C&O 355
Mathematical Programming
Fall 2010
Lecture 4

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Outline

- Solvability of Linear Equalities & Inequalities
- Farkas' Lemma
- Fourier-Motzkin Elimination
- Proof of Farkas' Lemma
- Proof of Strong LP Duality

Strong Duality

Primal LP:
$$\begin{aligned} \max \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

Dual LP:
$$\begin{aligned} \min \quad & b^\mathsf{T} y \\ \text{s.t.} \quad & A^\mathsf{T} y = c \\ & y \geq 0 \end{aligned}$$

Strong Duality Theorem:
Primal has an opt. solution x $\Leftrightarrow$ Dual has an opt. solution y.
Furthermore, optimal values are same: $c^\mathsf{T}x = b^\mathsf{T}y$.

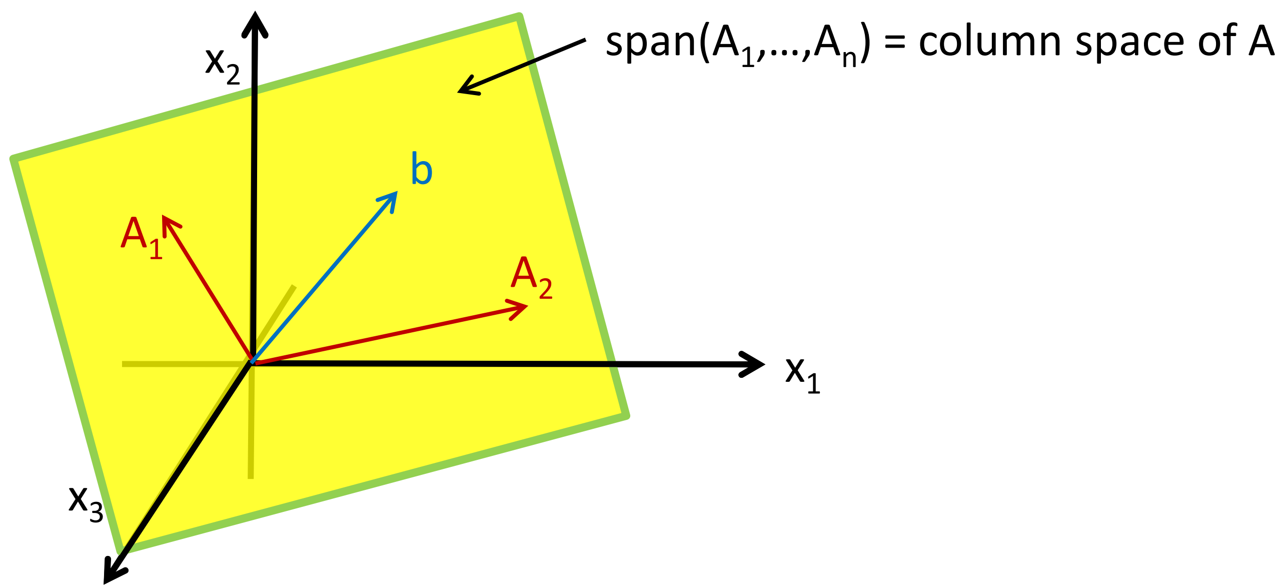
- **Our Goals:**
  - Understand when Primal and Dual have optimal solutions
  - Compute those optimal solutions
- **Combining Primal & Dual into a System of Inequalities:**
  x is optimal for Primal and y is optimal for Dual
  $\Leftrightarrow$ x and y are solutions to these inequalities:

$$Ax \leq b \qquad A^\mathsf{T} y = c \qquad y \geq 0 \qquad c^\mathsf{T} x \geq b^\mathsf{T} y$$

- Can we characterize when **systems of inequalities** are solvable?

