Plane Projective Geometry

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Contents

List of Figures

Chapter 1

Introduction

In thinking upon what to talk to you about, I decided on some *ideas* from my own school days that first fired up my mathematical interest, and made me realise that I wanted to become a Mathematician. The material for my talk to you on 19 June 1997 is selected from these Notes, but I thought that you would like to have a record of the mathematical details and additional examples that time constraints prevented me from discussing. If I have interested you, you will be able to read these notes at your leisure, and perhaps work through some of the problems. The material is elementary in that it uses no more that simple properties of determinants. However is quite sophisticated in the use of geometrical ideas. Indeed, the Notes are a discussion of *ideas*, rather than a dose of formulae; the technical manipulations are simple.

- The first question that you should ask, is whether a knowledge of *Plane Projective Geometry* is going to help you do things that would be very much harder in the traditional *Euclidean Geometry* that you see at school. To help you answer this question, I have selected some problems that are blisteringly hard by traditional means, but easier by the methods of projective geometry. However, you should judge for yourselves: try the problems by the methods you know, and see how far you get.

The problems come from scholarship examinations for admission to Cambridge University in England, and from Parts I and II of the Cambridge Mathematical Tripos (a tripos is a three-legged stool on which candidates for the degree sat during an oral examination: this is no longer practised, although the term remains in use!). The final degree depends on the candidate's performance in Part II. I have also included the classical theorem of Pappus, and the theorem of Pascal for you to prove as a test of your understanding of this material.

I hope that you enjoy the material as much as I did when I first saw it, more or less at your age.

Perhaps I should let you know a little of what I do. My research deals with discrete structure and its analysis through combinatorial means and through algebraic and analytic techniques, with applications to a variety of questions in topology, the theory of functions and to mathematical physics. Such things are only seen at the university level.

1.1 The problem of parallelism in Euclidean geometry

I am going to assume that you are familiar with elementary Euclidean geometry of the plane. If O is the origin of a rectangular coordinate system Oxy, and $P = (a, b)$, then the length of OP is $\sqrt{a^2 + b^2}$ by Pythagoras' Theorem. In this geometry, a pair of lines may either be *parallel* or may *intersect*, as you know. This distinction is present in the algebra as well. For example, the lines $x + 2y = 1$ and $x + 2y = 2$ are parallel, and the usual attempt to solve them gives $0 = 1$. This is an inconsistency, and it says that the assumption that they had as solution is false, so there is no solution.

The objection to the traditional view is that parallel lines need special treatment, and additional theorems are needed to deduce facts about them. All of this leads to a large accumulation of detailed case analysis. What we now aim to do is to legislate that all lines in the plane intersect, so that proofs can be made more uniform, and shorter.

1.2 Projective Geometry

The rules that we set up are

- two points define a unique line;
- two lines define a unique point;
- there are four points, no three of which are collinear.

Notice that the second rule does not hold for Euclidean geometry, for two lines may be parallel, with *no* point of intersection. The last rule ensures that the geometry is nontrivial.

1.3 The Triangle of Reference and the Unit Point

We implement these rules by what is at first a sleight of hand, but in fact has profound consequences on our thinking. We simply introduce another variable z, and rewrite the above equations as $x + 2y = z$ and $x + 2y = 2z$, thereby making each *homogeneous* of degree one. Thus $z = 0$. Their solutions is $(-2d, d, 0)$, for any value of d. But two lines meet in a unique point, so we declare that in projective geometry that (a, b, c) and (ka, kb, kc) are *the same point*.

We now have a new geometry to explore. In place of a *rectangular coordinate system*, we have apparently three lines $x = 0$, $y = 0$ and $z = 0$ that *must not* meet in a single point. This is called the *Triangle of Reference*. To recover the point in Euclidean geometry that corresponds to the point (a, b, c) in projective geometry, we normalise the third coordinate to 1, to get $(a/c, b/c, 1)$ if $c \neq 0$, and then declare that $(a/c, b/c)$ is the corresponding point in Euclidean geometry.

This device allows us to pass freely between Euclidean geometry and projective geometry, and to use the latter to prove results about the former, and in fact to derive Euclidean interpretations of theorems in projective geometry.

The points of the triangle of reference have coordinates $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ since they lie at the intersections of $y = 0, z = 0$ and $x = 0, z = 0$ and $x = 0, y = 0$.

Of course, this makes a mess of computing length; it can be done but it is part of a longer story. In fact, we seldom need the concept of length.

Finally, to fix the coordinate system, we select a point that is not on the triangle of reference to be the *unit point* $(1, 1, 1)$.

Notice that parallel lines meet at a point whose z-coordinate is zero (see the above example). Thus parallel lines are recognised as those lines in the projective plane that meet on $z = 0$, which we therefore call the *line at infinity*.

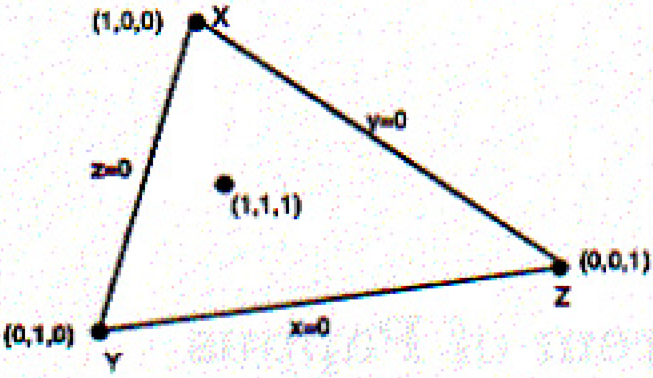

Figure 1.1: A Triangle of Reference, and the unit point

1.4 Review of some familiar results

I am going to assume that you are familiar with 2×2 and 3×3 determinants and their calculation.

