"NEWPACK" – a modified circle-packing scheme

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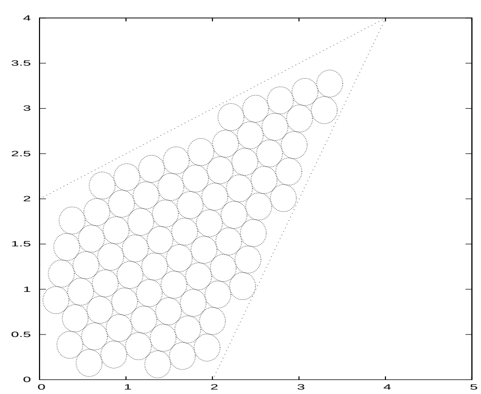
Overview

At this time, there are three main steps to this packing scheme – and a fourth to be added – which are outlined below. In subsequent sections, the details of the algorithm, including formulas, are presented.

In what follows, let $D \subset \mathbf{R^2}$ denoted a bounded region in the plane with boundary, to be denoted as C. We assume that the boundary ∂D is polygonal, i.e., a union of line segments, $C = \cup_{k=1}^{N} C_k$.

Step 1: Perform a hexagonal packing of the interior region D with circles of a prescribed radius R. This packing is easily accomplished by generating a set of lattice points separated by distance $2R$, starting at a reference point, $(\bar{x}, \bar{y})$, which is the center of one of the circles in the packing. This reference point may be viewed as the origin of a "body set of coordinates". (In the computations shown below, the origin this reference point was the centroid of region D but it doesn't have to be. In fact, it doesn't even have to lie in the region D.) The packing is also oriented at an angle θ to the horizontal axis.

During this packing step, identify all "boundary circles," i.e., the layer of circles which lie closest to the boundary C. An example of this construction is shown in the figure below.



Step 1: Hexagonal packing of a convex quadrilateral region D with circles of identical radius $R = 0.15$, starting at centroid. Orientation angle $\theta = \frac{\pi}{10}$ radian. This angle was chosen for demonstrative purposes only – it is not necessarily the optimal angle. Packing fraction $\phi = 0.70$. 79 Step 1 circles packed.

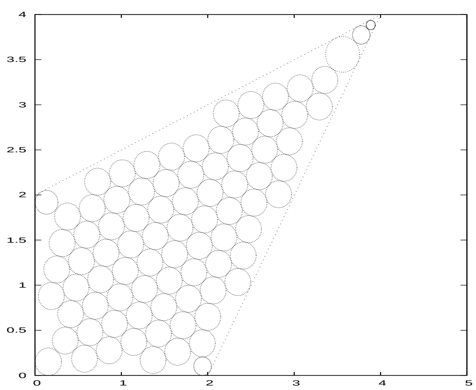
Here we mention that the packing achieved by the Step 1 scheme is identical to what would be achieved by the GGL (or anti-GGL) two-circle packing method. (In the GGL scheme, the initial placement of two touching circles would determine the position and orientation of the hexagonal packing.) The automatic packing of the Step 1 scheme bypasses the computations required in the GGL Details of this scheme are presented later in this report. method.

Step 2 "Corner packing": With reference to the upper right corner of the upper figure, it is possible that there are rather large unpacked regions near corners, espcially those with acute angles. There are a number of possible strategies to pack circles into these regions, e.g., 1) GGL circle-side packing and 2) GGL corner-packing algorithm. Currently (in program `newpack250.f`), we have adopted the following strategy. For each vertex P of the boundary, find the packed circle

C_P closest to P and then construct, if possible, the largest circle C which touches both boundary segments emanating from P as well as the packed circle C_P. Details of this scheme are presented later in this report.

This scheme can be iterated as many times as desired before moving to Step 3. There is also an option to exclude the packing of circles with radii smaller than a prescribed threshold radius, r_{min}. In all of the computations and steps shown below, we have used $r_{\text{min}} = 0.05$. In other words, no circles of radius less than 0.05 were employed in the packing.

Application of this corner packing method to the result of Step 1 shown above yields the packing shown below.



Step 2: Corner packing performed on the result of Step 1 shown above. Four iterations were performed resulting in an additional six circles being packed: 85 circles in total. Packing fraction $\phi = 0.74$.

Note that in the example shown above, the radius of the circle packed in the upper right corner is significantly larger than the radius R of the Step 1 circles. Such circles could be disallowed if desired, but some other method of packing the corner would then have to be used. We expect that such situations occur relatively infrequently, especially, as we discuss later, since we shall be considering a variety of reference points $(\bar{x}, \bar{y})$ as well as orientation angles θ.

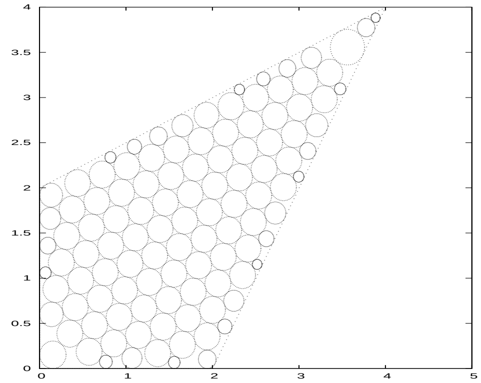
Step 3:

Use the boundary circles from Step 1 and the two-circle packing method from the GGL scheme to pack a second layer of smaller boundary circles which touch the boundary circles as well as the boundary. (This is accomplished analytically from a knowledge of the locations of the centres of the (touching) boundary circles and the line segment being used. The formulas are presented in a later section.)

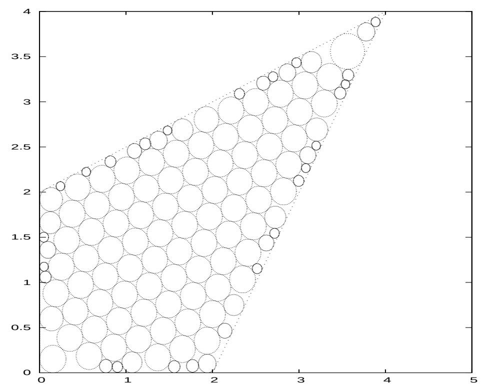
This step represents a significant deviation from any GGL-type schemes in that we do **not** use circles of prescribed radii. Here, Step 2 packed circles are "tailor made," i.e, their sizes are determined by the local geometry in an effort to increase packing efficiency.

Step 4: Pack a third layer of boundary circles which touch the boundary segments, using (not necessarily touching) boundary circles from Steps 1 and 2 and the two-circle packing scheme. This packing is **not** performed analytically. Instead, circles are grown from a minimum radius r_{min} in discrete increments. The procedure is stopped before the circles extend beyond the boundary. Details are presented in a later section.

This procedure may be iterated if desired since it is possible that some vacant spots are missed during the first application. In this particular case, no additional circles were packed.



Step 3: Packing a third set of boundary circles using the packed circles from Steps 1 and 2. 26 such circles were added for a total of 111 circles. Packing fraction $\phi = 0.84$, a significant increase from the previous packing.



Step 4: Packing a fourth set of circles using boundary circles from Steps 1-3. 15 circles were packed for a total of 126 circles. Packing fraction $\phi = 0.86$.

The results shown above were obtained from FORTRAN program `newpack250.f`. The input to this program includes: (1) the locations of the vertices of the polygonal boundary (in "lab frame" coordinates), (2) radius R of Step 1 circles, (3) angle of inclination θ, (3) control parameters to select whether the packing stops after Step 1, Step 2 or Step 3.

A later version of this program, `newpack350.f`, examines the packing over a number of discrete translations from the reference point (x_0, y_0) as well as discrete increments of the orientation angle from $\theta = 0$ to $\theta = \frac{\pi}{6}$, selecting the most efficient packing. This algorithm will be discussed later in this report.

Step 0: Initializing the problem

We first input the number N of vertices of the polygon which encloses the region D, followed by their coordinates $\mathbf{c}_k = (c_{k1}, c_{k2})$, $1 \leq k \leq N$ (in the "lab frame"). The vertices are ordered in a counterclockwise manner. We also compute the displacement vectors,

$$\mathbf{v}_k = \mathbf{c}_{k+1} - \mathbf{c}_k\,, \quad 1 \leq k \leq N \tag{1}$$

where

$$\mathbf{c}_{N+1} = \mathbf{c}_1\,. \tag{2}$$

These displacement vectors then define an oriented curve/path in the usual Vector Calculus sense, with the enclosed region D lying to the left of the curve. For the convex quadrilateral region shown in the figures above, $N = 4$ and the coordinates $\mathbf{c}_k$ are

$$\mathbf{c}_1 = (0,0) \quad \mathbf{c}_2 = (2,0) \quad \mathbf{c}_3 = (4,4) \quad \mathbf{c}_4 = (0,2)\,. \tag{3}$$

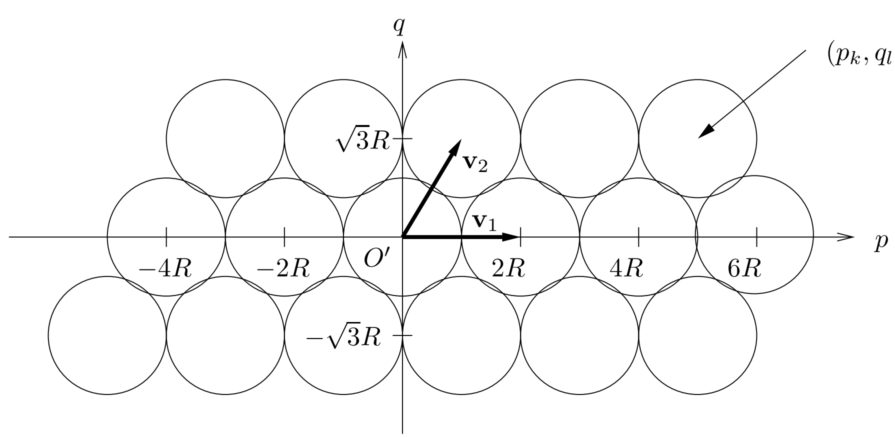
The displacement vectors are

$$\mathbf{v}_1 = (2,0) \quad \mathbf{v}_2 = (2,4) \quad \mathbf{v}_3 = (-4,-2) \quad \mathbf{v}_4 = (0,-2)\,. \tag{4}$$

Using Green's Theorem in the Plane, the area $A(D)$ of region D as well as its centroid $(\bar{x}_1, \bar{x}_2)$ may be computed from the $\mathbf{c}_k$ and $\mathbf{v}_k$, as shown in Appendix A.