Systems of **Eq**ualities

- **Lemma:** Exactly one of the following holds:
  - There exists x satisfying Ax=b (b is in column space of A)
  - There exists y satisfying $y^T A=0$ and $y^T b>0$
- **Geometrically…**

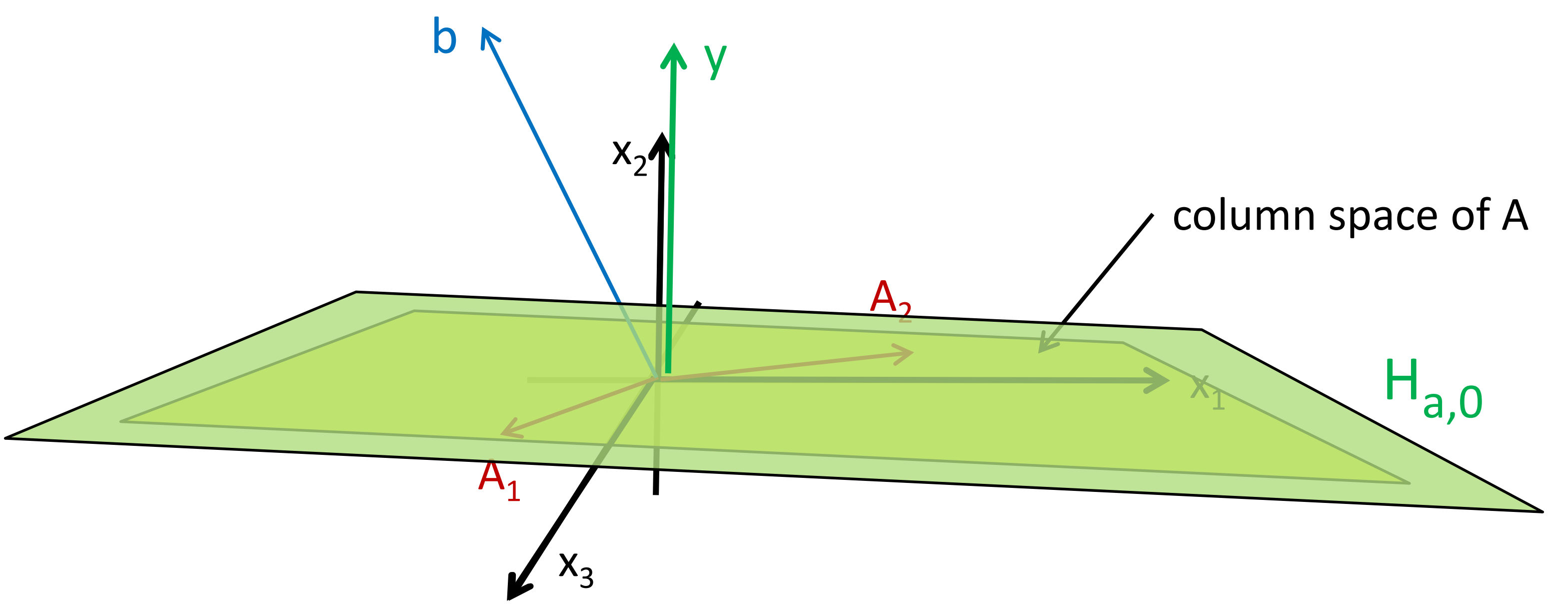


Systems of **Eq**ualities

- **Lemma:** Exactly one of the following holds:
  - There exists x satisfying Ax=b (b is in column space of A)
  - There exists y satisfying $y^T A=0$ and $y^T b>0$ (or it is not)
- **Geometrically…** col-space(A)$\subseteq H_{y,0}$ but $b \in H^{++}_{y,0}$

