \times 10) = Poi(40)$$

   (we will prove this in the next class)

2. Diana's children, Naomi and Isaac, each wake up at night according to independent Poisson processes. Naomi wakes up an average of once every 8 hours, and Isaac wakes up an average of once every 4 hours. Each time either of them wakes up, Diana wakes up too.

   a. Prove that the total number of times Diana wakes up follows a Poisson process as well, and determine the rate λ. Remember there are several different definitions of a Poisson process, so you can choose which one to use.

If $\{N(t)\}$ and $\{I(t)\}$ are Poisson processes, then $D(t) = N(t) + I(t)$ inherits the properties of stationary + independent increments and $D(0) = 0$.

$\{N(t)\}$ has rate $\frac{1}{8}$ and $\{I(t)\}$ has rate $\frac{1}{4}$

~~so~~ so $N(t) \sim Poi(\frac{1}{8}t)$ and $I(t) \sim Poi(\frac{1}{4}t)$

so $D(t) \sim Poi(\frac{1}{8}t + \frac{1}{4}t) = Poi(\frac{3}{8}t)$ since the sum of indep. Poi rvs is Poisson. So $\lambda = \frac{3}{8}$

   b. Find probability that Diana wakes up at least once between 11:00 PM and 7:00 AM (8 hours)

$X = D(8) \sim Poi(\frac{3}{8} \times 8) = Poi(3)$

$P(X \geq 1) = 1 - P(X=0)$

$= 1 - \frac{e^{-3}3^0}{0!} = \boxed{0.9502}$

   c. Find the probability that the second time Diana wakes up (after going to sleep at 11:00 PM) is after 3:00 AM but before 5:00 AM

We need $D(4) = 1$ and $D(6) \geq 2$ to both happen

so $P(D(4)=1)P(D(6)-D(4) \geq 1)$

$\left(\frac{e^{-1.5}(1.5)^1}{1!}\right)\left(1 - \frac{e^{-0.75}(0.75)^0}{0!}\right) = \boxed{0.1766}$

   d. If you know Diana has woken up 3 times between 11:00 PM and 7:00 AM (8 hours), what do you think is the distribution of the number of times she woke up between 11:00 PM and 3:00 AM (first 4 hours)?

$Bin(3, 0.5)$

(we will prove this in the next class too)

STAT 334 Fall 2017 Tutorial 10

Names: Solutions

IDs:

This tutorial will give you an introduction and motivation for Chapter 6: Brownian Motion

1. Consider the fair Random Walk, a Markov Chain with countably infinite state space and:
   $Z(0) = 0$
   $Z(t) = X_1 + X_2 + ... + X_t$ for integer t
   $X_t = \begin{cases} +1 \, with\, prob\, \frac{1}{2} \\ -1 \, with\, prob\, \frac{1}{2} \end{cases}$
   a. Using a coin to generate random moves (Heads = +1, Tails = –1) and the sheet of graph paper provided, plot a sample path of Z(t) for t = 0 to 10.
   b. What are the mean and variance of Z(10)? (Hint: find the mean and variance of Xt and use the results for sums of independent variables)
   $E[X_t] = 1 \times \frac{1}{2} + (-1) \times \frac{1}{2} = 0$
   $Var(X_t) = E[X_t^2] - 0^2 = 1^2 \times \frac{1}{2} + (-1)^2 \times \frac{1}{2} = 1$
   $Z(10) = \sum_{t=1}^{10} X_t$ so $E[Z(10)] = \sum_{t=1}^{10} E[X_t] = 0$, $Var(Z(10)) = \sum_{t=1}^{10} Var(X_t) = 10$
   since indep.
   c. What is the probability that Z(10) is strictly greater than 9?
   If Y = # of up steps (head) $Y \sim Bin(10, \frac{1}{2})$
   $P(Z(10) > 9) = P(Y = 10) = \binom{10}{10} (\frac{1}{2})^{10} (\frac{1}{2})^0 = \frac{1}{2^{10}} = 0.000977$
   d. In general for this process, what are the mean and variance of Z(t)?
   $E[Z(t)] = \sum_{i=1}^{t} E[X_i] = 0$
   $Var(Z(t)) = \sum_{i=1}^{t} Var(X_i) = t$