Step 1 Packing: Details

The structure of the unrotated hexagonal packing with respect to the origin O' (of "body system coordinates") which is situated at the reference point $(\bar{x}, \bar{y})$ (in "lab coordinates"), is shown below.



The centers (p_k, q_l) of the Step 1 circles of radius R are located at integer multiples of the basic tile vectors,

$$\mathbf{v}_1 = (2R, 0)\,, \qquad \mathbf{v}_2 = (R, \sqrt{3}R)\,, \tag{5}$$

i.e.,

$$(p_k, q_l) = k\mathbf{v}_1 + l\mathbf{v}_2 = (\,[2k + l]R, \sqrt{3}lR\,)\,. \tag{6}$$

All rotations will be performed with respect to the origin $(0,0)$ at an angle $\theta \in [0\frac{\pi}{6}$ with respect to the x-axis. The coordinates of the rotated circles will be

$$\begin{pmatrix} p'_k \\ q'_l \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} p_k \\ q_l \end{pmatrix}\,. \tag{7}$$

In the "lab coordinate" frame, the coordinates of these packed and rotated circles will be given by

$$(x_k, y_l) = (\bar{x}, \bar{y}) + (p'_k, q'_l)\,. \tag{8}$$

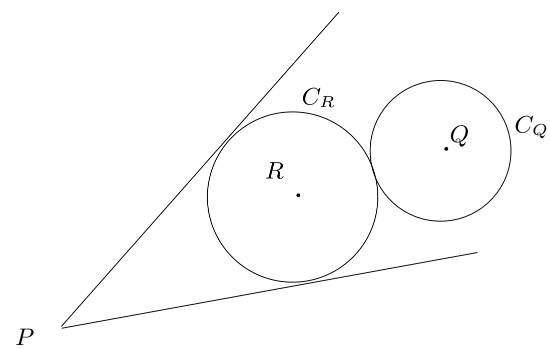
The generation of circle centers (x_k, y_l) can, of course, be performed easily in a loop but a couple of practical matters still remain:

1. Determining which Step 1 circles of radius R lie in region D and keeping them, discarding the others.
2. Determining which Step 1 circles are boundary circles, along with the problem of associating at least one boundary segment to each boundary circle.

These matters are discussed in Appendix B.

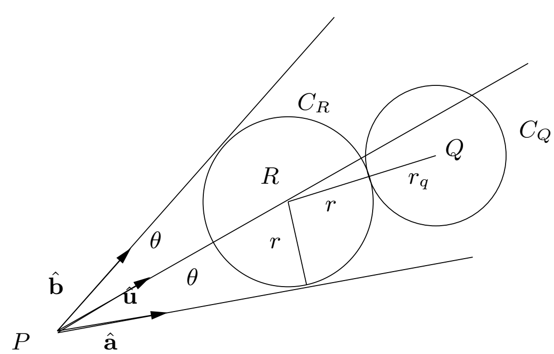
Step 2: "Corner packing"

The problem is sketched below. For a given vertex P situated at point $\mathbf{p} = (p_1, p_2)$, let circle C_Q, radius r_q and center $q = (q_1, q_2)$, be the packed circle in region D that lies closes to P. (It is possible that two or more circles lie equally close to P, in which case we simply pick one of them.) The goal is to construct the largest circle C_R centered at R with radius r which touches both boundary segments emanating from P as well as touching circle C_Q, as shown below.



Note that the circle C_R will touch both boundary segments only if the vertex P is convex. Later we shall also employ this strategy for the nonconvex situation. angles.

Any circle lying in D and touching the two boundary segments must have its center lying on the line which bisects the angle subtended by the segments. Let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be the unit vectors emanating from P in the directions of the two boundary segments, as sketched below.



Then define the following

$$\mathbf{u} = \frac{1}{2}\left(\hat{\mathbf{a}} + \hat{\mathbf{b}}\right)\,, \quad \hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = (\hat{u}_1, \hat{u}_2)\,. \tag{9}$$

Then the point

$$\mathbf{x} = \mathbf{p} + t\hat{\mathbf{u}}, \quad t \geq 0, \tag{10}$$

lies on the bisector of the angle subtended by the segments. In coordinate form,

$$(x, y) = (p_1, p_2) + t\,(\hat{u}_1, \hat{u}_2). \tag{11}$$

The distance from P to R is

$$\|\overline{PR}\| = |t| = t > 0. \tag{12}$$

The distance from R to Q is

$$\|\overline{RQ}\| = \|\mathbf{x} - \mathbf{q}\| = \sqrt{(x - q_1)^2 + (y - q_2)^2}. \tag{13}$$

The radius r of a circle centered at R is

$$r = t\,\sin\theta, \tag{14}$$

where $\theta \in (0, \pi)$ is the angle between $\hat{\mathbf{u}}$ and $\hat{\mathbf{a}}$ which can be easily computed as follows,

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta} \quad \text{where} \quad \cos\theta = \hat{\mathbf{u}} \cdot \hat{\mathbf{a}}. \tag{15}$$

(In the case that $\theta \geq \pi$ (nonconvex vertex), then the radius r is simply $\|\overline{PR}\| = t$.)

We now demand that circle C_R touches circle C_Q, i.e.,

$$\|\overline{RQ}\| = \|\mathbf{x} - \mathbf{q}\| = r + r_q, \tag{16}$$

which implies that

$$\|\mathbf{x} - \mathbf{q}\|^2 = (r + r_q)^2. \tag{17}$$

But

$$\begin{aligned} \mathbf{x} - \mathbf{q} &= \mathbf{p} + t\,\hat{\mathbf{u}} - \hat{\mathbf{q}} \\ &= \mathbf{w} + t\,\hat{\mathbf{u}}, \end{aligned} \tag{18}$$

where

$$\mathbf{w} = \mathbf{p} - \mathbf{q}. \tag{19}$$

Then

$$\begin{aligned} \|\mathbf{x} - \mathbf{q}\|^2 &= (\mathbf{w} + t\hat{\mathbf{u}}) \cdot (\mathbf{w} + t\hat{\mathbf{u}}) \\ &= \|\mathbf{w}\|^2 + (2\mathbf{w} \cdot \hat{\mathbf{u}})t + t^2. \end{aligned} \tag{20}$$

Furthermore

$$\begin{aligned} (r + r_q)^2 &= (t\sin\theta + r_q)^2 \\ &= t^2\sin^2\theta + (2r_q\sin\theta)t + r_q^2. \end{aligned} \tag{21}$$

Equating these two results, we obtain the following quadratic equation in t,

$$t^2(1 - \sin^2\theta) + 2(\hat{\mathbf{w}} \cdot \hat{\mathbf{u}} - r_q\sin\theta)t + (\|\mathbf{w}\|^2 - r_q^2) = 0. \tag{22}$$

The coefficient of t^2 is

$$1 - \sin^2\theta = \cos^2\theta = (\hat{\mathbf{u}} \cdot \hat{\mathbf{a}})^2, \tag{23}$$

so that the quadratic equation may be written as

$$(\hat{\mathbf{u}} \cdot \hat{\mathbf{a}})^2 t^2 + 2(\mathbf{w} \cdot \hat{\mathbf{u}} - r_q \sin\theta)t + (\|\mathbf{w}\|^2 - r_q^2) = 0\,. \tag{24}$$

Note that this equation is valid only in the case $0 < \theta < \frac{1}{2}$. We'll write it in the canonical form,

$$At^2 + Bt + C = 0\,, \tag{25}$$

with roots

$$t_{\pm} = -\frac{B}{2A} \pm \frac{1}{2A}\left[B^2 - 4AC\right]^{1/2}\,. \tag{26}$$

From geometry, $C > 0$. (The distance from P to Q must be greater than the radius of circle C_Q. Moreover, since $\mathbf{w}$ and $\hat{\mathbf{u}}$ point in opposite directions and $\sin\theta > 0$, it follows that $B < 0$. We must therefore select the root with the lower magnitude, i.e.,

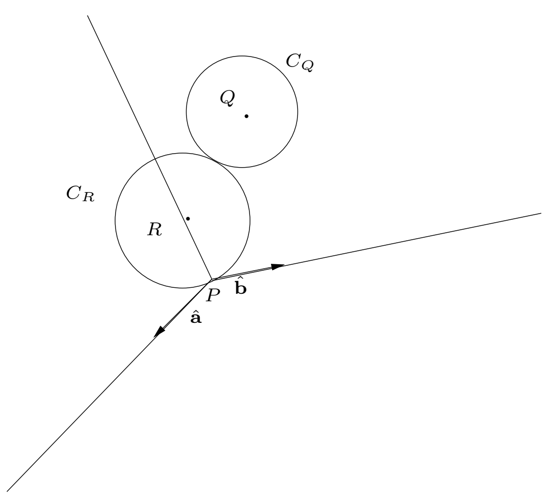
$$t = t_- = -\frac{B}{2A} - \frac{1}{2A}\left[B^2 - 4AC\right]^{1/2}\,. \tag{27}$$

The other root, $t_+ > t_-$, corresponds to a circle which lies on the other side of circle C_Q and touches it, i.e., it does not lie close to vertex P.