Below are some results that are known to you in the case when all of the z's are set equal to 1, so we are back in the Euclidean plane. I have simply restated them for the *projective plane.*

1) The *line through the points* (x_0, y_0, z_0) and (x_1, y_1, z_1) is

$$\begin{vmatrix} x & y & z \\ x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \end{vmatrix} = 0.$$

1a) The *point of intersection of the lines* $L_1 := l_1x + m_1y + n_1z = 0$ and $L_2 := l_2x + m_2y + n_2z = 0$ is the point

$$L_1 \wedge L_2 := \left\| \begin{matrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{matrix} \right\| := (m_1n_2 - m_2n_1, -(l_1n_2 - l_2n_1), l_1m_2 - l_2m_1).$$

I will use $\wedge$ to mean *intersect* a good deal, to save time.

2) The points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are *collinear* (lie on the same line) if

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

2a) The lines $l_1x + m_1y + n_1z = 0$, $l_2x + m_2y + n_2z = 0$ and $l_3x + m_3y + n_3z = 0$ are *concurrent* (meet at a single point) if

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$

These results will be used very many times, so I will not refer to them explicitly when they are used. If you see a determinant, and cannot work out what it is doing, check back in this section for instant enlightenment.

Chapter 2

The Theorem of Pappus

We now come to the first application of these ideas. It is quite easy because, surprisingly, in this geometry we do not have to do very much in this case.

2.1 The theorem

Theorem 2.1 *Let A, B, C be points on one line and let P, Q, R be points on another line. Let $F = BR \wedge QC$ $G = AR \wedge PC$ and $H = AQ \wedge PB$. Then F, G, H are collinear.*

Proof: Select the two lines to be $x = 0$ and $y = 0$. Select an arbitrary line, different from this in the diagram, to be the line $z = 0$. This completes the selection of the *Triangle of Reference*. The coordinates of the six points are as shown in Figure 2.1, where a, b, c, p, q, r are arbitrary numbers. Now AQ is the line $qx + ay - z = 0$. Then BP is the line $px + by - z = 0$. Then

$$H = AQ \wedge BP = \left\| \begin{matrix} q & a & -1 \\ p & b & -1 \end{matrix} \right\| = (-a + b, q - p, -ap + bq).$$

Similarly we can get G and F. Then

$$\left| \begin{matrix} H \\ G \\ F \end{matrix} \right| = \left| \begin{matrix} -a+b & q-p & -ap+bq \\ -a+c & r-p & -ap+cr \\ -b+c & r-q & -bq+cr \end{matrix} \right| = 0$$

by $\text{row}_1 \mapsto \text{row}_1 - \text{row}_2 + \text{row}_3$. Thus F, G, H are collinear, and the result follows. ∎

Of course, this theorem holds in the Euclidean plane, by our transformation from the projective plane to the Euclidean plane. Check that you believe it by drawing some pictures with a straight edge.

Now is the time for you to try to prove this theorem by standard methods in Euclidean geometry. It is possible but, by comparison, the proof is excruciating.

2.2 Mutatis mutandis

In the proof of Pappus' Theorem I computed $H = AQ \wedge BP$, and then asserted that F and G can be obtained in a similar way. Let us see how to get $G = AR \wedge CP$ from H without any work. The secret is to note that if Q is replaced by R and B is replaced by C, then H is changed into G. Thus the coordinates of H are transformed into the coordinates of G by replacing q by r and b by c.

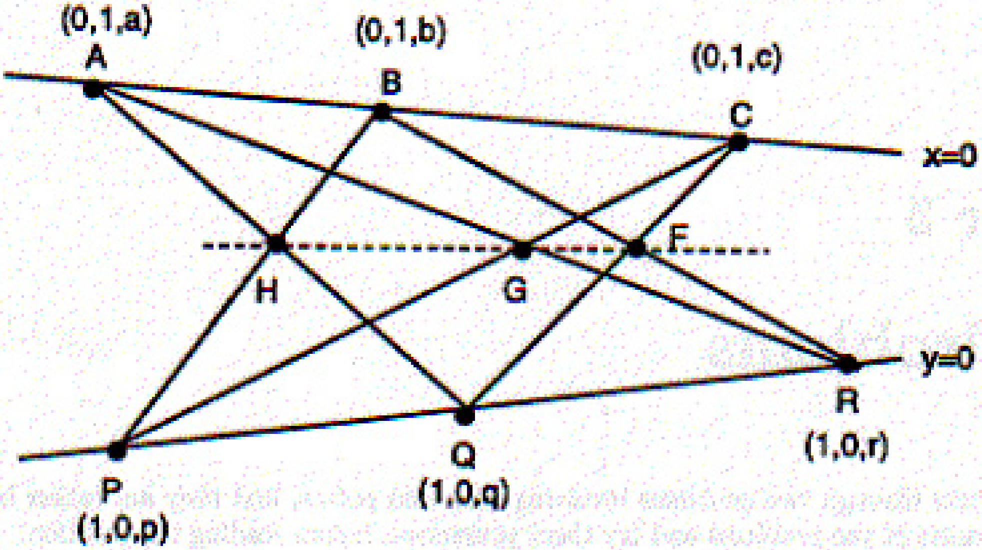

Figure 2.1: Pappus' Theorem

Thus $H = (-a + b, q - p, -ap + bq)$ is transformed into $G = (-a + c, r - p, -ap + cr)$. This greatly shortens the work by removing repetitive steps. I will summarise this process by saying that G and F are obtained from H *mutatis mutandis* (literally, the changeable symbols having been changed).

2.3 A homogeneous coordinate system

Now that you have seen one problem treated in detail, the following observation will be clearer. We are looking at questions in *plane projective geometry* by means of a *homogeneous system of coordinates*. This is comparable to looking at problems in *Euclidean geometry* by means of *rectangular Cartesian coordinates*.

Chapter 3

Two Problems

This chapter works through two problems involving lines and points, and they are rather harder. Read the statements of the problems and try them yourselves, before reading the solution.

3.1 Problem I (Cambridge Scholarship Examination)

The next problem is a more extensive exercise of this new approach.