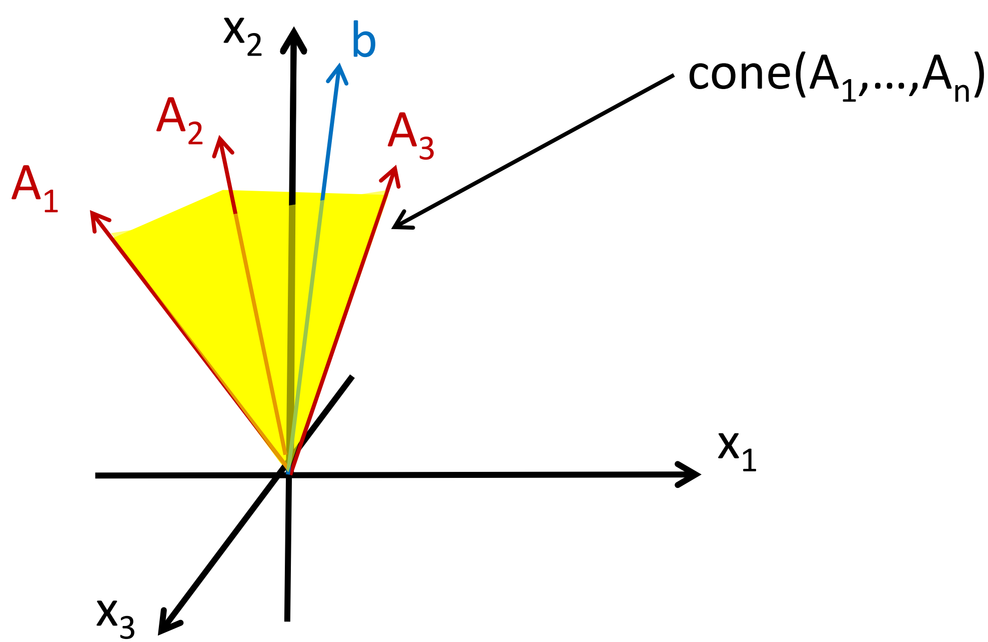
Hyperplane $H_{a,b} = \{ x \in \mathbb{R}^n : a^\mathsf{T} x = b \}$

Positive open halfspace $H^{++}_{a,b} = \{ x \in \mathbb{R}^n : a^\mathsf{T} x > b \}$



Systems of **Inequalities**

- **Lemma:** Exactly one of the following holds:
  - There exists $x \geq 0$ satisfying $Ax=b$ (b is in cone($A_1,\ldots,A_n$))
  - There exists y satisfying $y^T A \geq 0$ and $y^T b<0$
- **Geometrically…**
  Let cone($A_1,\ldots,A_n$) = $\{ \Sigma_i x_i A_i : x \geq 0 \}$ "cone generated by $A_1,\ldots,A_n$"
  (Here A_i is the i^{th} column of A)