Finally, as $\theta \to \pi^-$, the two boundary/circle touching points approach the vertex P. The above formulas apply in the special case $\theta = \pi$, where the two boundary/circle touching points have met at P.

Nonconvex case

Corner packing at a nonconvex vertex – which is not really "corner packing" but rather "vertex packing" – is sketched below. Actually, nonconvex vertices do not present as great a difficulty in packing as convex ones, particularly those that are acute. As such, this type of packing may be viewed as optional. We discuss it for the sake of completeness.



In this case, the circle touches the boundary only at vertex P. In contrast to Eq. (9), the unit vector $\hat{\mathbf{u}}$ which runs in the direction of the bisector of the angle into the packed region D must be defined as follows,

$$\hat{\mathbf{u}} = -\frac{1}{2}\left(\hat{\mathbf{a}} + \hat{\mathbf{b}}\right)\,, \quad \hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = (\hat{u}_1, \hat{u}_2)\,. \tag{28}$$

As before, the point

$$\mathbf{x} = \mathbf{p} + t\hat{\mathbf{u}}\,, \quad t \geq 0\,, \tag{29}$$

lies on the bisector of the angle subtended by the segments. In coordinate form,

$$(x, y) = (p_1, p_2) + t\,(\hat{u}_1, \hat{u}_2)\,. \tag{30}$$

The distance from P to R is

$$\|\overline{PR}\| = |t| = t > 0\,. \tag{31}$$

The distance from R to Q is

$$\|\overline{RQ}\| = \|\mathbf{x} - \mathbf{q}\| = \sqrt{(x - q_1)^2 + (y - q_2)^2}\,. \tag{32}$$

Unlike the convex case, however, the radius r of a circle centered at R is

$$r = t > 0\,. \tag{33}$$

Once again, we demand that circle C_R touches circle C_Q, i.e.,

$$\|\overline{RQ}\| = \|\mathbf{x} - \mathbf{q}\| = r + r_q\,, \tag{34}$$

which implies that

$$\|\mathbf{x} - \mathbf{q}\|^2 = (r + r_q)^2\,. \tag{35}$$

But

$$\begin{aligned} \mathbf{x} - \mathbf{q} &= \mathbf{p} + t\,\hat{\mathbf{u}} - \hat{\mathbf{q}} \\ &= \mathbf{w} + t\,\hat{\mathbf{u}}\,, \end{aligned} \tag{36}$$

where

$$\mathbf{w} = \mathbf{p} - \mathbf{q}\,. \tag{37}$$

Then

$$\begin{aligned} \|\mathbf{x} - \mathbf{q}\|^2 &= (\mathbf{w} + t\hat{\mathbf{u}}) \cdot (\mathbf{w} + t\hat{\mathbf{u}}) \\ &= \|\mathbf{w}\|^2 + (2\mathbf{w} \cdot \hat{\mathbf{u}})t + t^2\,. \end{aligned} \tag{38}$$

Once again in contrast to the convex case, we have

$$\begin{aligned} (r + r_q)^2 &= (t + r_q)^2 \\ &= t^2 + (2r_q)t + r_q^2\,. \end{aligned} \tag{39}$$

Equating these two results, we obtain the following linear equation in t,

$$2(\hat{\mathbf{w}} \cdot \hat{\mathbf{u}} - r_q)t + (\|\mathbf{w}\|^2 - r_q^2) = 0\,, \tag{40}$$

which we shall write in the canonical form,

$$Bt + C = 0\,. \tag{41}$$

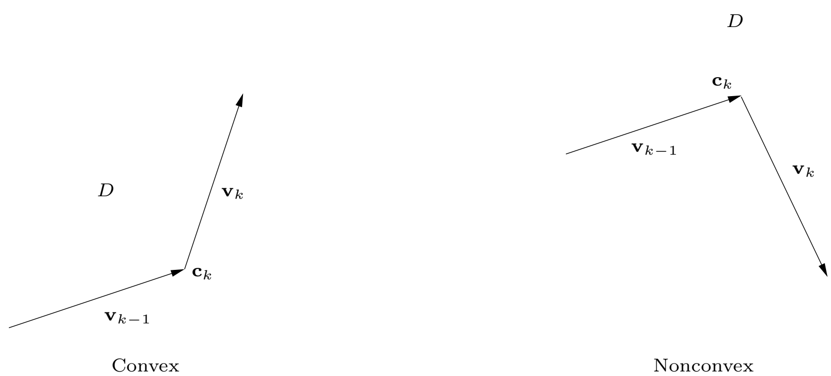
As in the convex case, $C > 0$. The solution of this equation is trivially

$$t = -\frac{C}{B} = -\frac{\|\hat{\mathbf{w}}\|^2 - r_q^2}{\hat{\mathbf{w}} \cdot \hat{\mathbf{u}} - r_q}\,. \tag{42}$$

Note: It is not necessarily the case that the vectors $\hat{\mathbf{w}}$ and $\hat{\mathbf{u}}$ point in the same direction.

Determining whether a vertex P is convex or nonconvex

Finally, we recall a simple test to determin whether or not a vertex P is convex. (This was discussed in some earlier notes on the GGL packing scheme.) In the figure below, we let the vertex P be given by coordinates $\mathbf{c}_k$, for some $k \in \{1, 2, \cdots, N\}$, as discussed in Step 0. The displacement vectors $\mathbf{v}_{k-1}$ and $\mathbf{v}_k$, also defined in Step 0, point, respectively, toward P and away from P. Recall that the region D enclosed by the polygonal boundary always lies **to the left** of its segments when travelled in a counterclockwise direction.



Since the entire boundary, hence all vectors $\mathbf{v}_k$, lie in the xy plane, the cross product

$$\mathbf{v}_{k-1} \times \mathbf{v}_k = a\,\mathbf{k}\,, \quad a \in \mathbf{R}. \tag{43}$$

It is easy to see that the vertex $\mathbf{c_k}$ is

$$\begin{array}{ll} \text{convex}, & a \geq 0\,, \\ \text{nonconvex(concave)}, & a < 0\,. \end{array} \tag{44}$$

Step 3: Two-circle packing touching boundaries

The idea, sketched in the figure below, is to use two adjacent (i.e., touching) boundary circles of radius R from Step 1, centered at A and B, to generate a third circle of radius $r < R$, centered at D (x, y) that touches each of them as well as the boundary segment $\overline{RS}$ so that the latter lies within the region D.

It is convenient to extend the line segment $\overline{RS}$ so that it intersects the line passing through the centers A and B at point O at an angle θ. The special case that $\theta = 0$, i.e., $\overline{AB} \parallel \overline{RS}$ will be treated later. The circle to be packed will touch the boundary segment $\overline{RS}$ at point T.

To be determined: The center (x, y) and radius r of the packed circle.

Let P be the midpoint of $\overline{AB}$ with coordinates,

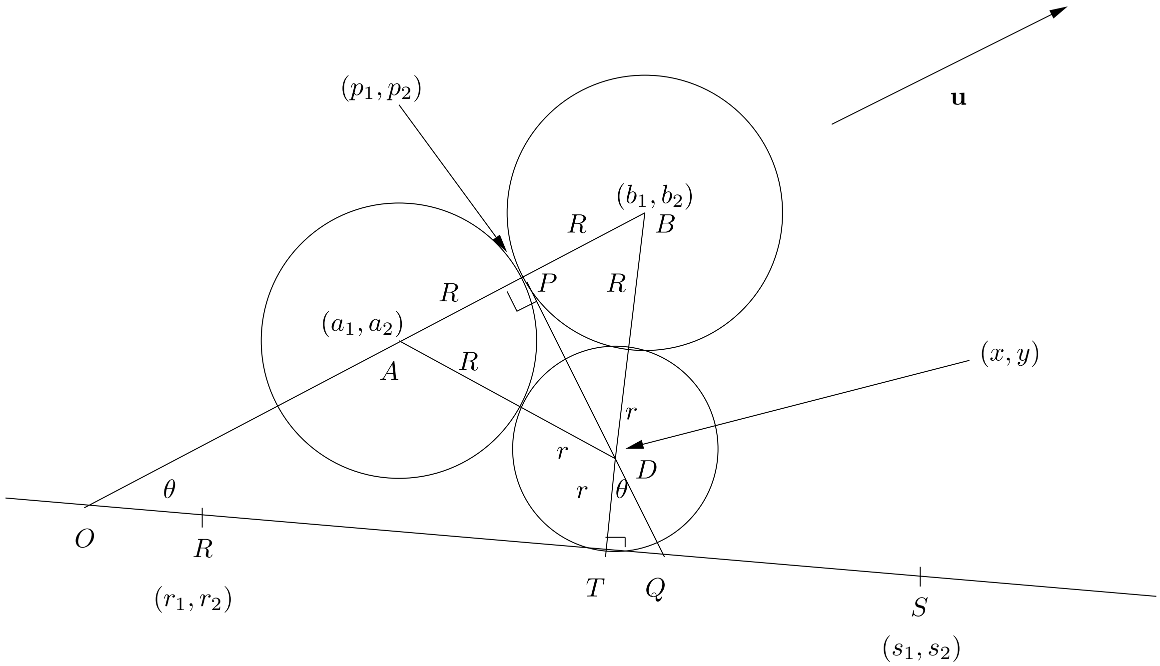
$$(p_1, p_2) = \left(\frac{1}{2}(a_1 + b_1), \frac{1}{2}(a_2 + b_2)\right)\,. \tag{45}$$

Let

$$\mathbf{u} = (u_1, u_2) = (b_1 - a_1, b_2 - a_2)\,. \tag{46}$$

The vector

$$\mathbf{v} = (-u_2, u_1) \tag{47}$$



is perpendicular to $\mathbf{u}$. (We could have also used $(u_2, -u_1)$. Define the unit vector,

$$\hat{\mathbf{v}} = (\hat{v}_1, \hat{v}_2) = \frac{\mathbf{v}}{\|\mathbf{v}\|}. \tag{48}$$

The center (x, y) of the packed circle will lie on the line $\overline{PQ}$ which is perpendicular to $\overline{AB}$ and therefore $\mathbf{u}$. Therefore,

$$(x, y) = (p_1, p_2) + s(\hat{v}_1, \hat{v}_2), \tag{49}$$

where the scalar s is to be determined. Note that

$$|s| = \sqrt{(R + r)^2 - R^2} = \sqrt{r(2R + r)}. \tag{50}$$

Of course, r is still unknown.