Problem 3.1 *ABC is a triangle. D is on BC, E is on CA, and F is on AB such that AD, BE, CF are concurrent. l is an arbitrary line. Let* $FE \wedge l = P$, $FD \wedge l = Q$, $DE \wedge l = R$. *Let* $AP \wedge BC = P'$, $BQ \wedge AC = Q'$ *and* $CR \wedge AB = R'$. *Prove that the points* P', Q' *and* R' *are collinear.*

Proof: I will concentrate on P', and then derive Q' and R' *mutatis mutandis*. We can therefore get away with drawing only the portion of the diagram that involves the construction of the point P'.

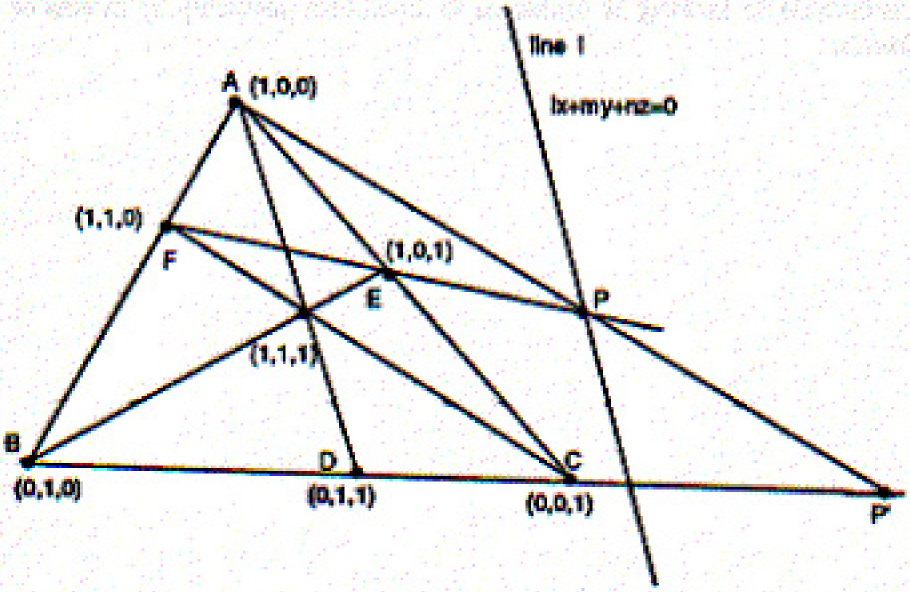

Figure 3.1: Construction of P' for Problem I

Select ABC as the *Triangle of Reference*. Select the point of concurrence of AD, BE, CF as the *unit point*. Then the coordinates of D, E, F are as indicated in Figure 3.1. For example, to obtain D, note that the line through A and the unit point is

$$\begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

so $y - z = 0$. This intersects the line BC (i.e. $x = 0$) at the point $(0,1,1)$, so this is D.

Moreover, the line EF is

$$\begin{vmatrix} x & y & z \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

This is $-x + y + z = 0$. Then

$$P := l \wedge EF = \begin{Vmatrix} l & m & n \\ -1 & 1 & 1 \end{Vmatrix} = (m - n, -l - n, l + m),$$

so AP is the line $(l+m)y + (l+n)z = 0$. Thus

$$P' := AP \wedge BC = \begin{Vmatrix} 0 & l+m & l+n \\ 1 & 0 & 0 \end{Vmatrix} = (0, l+n, -l-m).$$

Then Q' and R' are obtained *mutatis mutandis*, so

$$\begin{vmatrix} P' \\ Q' \\ R' \end{vmatrix} = \begin{vmatrix} 0 & l+n & -l-m \\ -m-n & 0 & l+m \\ m+n & -l-n & 0 \end{vmatrix} = 0$$

by $\text{row}_1 \mapsto \text{row}_1 + \text{row}_2 + \text{row}_3$. Thus the points P', Q', R' are collinear. |

You will now have some idea of the flavour of these methods. Since we can select *any three nonconcurrent lines* as the triangle of reference, and then select *any point not on the triangle of reference* as the unit point, we can certainly adapt the coordinate system more easily to fit problems.

3.2 Problem II (Mathematical Tripos, Part II)

This problem concerns triangles that are in *perspective*. We say that the triangles Δ^{PQR}_{UVW} are in perspective if PU, QV, RW are concurrent. The intersection is called the *point of perspectivity*. The triangles are stacked on top of each other to call attention to the lines that are to be concurrent: the are obtained from the three columns of this arrangement.

Problem 3.2 *The triangles Δ^{ABC}_{DEF} in the plane are in perspective. If the triangles Δ^{ABC}_{EFD} are in perspective, prove that the triangles Δ^{ABC}_{FDE} are in perspective.*

Proof: Select ABC as the *Triangle of Reference* and the *unit point* as the point of perspectivity of the triangles ABC and DEF. Then the coordinates of D, E, F are as shown in Figure 3.2, where a, b, c are arbitrary real numbers.

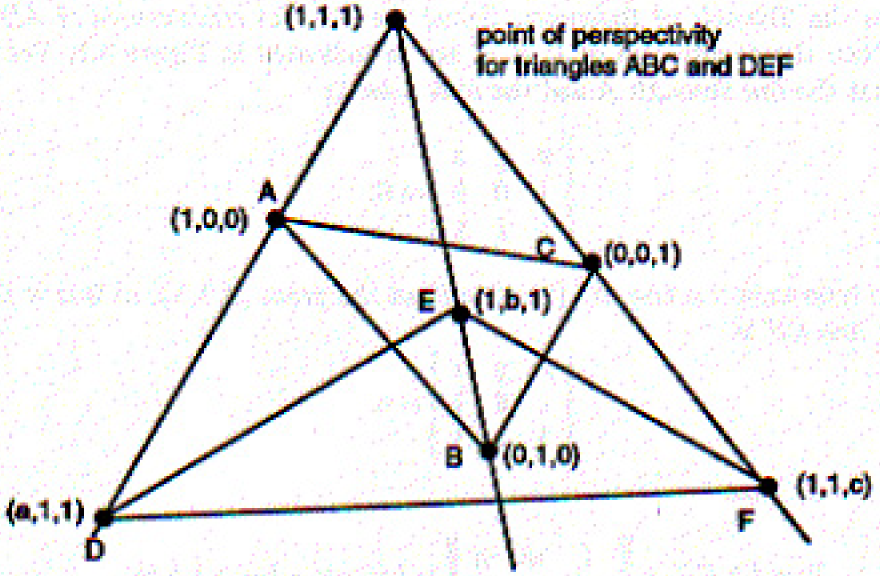

Figure 3.2: Triangles ABC, DEF in perspective for Problem II

For Δ^{ABC}_{EFD}, AE is the line $-y+bz=0$, BF is the line $cx-z=0$, and CD is the line $x-ay=0$. These triangles are in perspective (from some point) so

