In the problems below, G is a finite group and all representations are finite-dimensional. You may assume $F = \mathbb{C}$.

Q1. Determine the isotypic decompositions of the following $\mathbb{C}S_3$-modules. Write the decompositions in terms of $\operatorname{Irr}_{\mathbb{C}}(S_3) = \{V_{\text{triv}}, V_{\text{sgn}}, V_{\text{std}}\}$.

(a) V_{reg}.

(b) $(V_{\text{std}})^*$.

(c) $V_{\text{std}} \otimes V_{\text{std}}$.

Q2. Let $V = \mathbb{C}^2$ be the representation of $C_4 = \langle a \rangle$ in which a acts as a 90-degree counterclockwise rotation. Determine the isotypic decomposition of $W = \operatorname{Hom}(V, V)$ in terms of $\operatorname{Irr}_{\mathbb{C}}(C_4) = \{\chi_0, \chi_1, \chi_2, \chi_3\}$, where $\chi_j(a) = \exp(2\pi i j/4)$. [You should do this without writing down a single matrix!]

Q3. Let V be an FG-module. A bilinear form $B\colon V \times V \to F$ is said to be G**-invariant** if

$$B(gv, gw) = B(v, w) \text{ for all } g \in G \text{ and } v, w \in V.$$

Assume that V is irreducible and F is algebraically closed.

(a) Prove that if $B\colon V \times V \to F$ is a G-invariant bilinear form then there exists a scalar $c \in F$ such that $B(v, w) = cB(w, v)$ for all $v, w \in V$. [**Hint:** Use B to construct two maps $V \to V^*$.]

(b) Assume that there exists a non-zero G-invariant bilinear form $B\colon V \times V \to F$ and choose a scalar c as in part (a). Prove that if B' is another G-invariant bilinear form then $B'(v, w) = cB'(w, v)$ for all $v, w \in V$ and with the same scalar c. Thus, we may write c_V for the scalar c since it is independent of the choice of $B \neq 0$. Note that $c_V \neq 0$.

In the case where the only G-invariant bilinear form is the zero form, we set $c_V = 0$.

(c) Prove that $c_V \in \{0, \pm 1\}$ and $c_V = 0$ if and only if $V \not\cong V^*$ (as representations).