Now consider the point Q (q_1, q_2) which is the intersection of the extended line from $\overline{PD}$ and the boundary segment $\overline{RS}$.

$$(q_1, q_2) = (p_1, p_2) + t(\hat{v}_1, \hat{v}_2), \tag{51}$$

The point Q must lie on the boundary segment $\overline{RS}$, so its coordinates must satisfy the equation of this line.

1. **Case No. 1:** $\overline{RS}$ is not vertical. Then a point (x, y) on $\overline{RS}$ must satisfy the point-slope formula,

   $$y = m(x - r_1) + r_2, \quad \text{where} \quad m = \frac{s_2 - r_2}{s_1 - r_1}. \tag{52}$$

   From (51),

   $$p_2 + t\hat{v}_2 = m(p_1 + t\hat{v}_1 - r_1) + r_2. \tag{53}$$

   Solving for t:

   $$t = \frac{m(p_1 - r_1) + (r_2 - p_2)}{\hat{v}_2 - m\hat{v}_1}. \tag{54}$$

2. **Case No. 2:** $\overline{RS}$ is vertical so that $x = r_1$ for all points on this line. From (51),

$$p_1 + t\hat{v}_1 = r_1 \quad \Longrightarrow \quad t = \frac{r_1 - p_1}{\hat{v}_1} . \tag{55}$$

   Note that this result is also obtained by taking the limit $m \to \infty$ in (54).

Either of the above allows us to compute the coordinates (q_1, q_2) of Q, but we actually don't need them. And one final note: t can be positive or negative.

Now note that

$$\begin{aligned} T = |t| &= |\overline{PD}| + |\overline{DQ}| \\ &= \sqrt{(R+r)^2 - R^2} + r\sec\theta , \end{aligned} \tag{56}$$

so that

$$T - r\sec\theta = \sqrt{(R+r)^2 - R^2} . \tag{57}$$

Squaring both sides yields,

$$\begin{aligned} T^2 - 2rT\sec\theta + r^2\sec^2\theta &= (R+r)^2 - R^2 \\ &= 2Rr + r^2 . \end{aligned} \tag{58}$$

A rearrangement yields the following quadratic equation in r:

$$(\tan^2\theta)r^2 - 2(T\sec\theta + R)r + T^2 = 0 . \tag{59}$$

(We have used the identity $\sec^2\theta - 1 = \tan^2\theta$).) There are two cases to consider here:

1. **Case No. 1:** $\theta = 0$, i.e., when the lines $\overline{AB}$ and $\overline{RS}$ are parallel. In this case,

$$r = \frac{T^2}{2(R+T)} . \tag{60}$$

   This result is valid under the condition that $R \leq T$: Otherwise, each Step 1 circle of radius R would be intersecting the boundary $\overline{RS}$ at more than one point.

2. **Case No. 2:** $\theta \neq 0$. The two roots of (59) are

$$r_\pm = \frac{1}{\tan^2\theta}\left[(T\sec\theta + R) \pm \sqrt{(T\sec\theta + R)^2 - (\tan^2\theta)T^2}\right] . \tag{61}$$

   We choose the solution r_-, i.e.,

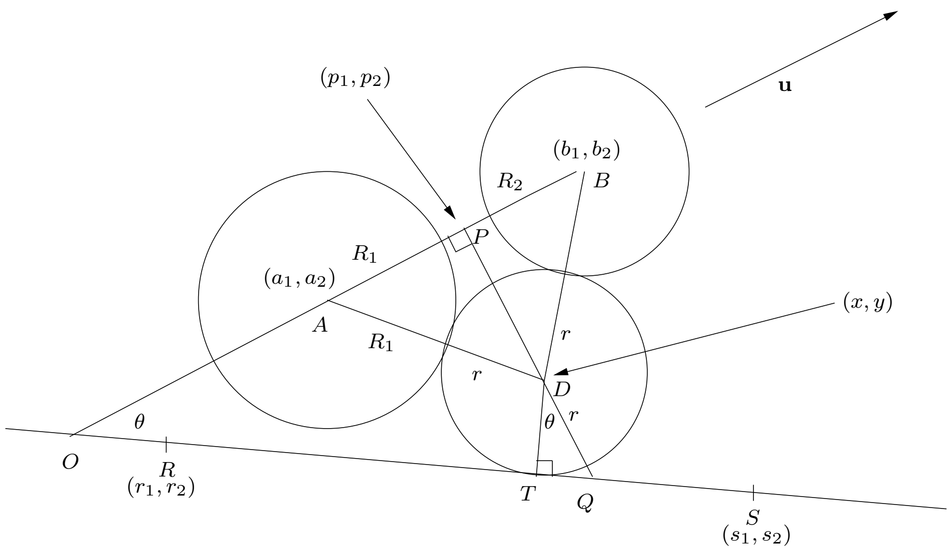
$$r = \frac{1}{\tan^2\theta}\left[(T\sec\theta + R) - \sqrt{(T\sec\theta + R)^2 - (\tan^2\theta)T^2}\right] . \tag{62}$$

   A little calculus (linear approximation of the square root) shows that this solution, in the limit $\theta \to 0$, becomes the solution in (60). The other solution $r_+ \to \infty$ as $\theta \to 0$.

From a knowledge of r, we can then compute (x, y), the center D of the packed circle.

Step 4: Two-circle packing, unequal circles, touching boundaries

The idea, sketched in the figure below, is to use two not-necessarily-touching boundary circles of not-necessarily-equal radii R_1 and R_2 from Steps 1 and 2, centered at A and B, to generate a third circle of radius $r < R$, centered at D (x, y) that touches each of them as well as the boundary segment $\overline{RS}$ so that the latter lies within the region D.



A neat, analytic solution does not seem to exist for the general case. As such, we must resort to approximative methods. One approach to is find estimates of the solution of the three equations that must be satisfied by the coordinates (x, y) of the center of the third circle and its radius r using, for example, the Newton-Rapson method. The first two equations which must be satisfied are

$$\begin{aligned} (x - a_1)^2 + (y - a_2)^2 &= (R_1 + r)^2 \qquad (|\overline{DA}| = R_1 + r) \\ (x - b_1)^2 + (y - b_2)^2 &= (R_2 + r)^2 \qquad (|\overline{DB}| = R_2 + r) \,. \end{aligned} \tag{63}$$

The third equation comes from the condition that the distance from (x, y) to the boundary segment $\overline{RS}$, i.e., $|\overline{TD}|$, is r. If $\overline{RS}$ is not vertical, then we have the equation,

$$(y - m(x - r_1) - r_2)^2 = r^2(1 + m^2)\,, \qquad \text{where} \;\; m = \frac{s_2 - r_2}{s_1 - r_1}\,. \tag{64}$$

If $\overline{RS}$ is vertical, i.e., $r_1 = s_1$, then we have the equation,

$$(x - r_1)^2 = r^2\,. \tag{65}$$

Eq. (64) comes from the easily-derived result that the distance d from the point (x, y) to the line $y = mx + b$ is

$$d = \frac{|y - mx - b|}{\sqrt{1 + m^2}}\,. \tag{66}$$

Instead of using the above approach, we employ an iterative method of "circle growing", starting with a circle of prescribed radius $r = r_{\text{min}}$ and pack it using two-circle packing, i.e., touching the two circles of radii R_1 and R_2 in the above figure. (The formulas for two-circle packing are given in Appendix C.) With reference to the above figure, we use

$$r_{\text{min}} = L - (R_1 + R_2) + \epsilon\,, \quad L = |\overline{AB}| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}\,, \tag{67}$$

where $\epsilon > 0$.

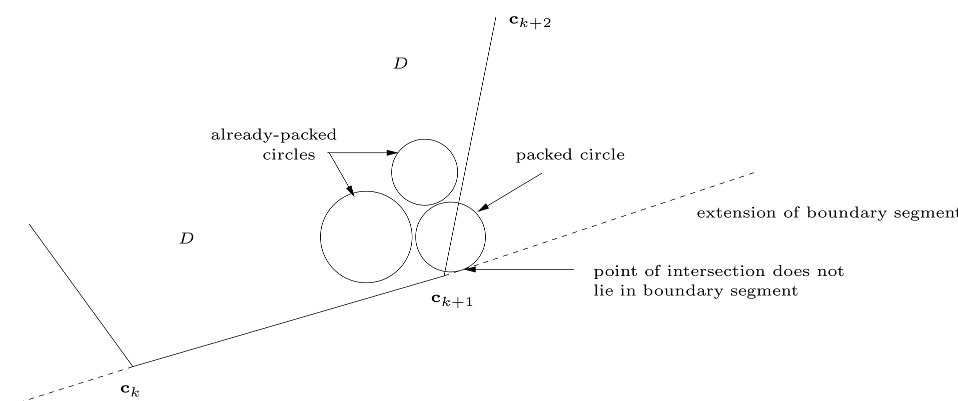
Using Eq. (64) or (65), we then check if the center (x, y) of this circle lies within a small perturbation of distance r from the boundary segment $\overline{RS}$. Most probably, it doesn't, so we increase the radius of the circle to be packed by some prescribed, small amount, Δr. The procedure is repeated until the distance d between the center (x, y) of the packed circle of radius r and the boundary segment $\overline{RS}$ satisfies the following condition

$$d - \Delta r < r \leq d\,. \tag{68}$$

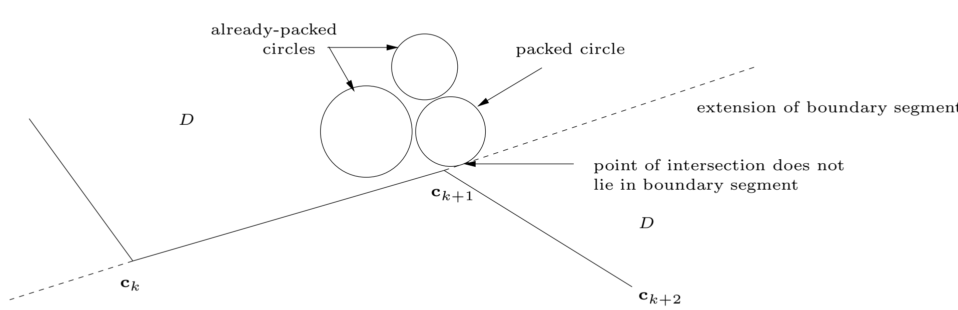
(The next increment Δr of the radius r produces a circle with points lying outside the region D.)