$$\begin{vmatrix} 0 & -1 & b \\ c & 0 & -1 \\ 1 & -a & 0 \end{vmatrix} = 0.$$

Thus $abc=1$.

For Δ^{ABC}_{FDE}, AF is the line $cy-z=0$, BD is the line $-x+az=0$, and CE is the line $bx-y=0$. But

$$\begin{vmatrix} 0 & c & -1 \\ -1 & 0 & a \\ b & -1 & 0 \end{vmatrix} = -1+abc=0,$$

so these triangles are indeed in perspective (from some point). This solves the problem. ∎

Chapter 4

Conics

So far I have concentrated on lines and points. I am now going to consider conics.

4.1 In Euclidean Geometry

In Euclidean geometry, a *conic* is a second degree curve and therefore has the form

$$ax^2 + by^2 + c + 2hxy + 2gx + 2fy = 0,$$

where a, b, c, f, g, h are numbers. Familiar examples are the *circle*, *parabola*, *ellipse*, *hyperbola* and *line pair*. For example, the equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

This simply states that

$$(x+g)^2 + (y+f)^2 + c - g^2 - f^2 = 0,$$

so the centre is $(-g, -f)$ and the radius is $\sqrt{g^2 + f^2 - c}$, by Pythagoras' Theorem (that uses the notion of distance).

4.2 In Projective Geometry

In Plane Projective geometry, the general equation of a conic is

$$S := ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz = 0.$$

The variable z has been introduced to make each term of degree two (homogeneous). For example, the equation of a circle is

$$x^2 + y^2 + 2gxz + 2fyz + cz^2 = 0,$$

obtained by inserting z into the Euclidean equation in a homogeneous way. One consequence of this is startling. It is that the points $I = (1, i, 0)$ and $J = (1, -i, 0)$ lie on *every circle*, where $i = \sqrt{-1}$. These points are therefore called the *circular points at infinity*. This gives us a way of recognising circles in projective geometry without using the concept of distance.

4.3 Pencils of conics

Let $S_1 = 0$ and $S_2 = 0$ be two conics, and consider $S := \lambda S_1 + \mu S_2$. Clearly, S is homogeneous, of degree 2, and is therefore a conic. Let P_1 be a point of intersection of S_1 and S_2. Then at P_1 we have $S_1 = 0$ and $S_2 = 0$ so $S = 0$, so P_1 is a point of S. Let P_2, P_3, P_4 be the other points of intersection of S_1 and S_2. Then these also lie on S by the same argument. Thus S is a conic passing through the four points of intersection of S_1 and S_2. The set of all conics S is called a *pencil* of conics generated by S_1 and S_2.

4.4 The Second Fundamental Form of a conic

Now select $S_1 := xz = 0$ and $S_2 := y^2 = 0$. This is certainly valid since these are homogeneous, of degree 2. S_1 is a pair of lines. S_2 is a pair of *repeated lines*. They are degenerate conics called *line pairs*.

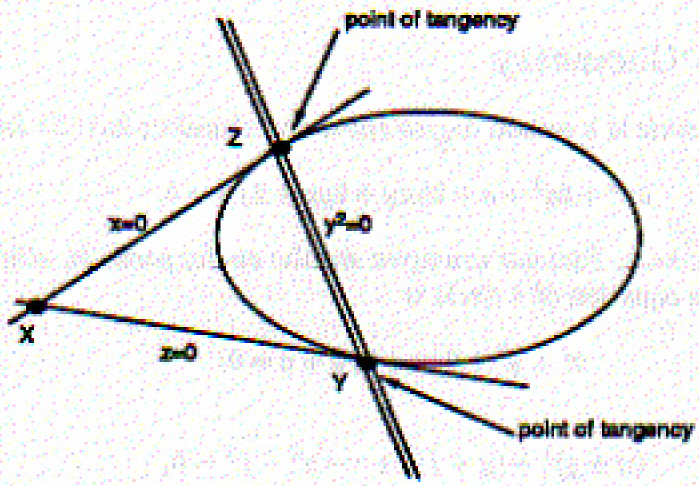

Figure 4.1: Selection of triangle of reference for a general conic

The conics of the pencil generated by this choice of S_1 and S_2 are tangent to $x = 0$ and $z = 0$ at Z and at X, since at these points there are two *identical* points of intersection. Thus $S = \lambda xz + \mu y^2$. Now let the unit point be on S. Then $\lambda + \mu = 0$, so $S = y^2 - xz$.

The conclusion is astounding. It is that, if the *Triangle of Reference* is selected as shown in Figure 4.1, then the general equation of a conic is simply $y^2 - xz = 0$. Moreover, any point of the conic has the form $(t^2, t, 1)$, for some number t. This is called a *parametrisation* of the conic, and t is called the *parameter* of the point $(t^2, t, 1)$ on S. It is called the *Second Fundamental Form* for a conic.

Chapter 5

Involutions on Conics

I am going to introduce you to the idea of an *involutions on a conic* through the following concrete problem.

5.1 Problem III (Mathematical Tripos Part I)

The following problem involves two general conics in the plane. In the Euclidean case, you would expect the solution to degenerate into a mess of symbols. In projective geometry something very delicate can be done, which reduces the algebra to a few subtle steps.