2. Now consider the same Random Walk, but for t = 0.25, 0.5, 0.75, 1, … (time steps Δt one quarter as long) and X_t = +0.5 or –0.5 (values half as large). This is still discrete time and discrete state space, just not integer-valued.
   a. Again using a coin, plot a sample path of Z(t) for t = 0 to 10. You will need 40 coin flips. What do you notice about the way the graph looks compared to 1a?
   The graph looks more jagged and the potential range is more spread out (±20 instead of ±10)
   b. What are the mean and variance of Z(10)? (Same hint, different X_t)
   $E[X_t] = 0.5 \times \frac{1}{2} + (-0.5) \times \frac{1}{2} = 0$
   $Var(X_t) = (0.5)^2 \times \frac{1}{2} + (-0.5)^2 \times \frac{1}{2} = 0.25$
   $Z(10) = \sum_{t=1}^{40} X_t$ so $E[Z(10)] = 40 \times 0 = 0$
   $Var(Z(10)) = 40 \times 0.25 = 10$ } same as in q1!

why $Y \geq 30$?
30 ups + 10 downs gives $Z(10) = 10$
29 ups + 11 downs gives $Z(10) = 9$

c. How could you find the probability that Z(10) is strictly greater than 9?

If Y = # of up steps (heads), $Y \sim Bin(40, \frac{1}{2})$

$$P(Z(10) > 9) = P(Y \geq 30)$$
$$= \sum_{y=30}^{40} \binom{40}{y} \left(\frac{1}{2}\right)^{40} = 0.00111$$

(you didn't have to find the value)

d. In general for this process, what are the mean and variance of Z(t)?

$$E[Z(t)] = \sum^{4t} E[X_i] = 0$$

$$Var(Z(t)) = \sum_{i=1}^{4t} Var(X_i) = 4t \times 0.25 = t$$

3. Imagine we continued shrinking the time steps Δt and the values of Xt (call them Δx) towards 0 in such a way that $(\Delta x)^2/\Delta t = 1$.

a. You do not need to graph it (we don't have time for an infinite number of coin flips!) but what do you think the graph of Z(t) would look like? What properties would it have? What kind of stochastic process would this be? (If you're not sure, try adding the first few values of Z(t) to your graph with Δt = 1/16 and Δx = 1/4.)

The graph would become a continuous stochastic process with a continuous state space. The line would be extremely jagged (and kind of look like a stock price!) This is Brownian Motion!

b. What is the distribution of Z(10), including its mean and variance? (Hint: what happens when you add up a LARGE number of independent variables?)

$E[X_t] = 0$, and $Var(X_i) = (\Delta x)^2 \frac{1}{2} + (-\Delta x)^2 \frac{1}{2} = (\Delta x)^2$

$Z(10) = \sum_{t=1}^{10/\Delta t} X_t$ so $E[Z(10)] = \frac{10}{\Delta t} \times 0$ $\quad Var(Z(10)) = \frac{10}{\Delta t}(\Delta x)^2 = 10.$

And by the Central Limit Theorem, the sum approaches a Normal distribution. So $Z(10) \sim N(0, 10)$.

c. How could you find the probability that Z(10) is strictly greater than 9?

$$P(Z(10) > 9) = P\left(\frac{Z(10) - 0}{\sqrt{10}} > \frac{9 - 0}{\sqrt{10}}\right)$$
$$= P(Z > 2.846)$$
$$= 1 - P(Z \leq 2.846) \leftarrow \text{look up in } N(0,1) \text{ table}$$
$$= 0.00221$$

d. In general for this process, what is the distribution of Z(t)?

Similarly, $\underline{Z(t) \sim N(0, t)}$

↑ this is the fundamental result for Brownian motion! ☺