An extremely important note regarding Step 3 or 4 packing of circles which touch boundaries

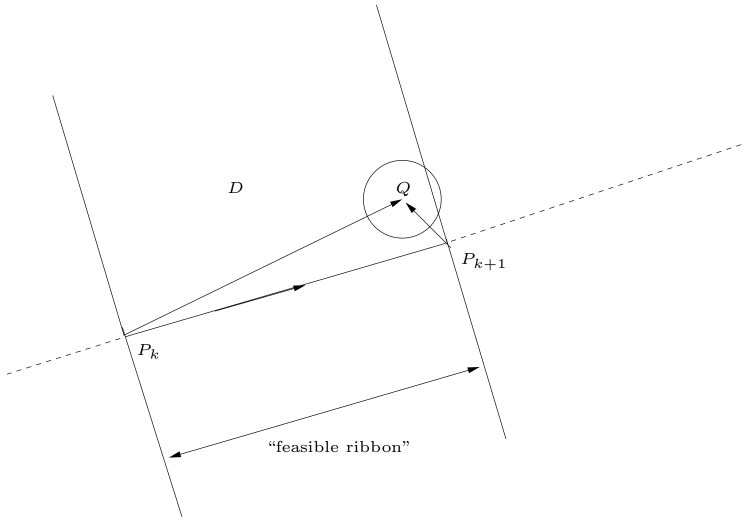
In the previous sections dealing with Step 3 and Step 4 circle-packing, it is possible (and, indeed, most probable) to encounter a situation in which a third circle touching two already-packed circles will **not** touch a boundary segment but rather an extension of it, either inside or outside the polygonal region D of interest. Such a situation is shown below.



In this situation, the packed circle would most probably be discarded because it lies outside region D. But in the case that the boundary segment from $\mathbf{c}_{k+1}$ to c_{k+2} pointed in the other direction, as shown below, the packed circle lies in region D. As such it would not be discarded for lying outside D. The question remains, however, whether this packed circle is acceptable since it does not touch a boundary segment. There may, for example, be a better way to pack a circle to touch the boundary segment from $\mathbf{c}_{k+1}$ to c_{k+2} using other circles.



The reason that such violations can occur is that in the determination of the third circle we simply employ the formula $y = mx + b$ (or $x = x_0$, if it is vertical) for the boundary segment, not imposing any constraints associated with the endpoints. As such, **it is necessary to check that the packed circle touches the boundary segment between its endpoints/vertices and not beyond them.** This can be done by demanding that the center $Q = (x, y)$ of the third circle lies in the "ribbon" which is perpendicular to the boundary segment, as shown below.



First of all, recall each boundary segment defines a vector $\overrightarrow{P_k P_{k+1}}$ which is oriented so that the region D lies to its left (counterclockwise orientation). For the center Q to lie within the "feasible ribbon" shown above, it is necessary that

$$\overrightarrow{P_k P_{k+1}} \cdot \overrightarrow{P_k Q} > 0 \quad \text{and} \quad \overrightarrow{P_k P_{k+1}} \cdot \overrightarrow{P_{k+1} Q} < 0\,. \tag{69}$$

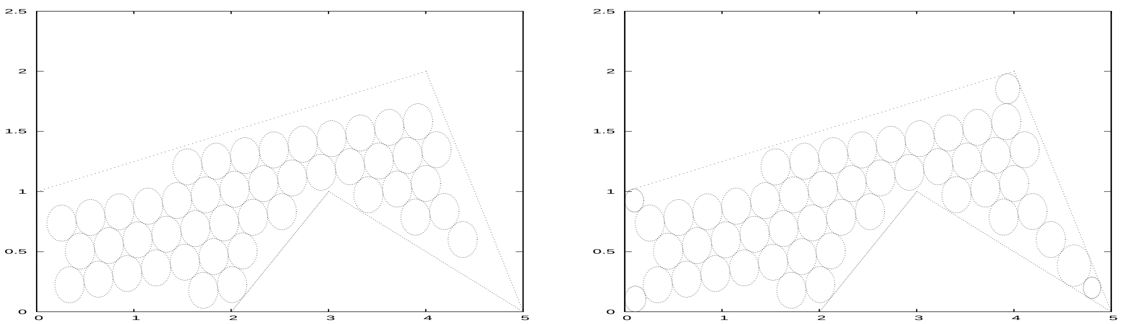
But it is also necessary that center Q lie to the left of boundary segment $P_k P_{k+1}$, i.e., the upper half of the ribbon, so that

$$\overrightarrow{P_k P_{k+1}} \times \overrightarrow{P_k Q} = A\mathbf{k}\,, \qquad \text{where} \quad A > 0\,. \tag{70}$$

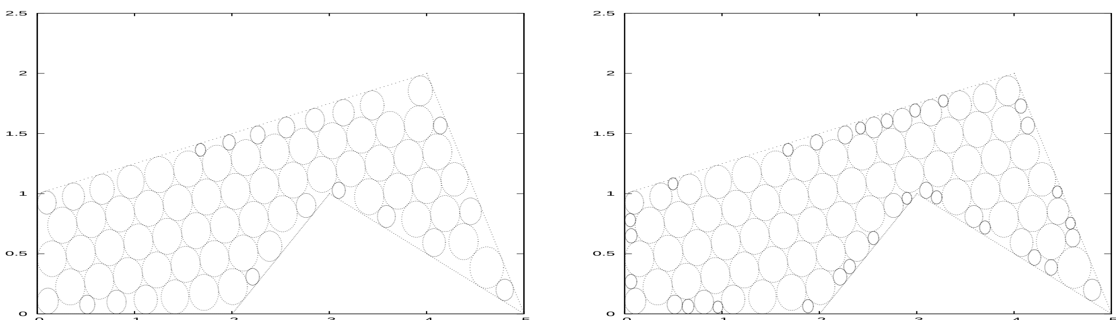
After the coordinates (x, y) of the center of the circle to be packed are determined from either Step 3 or Step 4, its feasibility can be determined from the above conditions since the coordinates of the vertices P_k are known.

Additional example - a nonconvex region

Here, we show some results for a more complicated nonconvex region composed of six sides, as shown in the figure below. As in the first section, we use Step 1 packing circles of radius $R = 0.15$. The reference point is once again the centroid of the region – in this case $(\bar{x}, \bar{y}) \cong (2.303, 0.879)$. The angle of inclination was once again chosen to be $\theta = \frac{\pi}{10}$ radian so that it would not coincide with the directions of any of the sides of the polygonal boundary. Unlike the previous case studied, a minimum radius threshold of $r_{\min}$ was imposed – no circles with radii less than $r_{\min}$ are packed.



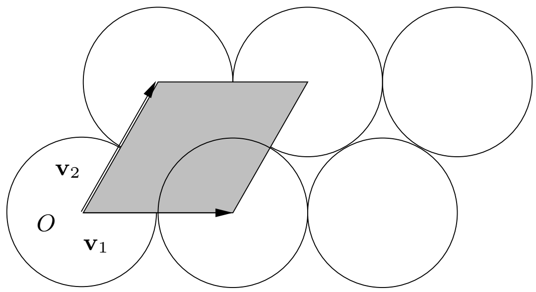
(Left) Step 1: 46 circles of radius $R = 0.15$. Packing fraction $\phi = 0.59$. **(Right)** Step 2: 46 circles from Step 1 along with 5 additional circles packed in corners. Packing fraction $\phi = 0.63$.



(Left) Step 3: An additional 24 boundary circles are packed. Packing fraction $\phi = 0.80$. **(Right)** Step 4: An additional 17 boundary circles are packed. Packing fraction $\phi = 0.85$.

Packings associated with fractional translations of origin from its fundamental position

In the previous version of this scheme outlined above, the origin O was, for convenience, chosedn to be the centroid $(\bar{x}, \bar{y})$ of the region D. All rotations were performed with respect to this origin. In this new version, FORTRAN program `newpack350.f`, we examine the packings yielded by translating the origin to a number of positions relative to the centroid. These positions lie within the basic parallelogram unit cell with sides of length $2R$ and lower left vertex situated at the centroid, as shown in the figure below.