Problem 5.1 *Let S and S' be conics in the plane that have a common point A. Let P be an arbitrary fixed point of the plane. A variable line l through P strikes S at X and Y. AX meets S' at X', and AY meets S' at Y'. Prove that $X'Y'$ passes through a fixed point as l varies.*

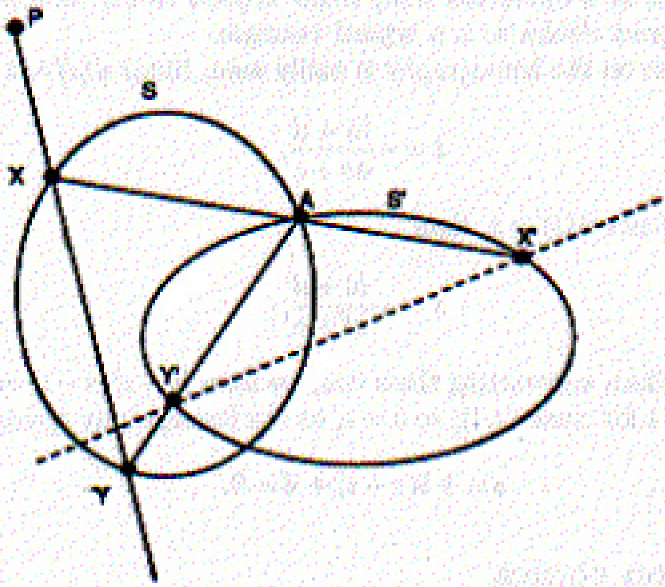

Figure 5.1: Two general conics for Problem III

To work on this problem we need another idea. It is simple, subtle and very effective. It is called *homography*.

5.2 Homography

We have just seen how to choose the *Triangle of Reference* so that the points on a general conic S can be parametrised by $(t^2, t, 1)$. This is a consequence of the second fundamental form discussed above. Suppose we have a one-to-one function ψ that acts on real numbers. Then ψ maps points on S to points on S. We call ψ a *one-to-one correspondence*.

Suppose now that, geometrically, ψ involves only intersections between conics and lines and points. Then ψ must be *algebraic*. A one-to-one algebraic correspondence is called a *homography*. It must therefore have the form

$$ast + bs + ct + d = 0,$$

where a, b, c, d are numbers, for then

$$t = -\frac{bs + d}{as + c}, \qquad s = -\frac{ct + d}{at + b}.$$

In this case

$$\psi(s) = -\frac{bs + d}{as + c},$$

so $\psi(s) = t$.

The very special case $s - t = 0$ means that the homography is $\psi(s) = s$. This maps the point P on the conic S onto P, for every P, so we call this particular ψ *trivial*.

5.3 Involution

Suppose now, that ψ has the additional property that if $\psi(s) = t$ then $\psi(t) = s$. This means that $\psi(\psi(s)) = s$, for all s. Such a ψ is called an *involution*.

Think of an involution as a operation that, when applied twice, returns to the initial state. Rotating through 180 degrees clockwise is a typical example.

The algebraic condition on the homography is easily seen. Since $\psi(s) = t$ then

$$t = -\frac{bs + d}{as + c},$$

so $ast + bs + ct + d = 0$. Since $\psi(t) = s$, then

$$s = -\frac{bt + d}{at + c},$$

so $ast + cs + bt + d = 0$. Then, subtracting these two, we have $(b - c)(s - t) = 0$. Now assume that ψ is nontrivial. Then $s \neq t$ for some (s, t), so $b = c$, so the homography specialises to

$$ast + b(s + t) + d = 0.$$

5.4 Involutions on conics

The following is a basic result about involutions on conics.

Lemma 5.2 *Let ψ be an involution on a conic S. If P, Q is a pair of points in involution, then the line PQ passes through a fixed point.*

Proof: Select the *Triangle of Reference* and the unit point so that the points of the conic S are parametrised by $(t^2, t, 1)$. Let P be the point $(s^2, s, 1)$ and let Q be the point $t^2, t, 1)$. Now PQ is the line

$$\begin{vmatrix} x & y & z \\ s^2 & s & 1 \\ t^2 & t & 1 \end{vmatrix} = 0,$$

so $(s-t)x - (s^2 - t^2)y + st(s-t)z = 0$. Thus PQ is the line

$$x - (s+t)y + stz = 0.$$

But P and Q are pairs in involution, so there are numbers a, b such that $ast + b(s+t) + 1 = 0$. But this implies that $(1, b, -a)$ is a point on PQ. But this point is independent of P and Q and is therefore fixed. The result follows. |

5.5 Solution of Problem III

This solution to Problem III can now be presented very quickly with the aid of this new concept.

Consider the mapping ψ that maps X' to Y' through the construction (see Figure 5.1)

$$X'A \wedge S = X, \quad PX \wedge S = Y, \quad YA \wedge S' = Y'.$$

Then

1. ψ is one-to-one;
2. ψ is algebraic (it involves only intersections of points, lines and conics);
3. ψ is involutory (do the construction twice).

Then by the involution lemma, $X'Y'$ passes through a fixed point. This completes the solution. |

Note that we are now starting to use some sophisticated results in projective geometry.

Chapter 6

Pascal's Theorem

6.1 The theorem

Theorem 6.1 *Let A, B, C, A', B', C' be points on a conic S. Let $AB' \wedge A'B = F$, $AC' \wedge A'C = E$, and $BC' \wedge B'C = D$. Then D, E, F are collinear.*

I thought that you would like to have a problem to work on, so I am going to leave the proof at you, after giving you a few hints.

