You may assume that all representations are over $F = \mathbb{C}$.

Q1. Tell me a bit about yourself.

(a) Have you taken any other 400-level (or equivalent) math courses? If so, which ones?
(b) What do you hope to get out of PMATH 445/745?

Q2. Let G be a finite group.

(a) Let X be a finite G-set and let V be the induced permutation representation. Show that $\dim V^G$ is equal to the number of G-orbits in X. [A **G-set** is a set with a G-action. A **G-orbit** is a subset of X of the form $\mathcal{O}_x = \{gx \colon g \in G\}$ for some $x \in X$.]
(b) Let $V = \mathcal{F}(G, F)$ be the regular representation of G. Show that V^G is the subspace of constant functions, and explain how $\dim V^G = 1$ is consistent with part (a).

Q3. Let $G = S_3$ and let $\sigma = (1\ 2)$ and $\tau = (1\ 2\ 3)$. Note that σ and τ generate G. Let $H = \langle \sigma \rangle$ be the subgroup of G generated by σ, and let $\rho \colon G \to GL(V)$ be the permutation representation induced by the action of G on G/H.

(a) Write down a basis for V consisting of coset representatives for G/H. Then write down the corresponding matrices for $\rho(\sigma)$ and $\rho(\tau)$.
(b) Prove that V is isomorphic to the defining representation of S_3.

Q4. The group $C_2 = \langle a \rangle$ has exactly two distinct one-dimensional representations over $\mathbb{C}$—namely, $\chi_\pm \colon C_2 \to \mathbb{C}^\times$ given by $\chi_\pm(a^i) = (\pm 1)^i$.

(a) Show that every $\mathbb{C}C_2$-module V can be decomposed as a direct sum $V = V_+ \oplus V_-$ of G-invariant subspaces where, for $v \in V_\pm$, $\rho(a^i)v = \chi_\pm(a^i)v$. [Hint: Eigenvectors?]
(b) The group C_2 acts on $V = M_n(\mathbb{C})$ by $a \cdot A = A^T$. (Note that this is a linear action.) The decomposition $V = V_+ \oplus V_-$ in this case is a familiar one. What is it? That is, describe as succinctly as possible the subspaces $V_\pm$ and the decomposition $A = A_+ + A_-$ of a matrix A into the sum of matrices $A_\pm \in V_\pm$.

Q5. Let V and W be finite-dimensional F-vector spaces.

(a) Let $z \in V \otimes W$. Prove that there exist linearly independent $v_1, \ldots, v_n \in V$ such that $z = \sum_{i=1}^n v_i \otimes w_i$ for some $w_i \in W$.
(b) Let $V_0 \subseteq V$ and $W_0 \subseteq W$ be subspaces. Show that there is a unique injective linear map $T \colon V_0 \otimes W_0 \to V \otimes W$ such that $T(v_0 \otimes w_0) = v_0 \otimes w_0$. [Note: The two occurrences of the symbol $v_0 \otimes w_0$ have two meanings. The first is an element of the tensor product space $V_0 \otimes W_0$; the second is an element of $V \otimes W$. A byproduct of this exercise is that there is no ambiguity here: The map T allows us to view $V_0 \otimes W_0$ as a subspace of $V \otimes W$ in a canonical way—so we can identify both instances of $v_0 \otimes w_0$ with each other.]
(c) True or false? Every subspace of $V \otimes W$ is of the form $V_0 \otimes W_0$ for some subspaces $V_0 \subseteq V$ and $W_0 \subseteq W$. [Note: Here $V_0 \otimes W_0$ is being viewed as a subspace of $V \otimes W$ via the canonical embedding given in part (a).]