The translations are fractional multiples of the principal horizontal and diagonal vectors that form the sides of the unit cell parallelogram

$$\mathbf{v}_1 = (2R, 0)\,, \qquad \mathbf{v}_2 = (R, \sqrt{3}R)\,. \tag{71}$$

In the computations reported below, we have examined the following 100 possible translations associated with the basic fractional unit 0.1:

$$(x_0, y_0) = (\bar{x}, \bar{y}) + \frac{n_1}{10}\mathbf{v}_1 + \frac{n_2}{10}\mathbf{v}_2\,, \quad 0 \le n_1, n_2 \le 9\,. \tag{72}$$

At each of these 100 positions of the origin, 10 angles of orientation between 0 and $\pi/6$ (rotations about the origin) were investigated, i.e., the orientation angles

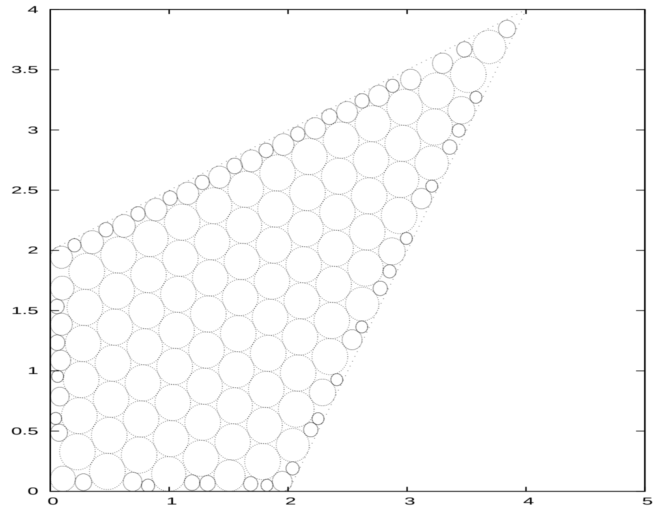
$$\theta_k = k\,\Delta\theta\,, \qquad 0 \le k \le 9\,, \quad \text{where } \Delta\theta = \frac{1}{10}\cdot\frac{\pi}{6}\,. \tag{73}$$

During each packing, "boundary circles," i.e., the layer of circles lying closest to the outer boundary, are identified.

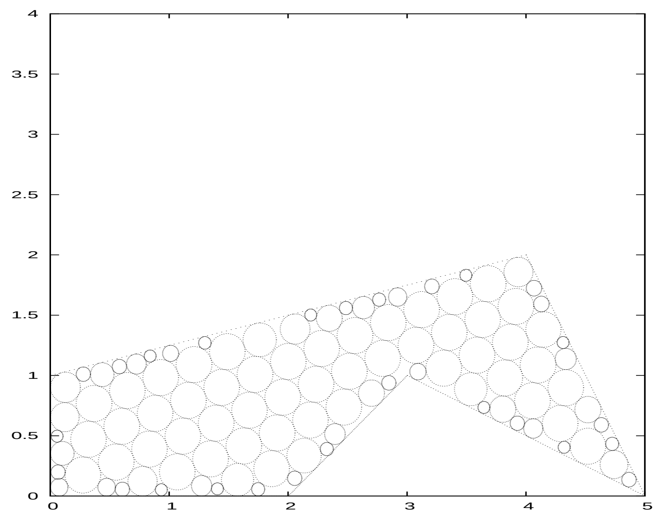
We then proceed with Steps 2-4 as outlined earlier:

- **Step 2:** "Corner packing."
- **Step 3:** Two-circle packing touching boundaries, using Step 1 boundary circles.
- **Step 4:** Two-circle packing touching boundaries, using Step 1-3 boundary circles.

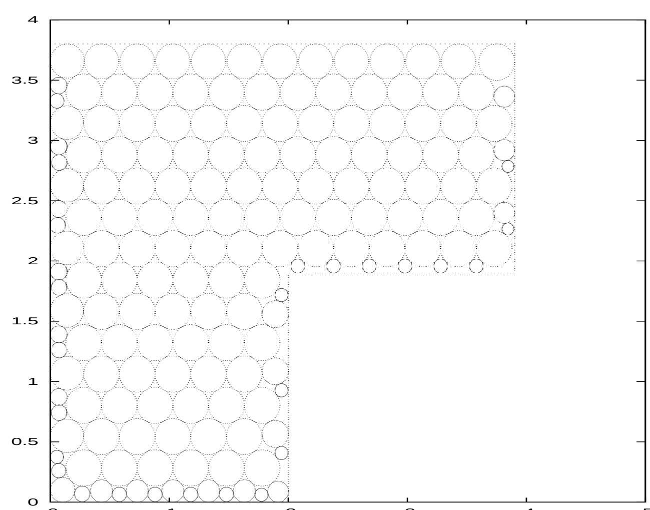
On the next page are shown the best packings for three given regions – packings that yield the largest covering fraction over all 100 positions of the origin and 10 angles of orientation. In all cases, the radii of the Step 1 circles were $R = 0.15$. In the next section, we shall consider a variety of values of R and extract the best packing from them.



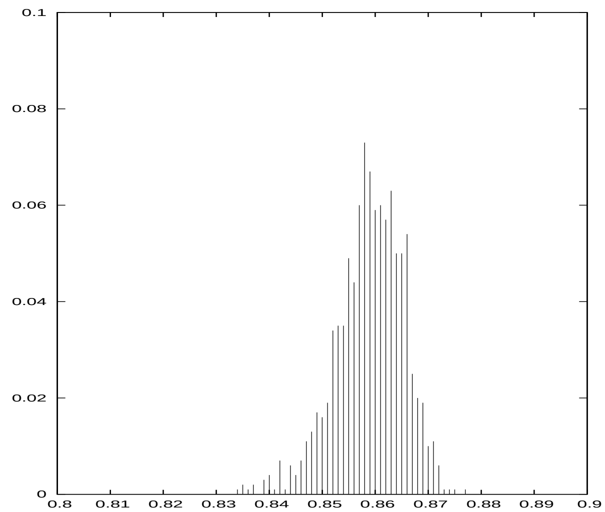
Trapezoidal region: 146 circles packed. Optimal angle of inclination $\theta = 0.471$ radian. Packing fraction $\phi = 0.88$, a slight improvement of result from Section 1. (Lowest packing fraction in ensemble: $\phi_{\min} = 0.83$.)



Nonconvex region: 102 circles packed. Optimal angle of inclination $\theta = 0.367$ radian. Packing fraction $\phi = 0.86$. Lowest packing fraction in ensemble: $\phi_{\min} = 0.79$.)



An L-shaped region with vertical side lengths that are incommensurate with R: 174 circles packed. Optimal angle of inclination $\theta = 0$. Packing fraction $\phi = 0.88$. (Lowest packing fraction: $\phi_{\min} = 0.84$.)



Histogram plot showing the probability distribution of the packing fractions ϕ for the 1000 packings of the trapezoidal region with $R = 0.15$.

For each region, we have also included the lowest packing fraction $\phi_{\min}$ encountered in the 1000 different initial conditions. In the trapezoidal region case, for example, $\phi_{\max} = 0.88$ and $\phi_{\min} = 0.83$ are not very different – less than 10% from each other, as is observed for the other cases. A histogram plot showing the (probability) distribution of the 1000 packing fractions computed for the trapezoidal region with $R = 0.15$ is presented in the figure above.

Packings associated with different R, the radius of the largest packing circles

We now examine the effect of changing the radius R of the largest (Step 1) circles on the packing of a given region.

In our first experiment, we consider the convex trapezoidal region examined in earlier sections. For each R value, the packing with the highest efficiency, i.e., the highest fraction ϕ of area covered by the packed circles is chosen from 1000 possible starting conditions – 100 fractional translations of the origin over a basic unit cell (as described in the previous section) and 10 values of the orientation angle between 0 and $\pi/6$ radians. Using FORTRAN program `newpack350.f`, the following criteria were employed:

- Circles of radius $r < 0.05$ were not allowed in the packing, i.e., $r_{\min} = 0.05$. This is a rather arbitrary, yet seemingly reasonable, value chosen to simulate the effect of manufacturing limits.
- A maximum of **four** Step 2 corner-packing iterations was used.
- Only **one** application of the Step 3 two-circle packing scheme was considered. This is suboptimal since the boundary circles of radius R were chosen in a definite order – other orders could yield better packing.
- A maximum of **three** Step 4 two-circle packing iterations was used. This is a quite time-consuming part of the program. Three applications of Step 4 were presumed sufficient since they generally produce circles with radii that are smaller than $r_{\min}$.

Packings for the convex trapezoidal region using the values $R = 0.1, 0.15, 0.2, 0.3, 0.4$ and 0.5 are shown in the figures below. In the caption for each packing is presented its packing fraction ϕ along with the total number of circles employed in the packing. Perhaps most noteworthy is the fact that the packing fractions are all virtually the same, around 0.88. This might seem counterintuitive since one might think that packing efficiency should increase with smaller circle sizes. After all, the area $A(R)$ of the unpacked region enclosed by three circles of radius R that touch each other (their centers lie on an equilateral triangle with sidelength $2R$) decreases with R. But as R decreases, the number of such unpacked areas increases. As a result, the net unpacked area by these circles is (roughly) constant.

The roughly constant packing fraction is also a testimony to the effectiveness of this new packing scheme in which boundary circles are "tailor made" to fit unpacked regions between previously packed boundary circles and the boundary of the region.

As discussed early in the project, there are at least two problems associated with smaller tube sizes: (i) more tubes are needed and (ii) the resistance of a tube increases with decreasing radius r. In an effort to quantitatively assign some kind of penalty associated with smaller tubes, we have computed the following very simple penalty function asssociated with a packing that employs Step 1 circles of radii R:

$$P(R) = \sum_{k=1}^{N} \frac{1}{r_k^4}, \qquad r_k \geq r_{\min}. \tag{74}$$

where the summation runs over all packed circles. (Note that we have omitted the inequality $r_k \leq R$ since it is possible that circles of radii greater than R are packed.) The caption of each figure below also presents penalty function computed for that packing.

Apart from a few "blips," the penalty function values of the packings are seen to decrease with R. As such, packings with larger R values could be viewed as more favorable from the point of view of resistance. But as R increases, the number of Step 1 circles that can be fitted into the region must decrease.