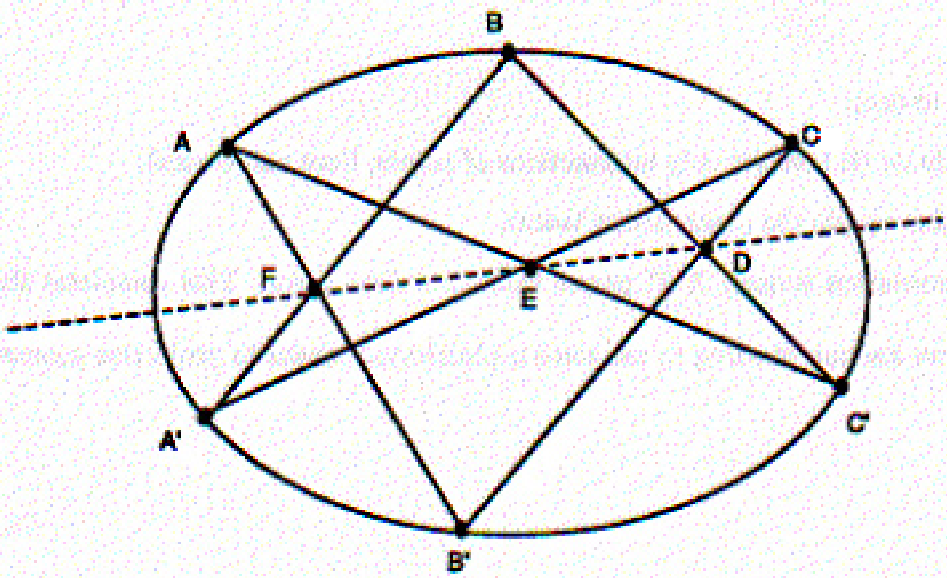

Figure 6.1: Pascal's Theorem

Proof: (hints) Select the *Triangle of Reference* and the unit point so that points on the conic are parametrised by $(t^2, t, 1)$. Let $t_1, t_2, t_3, s_1, s_2, s_3$ be the parameters of A, B, C, A', B', C' respectively. Show that AB' is the line

$$x - (t_1 + s_2)y + t_1 s_2 z = 0,$$

and that

$$F = \left\| \begin{matrix} 1 & -(t_1 + s_2) & t_1 s_2 \\ 1 & -(t_2 + s_1) & t_2 s_1 \end{matrix} \right\|.$$

Determine D and E *mutatis mutandis*, and show that the determinant

$$\begin{vmatrix} D \\ E \\ F \end{vmatrix}$$

is equal to zero.

This is quite a task, so check that you can set $t_1 = 0$, so A is $(0,0,1)$; and $t_2 = 1$ so B is $(1,1,1)$; and $t_3 = \infty$, so C is $(0,0,1)$. To deal with infinities, recall that $(t_3^2, t_3, 1)$ and $(1, 1/t_3, 1/t_3^2)$ are the same point, since one is a multiple of the other. Put $t_3 = \infty$ in the second form of the point to get $(1,0,0)$, as indicated above. With these settings, which are made without loss of generality, you will obtain the determinant

$$\begin{vmatrix} -s_1 s_2 & -s_1 & -1 - s_1 + s_2 \\ -s_1 s_3 & -s_1 & -1 \\ -s_2 - s_2 s_3 + s_3 & -s_2 & -1 \end{vmatrix}.$$

This can be computed easily, and found to be zero. This proves that the points D, E, F are concurrent. You can now fill in the details, and justify each step. |

It is convenient to refer to the line D, E, F as the *cross-axis* of the triangles $\Delta^{ABC}_{A'B'C'}$.

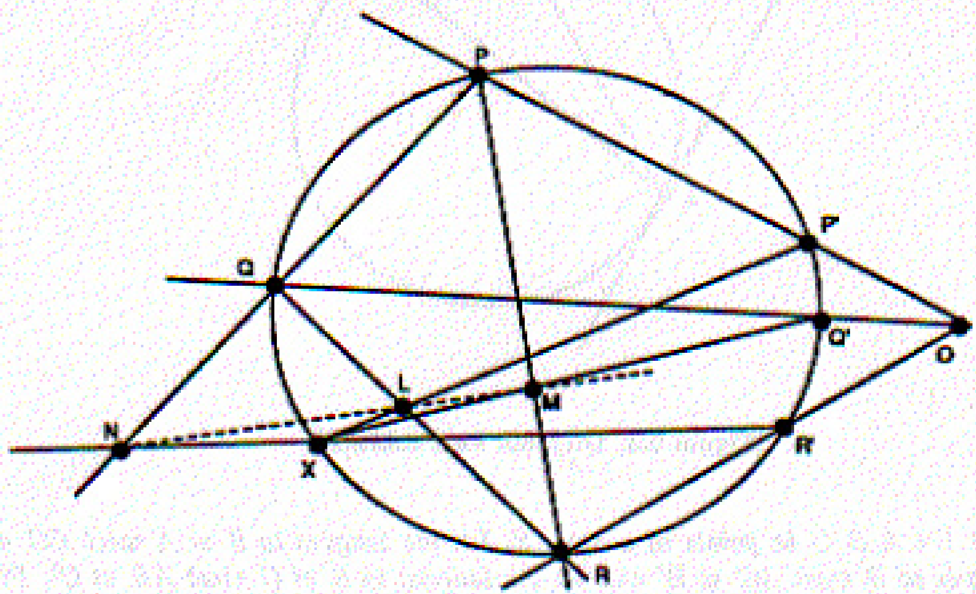

Figure 6.2: Diagram for Problem IV

6.2 Problem IV (Cambridge Scholarship Examination)

The following is an application of Pascal's Theorem.

Problem 6.2 *Let PP', QQ', RR' be chords of a conic S, and suppose that these chords pass through a point O. Let X be any other point on S. Let $L = QR \wedge XP'$, $M = RP \wedge XQ'$ and $N = PQ \wedge XR'$. Prove that L, M, N, O are collinear.*

Proof: Consider the points P, Q, X and Q', P', R on S. Then, from Pascal's Theorem, the points $PP' \wedge QQ'$, $QR \wedge P'X$, $PR \wedge Q'X$ are collinear, on the cross-axis of $\Delta^{PQX}_{Q'P'R}$. But these are precisely the points, O, L, M respectively.

