In the problems below, G is a finite group and all representations are finite-dimensional.

Q1. The **character** of a $\mathbb{C}C_n$-module (V, ρ) is the function $\chi_\rho \colon G \to \mathbb{C}$ defined by $\chi_\rho(g) = \operatorname{tr}(\rho(g))$. Prove that two $\mathbb{C}C_n$-modules (V, ρ) and (W, σ) are isomorphic if and only if $\chi_\rho = \chi_\sigma$ as functions. [Hint: Use the isotypic decomposition of V to diagonalize $\rho(a)$. Warm up with $n = 2$.]

In next week's lectures, we will prove that this (incredible) result holds for *all* finite groups G. It's instructive to try to do it "by hand" for $G = C_n$.

Q2. Assume $n \geq 3$. In this problem you will determine the isotypic decomposition of the defining representation $V = \mathbb{C}^n$ of S_n. Let

$$U = \operatorname{span}\{(1, 1, \ldots, 1)\} \quad \text{and} \quad W = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \colon x_1 + \cdots + x_n = 0\}.$$

(a) Show that U and W are S_n-invariant and that $V = U \oplus W$.

(b) Let $e_1, \ldots, e_n$ denote the standard basis vectors of $\mathbb{C}^n$. Let w be a nonzero vector in W. Show that $e_1 - e_i \in \operatorname{span}\{\pi w \colon \pi \in S_n\}$ for all $i = 2, \ldots, n$. Deduce that W has no proper nonzero S_n-invariant subspaces.

The irreducible representation W is called the **standard representation of S_n**.

Q3. In this problem you will determine $\operatorname{Irr}_{\mathbb{C}}(S_4)$.

(a) Prove that the trivial representation and the alternating representation are the only one-dimensional representations of S_4.

(b) Prove that the tensor product of an irreducible representation and a one-dimensional representation is irreducible. (Compare A2 Q4.)

(c) Determine the number and dimensions of the irreducible representations of S_4.

(d) Figure out (or look up) how to realize S_3 as a quotient of S_4. Use this to produce an irreducible two-dimensional representation of S_4.

(e) Describe $\operatorname{Irr}_{\mathbb{C}}(S_4)$.

Q4. Bonus. Suppose $\rho \colon G \to GL_2(\mathbb{C})$ is a representation of G such that, for each $g \in G$,

$$\rho(g) \in \left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \mathbb{C} \right\}.$$

Prove that ρ cannot be irreducible.