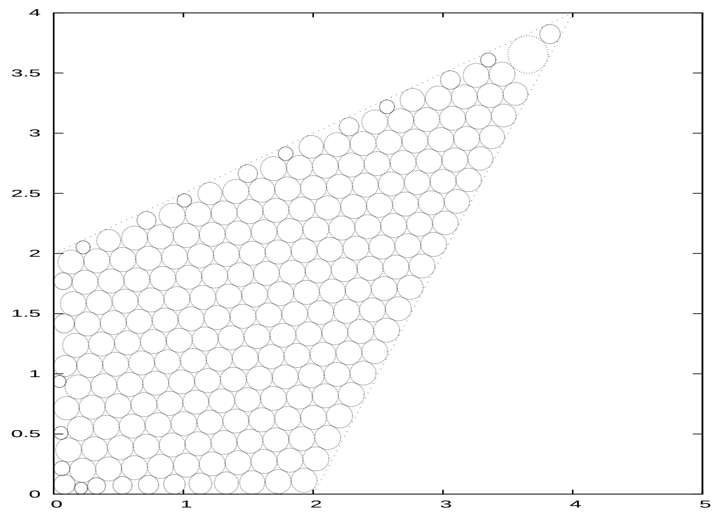
To illustrate, the packing associated with $R = 0.4$ uses 8 Step 1 circles and one circle with $r > 0.4$ produced by cornerpacking. When $R = 0.5$, four Step 1 circles are packed into the region along with one cornerpacked circle of greater radius. And when $R = 0.6$, the optimal packing employs only one Step 1 circle . The other large circles – one much larger and two slightly smaller – are constructed by the cornerpacking scheme.

For higher R values, the situation is rather uncertain. Even if a Step 1 circle of radius R can be packed into a region, it is not guaranteed that this circle will be included in the optimal packing of the region, where all 1000 fractional translations of the circle (and possibly neighbours) over a parallelogram unit cell are considered. Some translations and rotations could produce situations where no Step 1 circles can be packed into the region and yet an optimal packing is produced. This is seen in the case $R = 0.7$ shown below. There are no circles of radius $r = 0.7$ in the optimal packing – the circle in the lower left corner has radius $r \sim 0.6667$. The same phenomenon – no Step 1 circles in the optimal packing – is observed for $R = 0.8$ and $R = 0.9$. Technically speaking, if there is no Step 1 circle at the start of a packing, Step 2 cornerpacking should not take place since there is no "closest Step 1 circle" to a corner. The program presumably uses a circle from a previous packing.

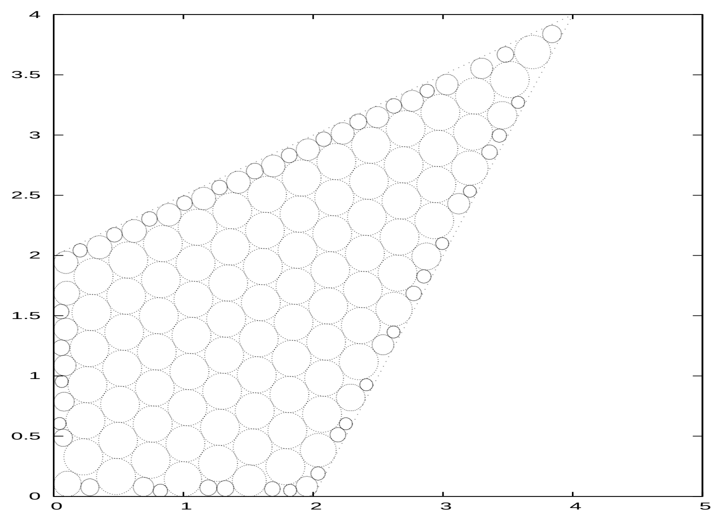
What is remarkable, however, is that the packing fractions for these larger R packings are quite high – higher, in fact, that packing fractions for lower R values! This leads to the question of using other strategies to start the packing of a region with larger circles. For example, we may wish to insert at least one large Step 1 circle into the region and not consider translations of it.

But this all being said, it may be necessary to limit the size of the largest circles because of connectivity issues. The question of whether a tube of radius r_a can be endcap- or merge-connected to a tube of radius r_b when the two radii have significantly different values remains open. It may be necessary to place upper bounds on the ratio $\frac{r_a}{r_b}$, where $r_a > r_b > 0$.

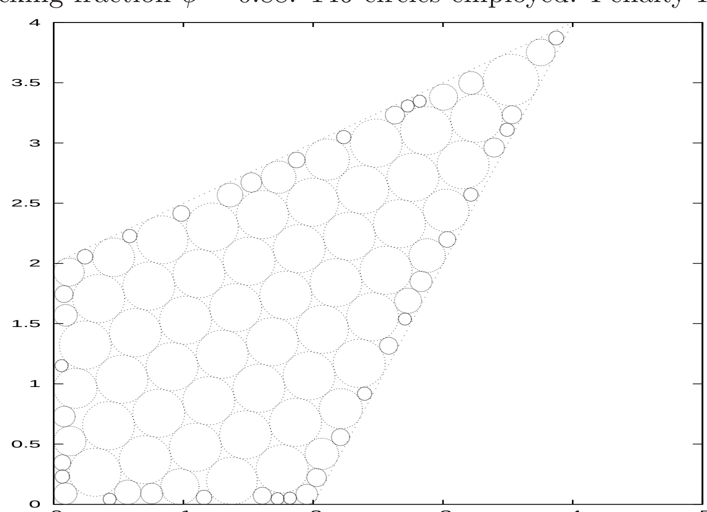
Packings produced with Step 1 circles of different R-values



$R = 0.1$. Packing fraction $\phi = 0.87$. 232 circles employed. Penalty $P = 3.5 \times 10^6$.

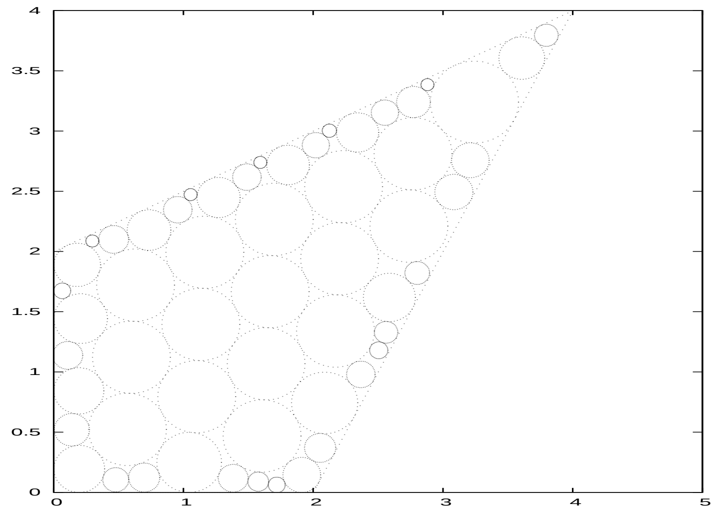


$R = 0.15$. Packing fraction $\phi = 0.88$. 146 circles employed. Penalty $P = 4.0 \times 10^6$.

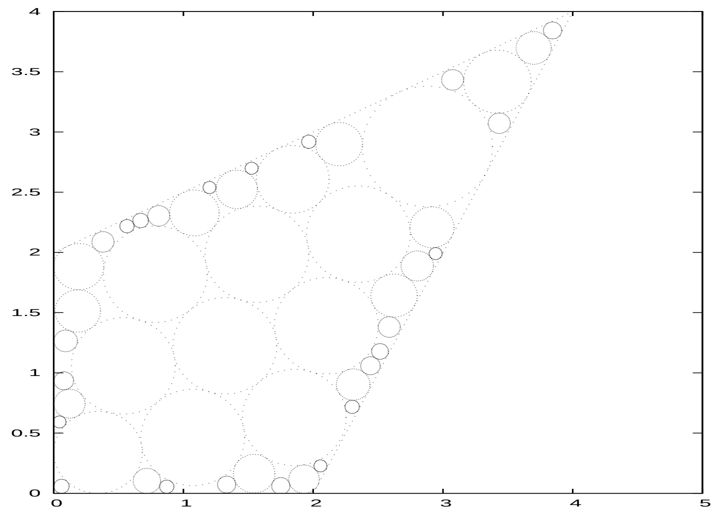


$R = 0.2$. Packing fraction $\phi = 0.88$. 96 circles employed. Penalty $P = 2.7 \times 10^6$.

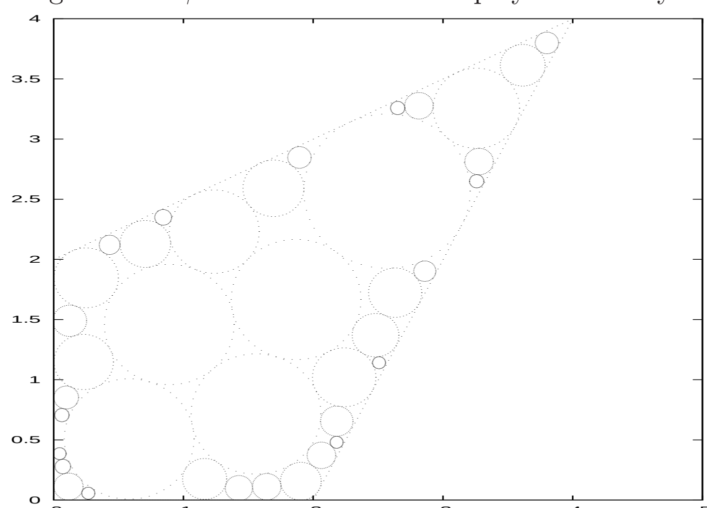
Packings produced with Step 1 circles of different R-values (cont'd)



$R = 0.3$. Packing fraction $\phi = 0.87$. 56 circles employed. Penalty $P = 1.1 \times 10^6$.