Now consider the points Q, R, X and R', Q', P on S. Then, again from Pascal's theorem, the points $QQ' \wedge R'R$, $RP \wedge Q'X$, $QP \wedge R'X$ are collinear, on the cross-axis of $\Delta^{QRX}_{R'Q'P}$. But these are precisely the points O, M, N respectively.

We conclude therefore that L, M, N, O are collinear, since the two cross-axes have O, M in common. |

6.3 Problem V

We conclude with another, perhaps surprising, application of Pascal's Theorem.

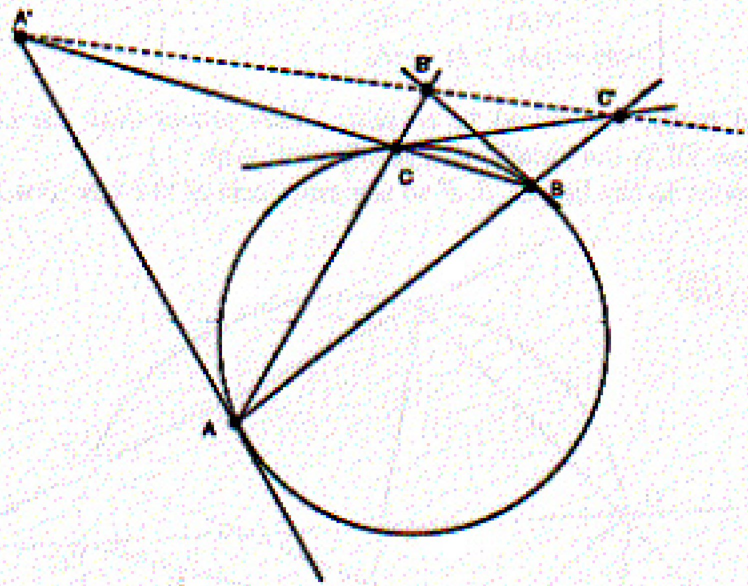

Figure 6.3: Diagram for Problem V

Problem 6.3 *Let A, B, C be points of a conic S. Let the tangent to S at A meet BC at A', let the tangent to S at B meet AC at B' and let the tangent to S at C meet AB at C'. Prove that A', B', C' are collinear.*

Proof: Consider the cross-axis of Δ^{ABC}_{CAB}, in which each point is considered twice. By Pascal's Theorem, this contains the points $AA \wedge BC$, $BB \wedge AC$, and $CC \wedge AB$. But AA, BB, CC denote the tangents to S at A, B, C, respectively. The three points are identified therefore as A', B', C', respectively. The result now follows. |

Chapter 7

Poles and Polars

I am going to close these notes with a brief mention of two very important and powerful concepts, namely, those of *pole* and *polar*.

Let S be a conic in the plane and let P be a point of the plane. There are two tangents to S passing through P, with two points of tangency. The line joining these two points is called the *polar* of P with respect to S. If P is on S then the polar is the tangent to S at P.

This definition does not look after the case when P is inside the conic S, so a more indirect definition is used in practice. (It requires the introduction of the *cross-ratio* as a projective invariant!!!!) However, the one that I have just given is adequate for the present purpose.

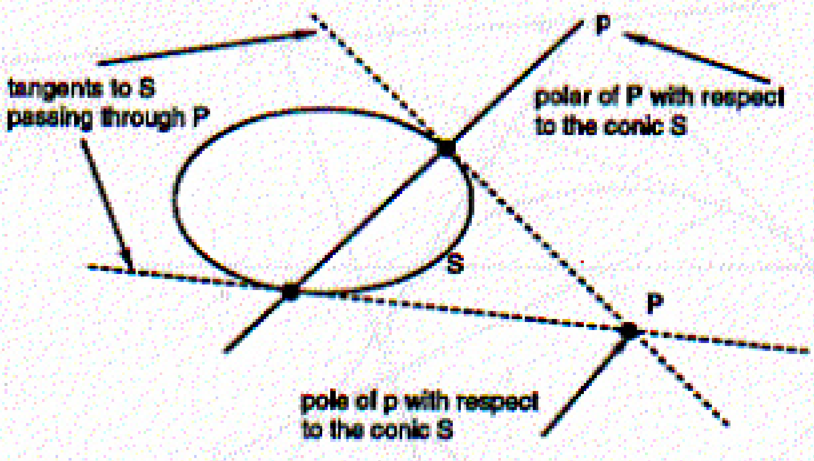

Figure 7.1: Construction of the pole and polar

7.1 The reciprocity theorem

The next result is important and is often used in constructions.

Theorem 7.1 *If the polar of P with respect to the conic S passes through the point Q, then the polar of Q with respect to S passes through P.*

Proof: We select the *Triangle of Reference* so that the conic is parametrised by $(t^2, t, 1)$. The line joining the points with parameters s and t is $x - (s+t)y + stz = 0$. This has been proved earlier.

Thus the tangent at a point with parameter u is $x - 2uy + u^2z = 0$. So see this set $u = s = t$ for double contact on S at the point with parameter u. Now this tangent is to pass through the point $P = (a, b, c)$. Then

$$a - 2bu + cu^2 = 0.$$

This is a quadratic equation, with roots α and β, which are parameters to the two points of contact of tangents to S from P. From the theory of equations $\alpha + \beta = 2b/c$ and $\alpha\beta = a/c$. The line joining the points with parameters α and β is $x - (\alpha + \beta)y + \alpha\beta z = 0$. But, this is $cx - 2by + az = 0$, by substituting the expressions for $\alpha + \beta$ and $\alpha\beta$. This is therefore the equation of the pole of P.

If $Q = (A, B, C)$ lies on this line then

$$cA - 2bB + aC = 0.$$

But the polar of Q is $Cx - 2By + Az = 0$, by the same argument, and P lies on this if

$$Ca - 2Bb + Ac = 0.$$

But this is precisely the condition that Q lies on the polar of P. This completes the proof. |

We can use the Reciprocity Theorem to construct the polar of a point P inside S. Draw any two lines a and b through P and construct the pole A of a and the pole B of b. Now the polar of A is a, that passes through P, so the polar of P passes through A by reciprocity. Similarly, the polar of P passes through B. Then the polar of P is the line AB.

