Perspectives on the moduli space of torsion-free G_2-structures

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G_2 preliminaries

Definition

Let M^7 be a smooth 7-manifold. A G_2-structure on M is a smooth 3-form φ on M such that at each point $p \in M$, there exists a linear isomorphism $T_pM \cong \mathbb{R}^7$ with respect to which $\varphi_p \in \Lambda^3(T_p^*M)$ corresponds to the associative form $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$ given by $\varphi_0(a, b, c) = \langle a \times b, c \rangle$.

- The G_2-structure induces a Riemannian metric g_φ and an orientation, which gives us a Hodge star operator $\star_\varphi$ and dual 4-form $\psi = \star_\varphi \varphi$.
- A G_2-structure exists iff M is orientable and spinnable.

G_2 preliminaries

Definition

Let ∇ be the Levi-Civita connection of g_φ. We say that a manifold (M, φ) with G_2-structure is a G_2-manifold if φ is torsion-free. That is, if $\nabla\varphi = 0$.

Theorem (Fernández-Gray, 1982)

(M, φ) is torsion-free iff $d\varphi = 0$ (closed) and $d\psi = 0$ (co-closed).

On (M, φ), the G_2-structure φ induces the following decomposition on the space of differential forms:

$$\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14}$$
$$\Omega^3 = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27},$$

where Ω^k_l has dimension l.

The moduli space of compact torsion-free G_2-structures

Assume that M is compact and let $\mathcal{X}$ be the space of torsion-free structures on M. Then, if $\mathcal{D}$ is the group of diffeomorphisms isotopic to the identity, we define the moduli space of torsion-free G_2-structures on M to be the quotient space

$$\mathcal{M} = \mathcal{X}/\mathcal{D}$$

where the action of $\mathcal{D}$ on $\mathcal{X}$ is given by

$$\varphi \mapsto \Psi^*(\varphi),$$

for $\Psi \in \mathcal{D}, \varphi \in \mathcal{X}$.

The moduli space of compact torsion-free G_2-structures

Theorem (Joyce, 1994)

The moduli space $\mathcal{M}$ is a smooth manifold of dimension $b^3(M)$, and is locally diffeomorphic to an open subset of the vector space of $H^3(M, \mathbb{R})$ through the map $[\varphi]_{\mathcal{M}} \mapsto [\varphi]_{dR}$.

Sketch of Proof.

We construct a "slice" S_φ for the action of $\mathcal{D}$ on $\mathcal{X}$, which is a submanifold of $\mathcal{X}$ containing φ locally transverse to the nearby orbits of φ in $\mathcal{D}$. Then, $\mathcal{M}$ is locally homeomorphic in a neighbourhood of $[\varphi]_{\mathcal{M}} \in \mathcal{M}$ to S_φ. As $\varphi \in \mathcal{X}$ is arbitrary, $\mathcal{M}$ is a smooth manifold. □

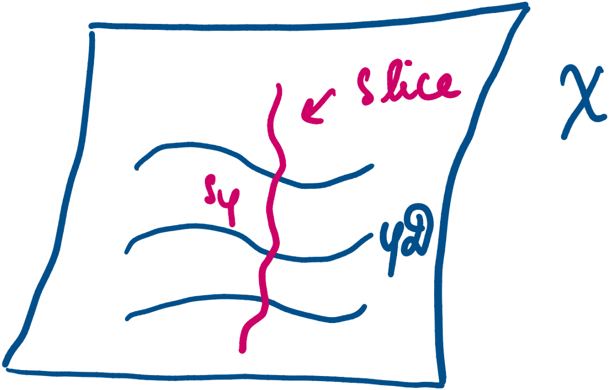

Figure: Slice

Gauge transformations on torsion-free G_2-structures

We will now consider the infinitesimal characterization of this moduli space using a different approach, which is by exploring the action of gauge transformations of the form e^{tA}, where A is a 2-tensor, on the space of torsion-free G_2-structures.

