University of Waterloo

PMATH 445/745 SAMPLE Test 1

Winter 2025

Date:	Wednesday, Feb 12, 2025
Exam period:	4:30 PM to 5:20 PM
Duration:	50 minutes

Additional instructions

1. Write your answers in the space provided. If you need more room, use the space available on the last page, and indicate this on the question page where you ran out of room.
2. *Do NOT detach any paper from this test.*
3. On this test, **all groups G are finite, all vector spaces V are finite-dimensional, and the underlying field is $F = \mathbb{C}$.**

Q1. Short answer; no justification required.

(a) If V is an irreducible $\mathbb{C}G$-module, what is $\dim \operatorname{Hom}(V,V)^G$?

(b) **T/F?** All two-dimensional representations of C_2 are isomorphic.

(c) **T/F?** The vector space $\mathbb{C}^2 \otimes \mathbb{C}^2$ contains a subspace of dimension 3.

(d) If there is a $V \in \operatorname{Irr}_{\mathbb{C}}(G)$ such that $\dim V = |G|$, what is G?

(e) Write down the isotypic decomposition of the $\mathbb{C}S_3$-module $V_{\text{reg}} \otimes V_{\text{sgn}}$.

(f) If $V = \mathbb{C}^n$ is a C_2-module, provide a formula for a C_2-invariant inner product on V:

$\langle x, y \rangle =$

Q2. Short answer; give brief justification.

(a) Let V_{reg} be the regular representation of C_3. Describe the isotypic decomposition of $\text{Hom}(V_{\text{reg}}, V_{\text{reg}})$.

(b) Let $V \in \text{Irr}_{\mathbb{C}}(G)$. Prove that $V \otimes V^*$ cannot be irreducible if $\dim V > 1$.

(c) Let χ be the character of the defining representation of S_n on $\mathbb{C}^n$ (i.e. the representation induced by *the* action of S_n on $\{1, 2, \ldots, n\}$). Determine $\sum_{\pi \in S_n} \chi(\pi)$.

(d) Determine the number of pairs $(A, B) \in GL_4(\mathbb{C})$ (up to conjugacy*) such that $A^2 = B^3 = (AB)^2 = I$.

[*: Two pairs (A, B) and (A', B') are *conjugate* if there is an $S \in GL_4(\mathbb{C})$ such that $SAS^{-1} = A'$ and $SBS^{-1} = B'$.]

Q3. Let $g, h \in G$. Show that if $\chi(g) = \chi(h)$ for all irreducible characters χ of G then g and h must be conjugate in G.

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You may use this space if you run out of room for a particular question.

If you do, be sure to indicate this clearly here on this page and also on the question page.