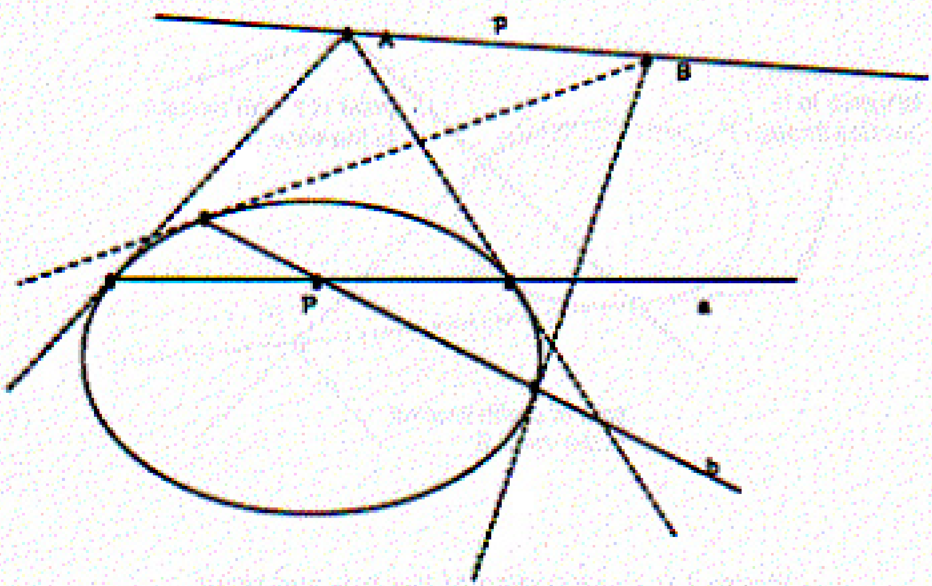

Figure 7.2: Construction of the polar of a point inside a conic

7.2 Problem VI (Cambridge Scholarship Examination)

The final problem is going to draw together all of the ideas that have been developed in these Notes.

Problem 7.2 *Tangents are drawn from a given point O to a given conic, meeting it at A and B. A variable line l is drawn through a given point C on AB, cutting OA, OB at P, Q respectively. Prove that the locus of the point of intersection L of the other tangents from P and Q (as l varies) is a straight line.*

Proof: We set up a correspondence ψ between points on the conic. Given A' on S, we construct $\psi(A')$ as follows. The tangent to S at A' meets OA at P. Let $PC \wedge OB = Q$. Let the other tangent from Q to S touch S at B'. Then let $\psi(A') = B'$.

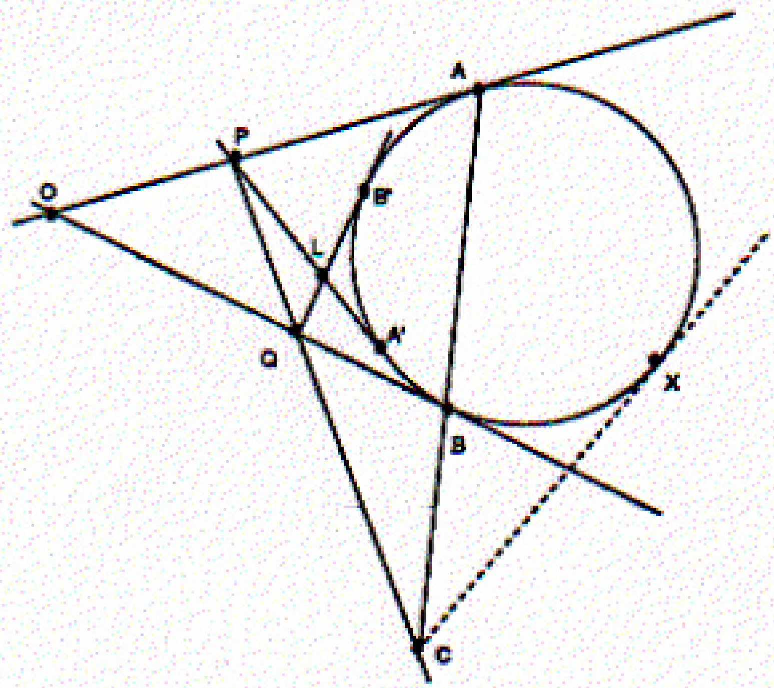

Figure 7.3: Diagram for Problem VI

Now consider ψ.

- ψ is a one-to-one correspondence on S;
- ψ is algebraic (B' is computed from A' by intersections of points lines and conics);
- $\psi(B') = A'$.

Then ψ is an involution on S and, from the involution lemma, we conclude that $A'B'$ passes through a fixed point, T, say.

We now identify T. To do this we need compute other images of ψ. Let X be a point of contact of a tangent to S from C. By working through the construction, it is clear that $\psi(X) = X$. Thus X, X are paired in the involution, so T lies on XX, the tangent to S at X.

In addition, by working through the construction, it is clear that $\psi(A) = B$. Thus A, B are paired in the involution, to T lies on the line AB.

It follows from this that T is the point of intersection of the tangent to S at X and the line AB. But this point is C. We have therefore identified the fixed point T as C.

Now $A'B'$ is the polar of L with respect to S. But $A'B'$ passes through C, so the polar of L passes through C. Then, by the Reciprocity Theorem, L lies on the polar of C. We conclude that the locus of L (as l varies) is the polar of C. But this is a fixed line, and the result follows. ∎

7.3 Closing comments

This completes my brief introduction for you to the ideas of plane projective geometry. Even in this very brief glimpse, you have before you evidence that some very elegant ideas can greatly simplify hard problems. In fact, throughout these Notes, it is the ideas that have dominated, rather than the algebraic manipulation, of which there has been little. What there has been, has been deft and effective.