In the problems below, G is a finite group. You may assume $F = \mathbb{C}$.

Q1. (Copy of **A1Q5**.) Let V and W be finite-dimensional F-vector spaces.

(a) Let $z \in V \otimes W$. Prove that there exist linearly independent $v_1, \ldots, v_n \in V$ such that $z = \sum_{i=1}^n v_i \otimes w_i$ for some $w_i \in W$.

(b) Let $V_0 \subseteq V$ and $W_0 \subseteq W$ be subspaces. Show that there is a unique injective linear map $T\colon V_0 \otimes W_0 \to V \otimes W$ such that $T(v_0 \otimes w_0) = v_0 \otimes w_0$.

(c) True or false? Every subspace of $V \otimes W$ is of the form $V_0 \otimes W_0$ for some subspaces $V_0 \subseteq V$ and $W_0 \subseteq W$. [Note: Here $V_0 \otimes W_0$ is being viewed as a subspace of $V \otimes W$ via the embedding given in part (b).]

Q2. Let X and Y be finite G-sets. In this problem, view $X \times Y$ as a G-set with action given by $g(x, y) = (gx, gy)$.

(a) Establish the following isomorphisms of G-modules:

(i) $F\langle X \times Y\rangle \cong F\langle X\rangle \otimes F\langle Y\rangle$.

(ii) $F\langle X\rangle^* \cong F\langle X\rangle$.

(iii) $\operatorname{Hom}(F\langle X\rangle, F\langle Y\rangle) \cong F\langle X \times Y\rangle$.

(b) Deduce that $\dim \operatorname{Hom}_G(F\langle X\rangle, F\langle Y\rangle)$ is equal to the number of G-orbits in $X \times Y$.

(c) Determine $\dim \operatorname{Hom}_{S_n}(V, V)$, where V is the defining representation of S_n.

Q3. Let V be an FG-module and let $\chi\colon G \to F^\times$ be a one-dimensional representation of G. Put $V_\chi = \{v \in V : gv = \chi(g)v \text{ for all } g \in G\}$ and note that this is a G-submodule of V. Let F_χ denote the G-module determined by χ; that is, $F_\chi = F$ and the action of $g \in G$ on $a \in F$ is given by $g \cdot a = \chi(g)a$.

(a) Show that $\operatorname{Hom}_G(F_\chi, V) \cong V_\chi$ as vector spaces.

(b) Determine $\dim \operatorname{Hom}_G(F_\chi, V_{\text{reg}})$ if χ is the trivial representation of G.

Q4. Let V and W be finite-dimensional G-modules. Prove or disprove:

(a) V^* is irreducible if and only if V is irreducible.

(b) $V \otimes W$ is irreducible if and only if V and W are irreducible.

Q5. Bonus. Let $A \in M_n(F)$ and $B \in M_m(F)$. Prove that $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A)\operatorname{tr}(B)$ using the characterization of trace given in Theorem 5.4. [I am looking for an argument along the lines of the solution to Exercise 5.5. Submit your solution directly by email to me.]