- The space $\mathcal{T}^2$ of 2-tensors decomposes as

$$\mathcal{T}^2 \cong \Omega^0 \oplus \mathcal{S}^2_0 \oplus \Omega^2_7 \oplus \Omega^2_{14},$$

  which allows us to write $A \in \mathcal{T}^2$ as

$$A = \frac{1}{7}(\operatorname{tr} A)g + A_{27} + A_7 + A_{14},$$

  where A_{27} is the traceless symmetric part of A.
- We take $P = e^{tA}$, and let

$$\widetilde{\varphi} = P^*\varphi.$$

Gauge transformations on torsion-free G_2-structures

- For k-forms $\sigma \in \Omega^k$ and 2-tensors $A = A_{ij} dx^i \otimes dx^j \in \mathcal{T}^2$, we define the diamond operator as

$$(A \diamond \sigma)_{i_1 i_2 \cdots i_k} = A_{i_1 p} \sigma_{p i_2 \cdots i_k} + A_{i_2 p} \sigma_{i_1 p i_3 \cdots i_k} + \cdots + A_{i_k p} \sigma_{i_1 i_2 \cdots i_{k-1} p}.$$

  Taking $\sigma = \varphi$, we have

$$(A \diamond \varphi)_{ijk} = A_{ip} \varphi_{pjk} + A_{jp} \varphi_{ipk} + A_{kp} \varphi_{ijp}.$$

- Using this framework of gauge transformations, we will show that infinitesimally, there is a relation between the torsion-free condition and the 3-form $A \diamond \varphi$ being harmonic (closed and co-closed).

Harmonicity of $A \diamond \varphi$

Proposition

Let (M, φ) is a compact G_2-manifold. Suppose that $\gamma = A \diamond \varphi$ is a 3-form on M where $A \in \mathcal{T}^2$. Then, γ is harmonic if and only if

$$\nabla_i A_{ip} \varphi_{pjk} + \nabla_i A_{jp} \varphi_{ipk} + \nabla_i A_{kp} \varphi_{ijp} = 0$$

and

$$\begin{aligned} &\nabla_i A_{pq} \varphi_{pqa} + \nabla_p A_{iq} \varphi_{paq} + \nabla_j A_{kp} \varphi_{jkp} g_{ia} \\ &- \nabla_j A_{ka} \varphi_{ijk} - \nabla_p A_{kp} \varphi_{aik} - \nabla_j (\operatorname{tr} A) \varphi_{aji} = 0. \end{aligned}$$

With respect to the decomposition $\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14}$ and $\Omega^4 = \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27}$, we have the explicit equations

$$\boxed{(d^*\gamma)_7 = 0 \iff 2\,\mathrm{div}\, A + \nabla\, \mathrm{Tr}\, A - \langle \nabla A, \psi \rangle = 0}$$

$$\boxed{(d\gamma)_1 = 0 \iff \mathrm{div}(\mathrm{V}A) = \nabla_a A_{pq} \varphi_{pqa} = 0}$$

$$\boxed{(d\gamma)_7 = 0 \iff 2\,\mathrm{div}\, A^T - 2\nabla\, \mathrm{Tr}\, A + \langle \nabla A, \psi \rangle = 0,}$$

where $(\mathrm{div}\, A)_m = \nabla_i A_{im}$, $(\mathrm{V}A)_k = A_{ij}\varphi_{ijk}$ and $\langle \nabla A, \psi \rangle_m = \nabla_i A_{pq} \psi_{ipqm}$.

Linearization of the torsion-free condition

Let $\widetilde{g} = P^*g = (e^{tA})^*\varphi$ and let $\widetilde{\nabla}$ denote the Levi-Civita connection of $\widetilde{g}$. Then, infinitesimally the torsion-free condition is given by vanishing of the linearization

$$\frac{d}{dt}\bigg|_{t=0} \widetilde{\nabla}\widetilde{\varphi}.$$

Lemma

For any vector field X on M, $K_X = \frac{d}{dt}|_{t=0}\widetilde{\nabla}_X\widetilde{\varphi}$ lies in Ω^3_7 with respect to the G_2-structure φ.