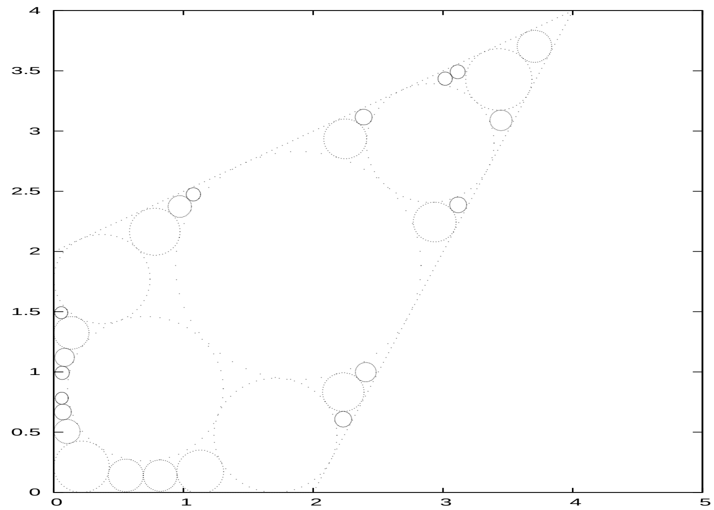


$R = 0.4$. Packing fraction $\phi = 0.88$. 49 circles employed. Penalty $P = 1.8 \times 10^6$.

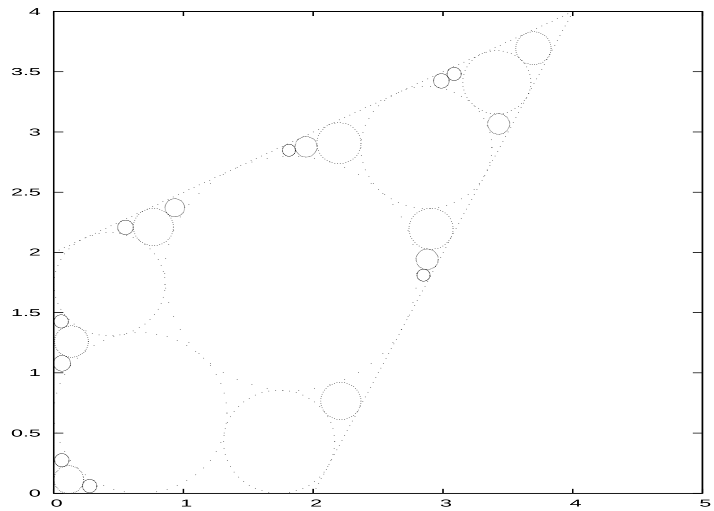


$R = 0.5$. Packing fraction $\phi = 0.88$. 39 circles employed. Penalty $P = 1.2 \times 10^6$.

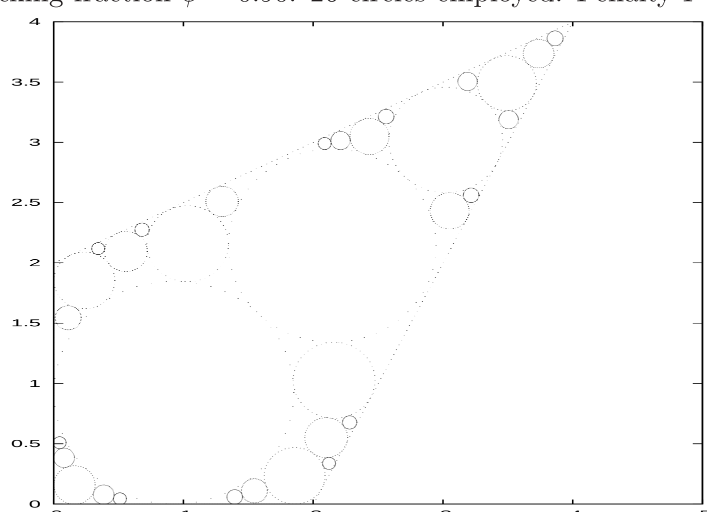
Packings produced with Step 1 circles of different R-values (cont'd)



$R = 0.6$. Packing fraction $\phi = 0.89$. 31 circles employed. Penalty $P = 1.1 \times 10^6$.

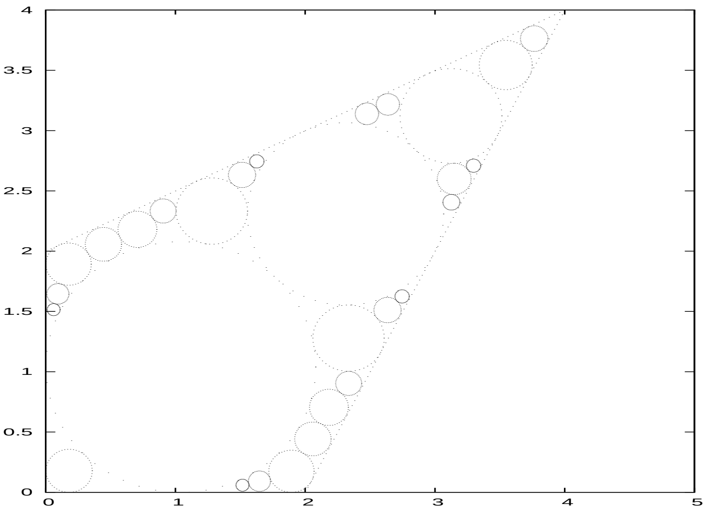


$R = 0.7$. Packing fraction $\phi = 0.90$. 26 circles employed. Penalty $P = 1.1 \times 10^6$.



$R = 0.8$. Packing fraction $\phi = 0.90$. 33 circles employed. Penalty $P = 1.5 \times 10^6$.

Packings produced with Step 1 circles of different R-values (cont'd)



$R = 0.9$. Packing fraction $\phi = 0.90$. 29 circles employed. Penalty $P = 8.3 \times 10^5$. No Step 1 circles employed are employed in the optimal packing.

(75)

Appendix A: Using Green's Theorem to compute areas and centroids of polygonal regions

Green's Theorem (in the plane): Let $D \subset \mathbf{R}^2$ be a bounded and simply connected region enclosed by a (boundary) curve C which is simple and piecewise smooth. (Polygonal regions satisfy this condition: C is a union of line segments.) Let $\mathbf{F}(x_1, x_2) = (F_1(x_1, x_2), F_2(x_1, x_2))$ be a C^1 (planar) vector field defined over D. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int\int_D \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dA \,. \tag{76}$$

In order to compute the area of D, to be denoted as $A(D)$, the integrand of the double integral on the RHS must be 1. Here, we shall use the convenient vector field $\mathbf{F} = (0, x_1)$. (We could also have used, for example, $\mathbf{F} = (-x_2, 0)$.

Recall that if the curve C may be parametrized as $\mathbf{x}(t) = (x_1(t), x_2)(t)$, $a \le t \le b$, then the line integral on the left of (76) may be evaluated as follows,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b F(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \, dt \,. \tag{77}$$

In the case where curve C is polygonal, the line integral becomes

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \sum_{k=1}^{n_s} \int_{C_k} \mathbf{F} \cdot d\mathbf{r} \,, \tag{78}$$

where the integrations are performed over all line segments C_k comprising C. Each line segment C_k may be parametrized as follows,

$$\mathbf{x}_k(t) = \mathbf{c}_k + t\mathbf{v}_k \,, \quad 0 \le t \le 1 \,, \tag{79}$$

where the $\mathbf{c}_k$ denote the vertices of the polygon and the $\mathbf{v}_k$ are the displacement vectors defined in the main text. Therefore, for the case $\mathbf{F} = (0, x)$,

$$\begin{aligned} \int_{C_k} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{x}_k(t)) \cdot \mathbf{v}_k(t) \, dt \\ &= \int_0^1 (0, x_{k1}(t)) \cdot (v_{k1}, v_{k2}) \, dt \\ &= \int_0^1 x_{k1}(t) v_{k2} \, dt \\ &= \int_1^1 (c_{k1} + t v_{k1}) v_{k2} \, dt \\ &= \left[c_{k1} + \frac{1}{2} v_{k1} \right] v_{k2} \,. \end{aligned} \tag{80}$$

Summation over all line segments yields the area, $A(D)$,

$$A(D) = \sum_{k=1}^{n_s} \left[c_{k1} + \frac{1}{2} v_{k1} \right] v_{k2} \,. \tag{81}$$

For the quadrilateral region shown in the overview, with coordinates and displacement vectors given in Eqs. (3) and (4), respectively, we compute $A(D)$ to be

$$A(D) = (0)(0) + (2+1)(4) + (4-2)(-2) + (0)(-2) = 8 \,. \tag{82}$$

The coordinates $(\bar{x}_1, \bar{x}_2)$ of the centroid of region D are given by

$$\bar{x}_1 = \frac{1}{A(D)} \int\int_D x_1 \, dA \, , \quad \bar{x}_2 = \frac{1}{A(D)} \int\int_D x_2 \, dA \, . \tag{83}$$

Using Green's Theorem, the two double integrals may be evaluated by choosing appropriate forms of the vector field $\mathbf{F}$. For the integral $\int\int_D x_1 \, dA$, one can use the vector field $\mathbf{F} = (0, \frac{1}{2}x_1^2)$. For the integral $\int\int_D x_2 \, dA$, one can use the vector field $\mathbf{F} = (0, x_1 x_2)$. Here we simply state the final results,

$$\bar{x}_1 = \frac{1}{2A(D)} \sum_{k=1}^{N} \left[c_{k1}^2 + c_{k1} v_{k1} + \frac{1}{3} v_{k1}^2 \right] v_{k2} \, , \tag{84}$$

and

$$\bar{x}_2 = \frac{1}{A(D)} \sum_{k=1}^{N} \left[c_{k1} c_{k2} + \frac{1}{2} \left(c_{k2} v_{k1} + c_{k1} v_{k2} \right) + \frac{1}{3} v_{k1} v_{k2} \right] v_{k2} \, . \tag{85}$$

For the quadrilateral region, the centroid is found to be located at

$$(\bar{x}_1, \bar{x}_2) = \left(\frac{5}{3}, \frac{5}{3} \right) . \tag{86}$$