As $\Omega^3_7 = \{X \lrcorner \psi \mid X \in \mathfrak{X}(M)\}$, we have $K_X = K(X) \lrcorner \psi$ for some unique vector field $K(X) \in \mathfrak{X}$.

Linearization of the torsion-free condition

Proposition

Let K be the 2-tensor defined by the equation $K_X = K(X) \lrcorner \psi$, where $K_X = \frac{d}{dt}\big|_{t=0} \widetilde{\nabla}_X \widetilde{\varphi}$.

$$K_{ia} = 0 \iff -\nabla_i A_{pq} \varphi_{apq} + \nabla_p A_{qi} \varphi_{pqa} - \nabla_p A_{iq} \varphi_{paq} = 0$$

Moreover, we have

$$\boxed{K_7 = 0 \iff -2 \operatorname{div} A^T + 2\nabla \operatorname{tr} A - \langle \nabla A, \psi \rangle = 0}$$

$$\boxed{K_1 = 0 \iff \nabla_a A_{pq} \varphi_{apq} = 0,}$$

with respect to the decomposition $\mathcal{T}^2 \cong \Omega^0 \oplus \mathcal{S}^2_0 \oplus \Omega^2_7 \oplus \Omega^2_{14}$.

Gauge-fixing condition

So far we have that

$$\boxed{K_1 = 0 \iff (d\gamma)_1 = 0}$$

and

$$\boxed{K_7 = 0 \iff (d\gamma)_7 = 0.}$$

Furthermore, since we want tangent directions to our "slice" of torsion-free G_2-structures to be L^2-orthogonal to the infinitesimal diffeomorphisms $\mathcal{L}_W\varphi$, our gauge-fixing condition is given as

$$\langle A \diamond \varphi, \mathcal{L}_W\varphi\rangle_{L^2} = 0,$$

which, by expanding the Lie derivative and using the fact that φ is torsion-free, can be shown to be equivalent to

$$\langle A \diamond \varphi, \nabla W \diamond \varphi\rangle_{L^2} = 0$$

for all $W \in \mathfrak{X}(M)$.

Gauge-fixing condition

It turns out the gauge-fixing condition is equivalent to the equation

$$2\,\mathrm{div}\,A + \nabla\,\mathrm{tr}\,A - \langle \nabla A, \psi \rangle = 0,$$

which gives us

$$\boxed{\text{gauge-fixing condition} \iff (d^*\gamma)_7 = 0}$$

In addition, we also have

$$\boxed{K_{27} = 0 \iff (d\gamma)_{27} = 0.}$$

If we also had

$$K_{14} = 0 \iff (d^*\gamma)_{14} = 0,$$

then we would have

$$A \diamond \varphi \text{ is harmonic } \iff (K = 0 + \text{G.F. condition}).$$

Gauge-fixing condition

We have thus obtained the following theorem

Theorem

Let $A \in \mathcal{T}^2$ and K be a 2-tensor defined by the equation $K_X = K(X) \lrcorner \psi$, where $K_X = \frac{d}{dt}|_{t=0} \widetilde{\nabla}_X \widetilde{\varphi}$. Then, our gauge-fixing (G.F.) condition is given by the equation

$$2 \operatorname{div} A + \nabla \operatorname{tr} A - \langle \nabla A, \psi \rangle = 0.$$

Then, if $\gamma = A \diamond \varphi$, we have that

$$\begin{aligned}
(d\gamma)_1 = 0 &\iff K_1 = 0 \\
(d^*\gamma)_7 = 0 &\iff \text{G.F.} = 0 \\
(d\gamma)_7 = 0 &\iff K_7 = 0 \\
(d\gamma)_{27} = 0 &\iff K_{27} = 0.
\end{aligned}$$

Future questions

- One could explore if this method can be used in the non-infinitesimal case to give an alternate proof of the fact that the moduli space of G_2-structures forms a non-singular smooth manifold.
- It could prove fruitful to use this method to prove analogous results for the moduli space formed by structures on manifolds with different holonomy groups such as Spin(7) and $\mathrm{U}(m)$.
- In particular, this point of view could give us a differential geometric explanation for why the Kähler moduli space is not smooth in general.

Thank You!