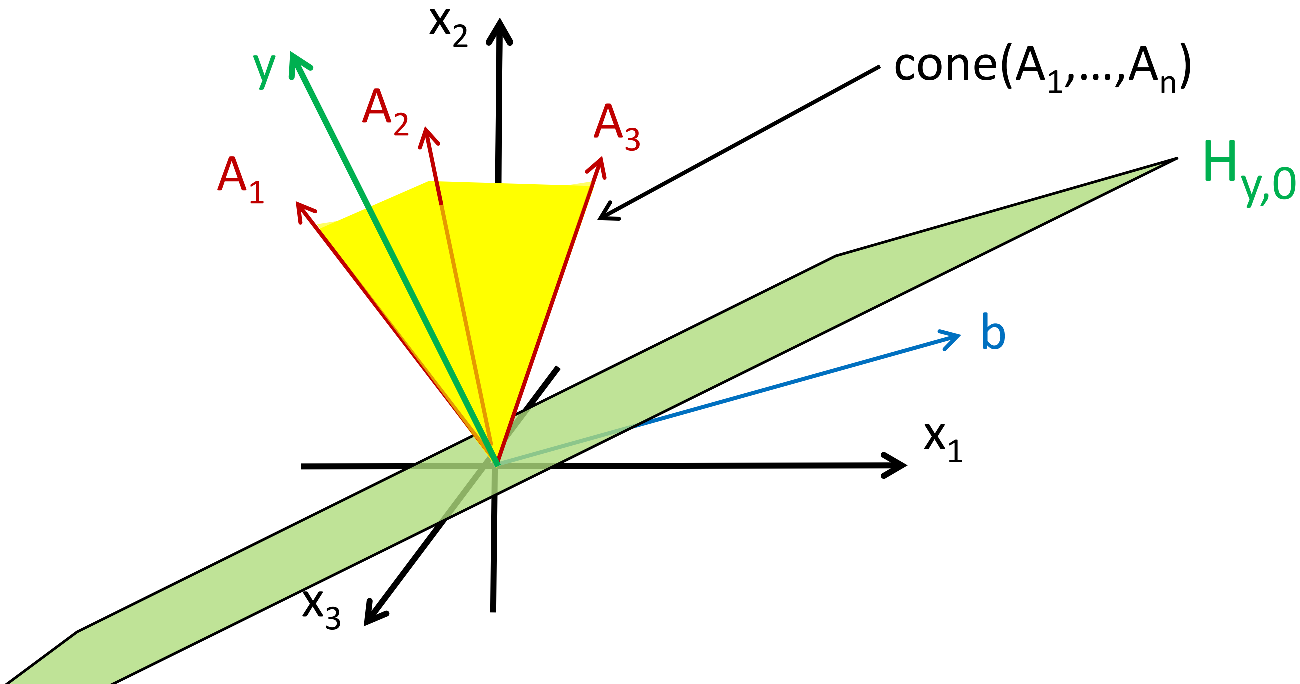


Systems of **Inequalities**

- **Lemma:** Exactly one of the following holds:
  - There exists $x \geq 0$ satisfying $Ax=b$ (b is in cone($A_1,\ldots,A_n$))
  - There exists y satisfying $y^T A \geq 0$ and $y^T b < 0$ (y gives a **separating hyperplane**)
- **Geometrically…** $\text{cone}(A_1,\ldots,A_n) \in H^{+}_{y,0}$ but $b \in H^{--}_{y,0}$

Positive closed halfspace $H^{+}_{a,b} = \{ x \in \mathbb{R}^n : a^{\mathsf{T}} x \geq b \}$

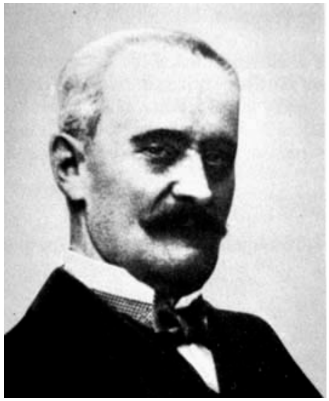
Negative open halfspace $H^{--}_{a,b} = \{ x \in \mathbb{R}^n : a^{\mathsf{T}} x < b \}$



Systems of **Inequalities**

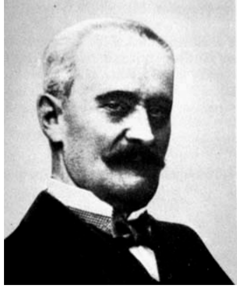
- **Lemma:** Exactly one of the following holds:
  - There exists $x \geq 0$ satisfying $Ax=b$ (b is in cone($A_1,\ldots,A_n$))
  - There exists y satisfying $y^T A \geq 0$ and $y^T b < 0$ (y gives a "separating hyperplane")

- This is called **"Farkas' Lemma"**
  - It has many interesting proofs.
    - There are 3 proofs in Ch. 6 of Matousek-Gartner
  - It is "equivalent" to strong duality for LP.
  - There are several "equivalent" versions of it.



Gyula Farkas

Variants of Farkas' Lemma



Gyula Farkas

The System	$Ax \leq b$	$Ax = b$
has **no** solution $x \geq \mathbf{0}$ iff	$\exists y \geq 0,\ A^T y \geq \mathbf{0},\ b^T y < 0$	$\exists y \in \mathbb{R}^n,\ A^T y \geq \mathbf{0},\ b^T y < 0$
has **no** solution $x \in \mathbb{R}^n$ iff	$\exists y \geq 0,\ A^T y = \mathbf{0},\ b^T y < 0$	$\exists y \in \mathbb{R}^n,\ A^T y = \mathbf{0},\ b^T y < 0$

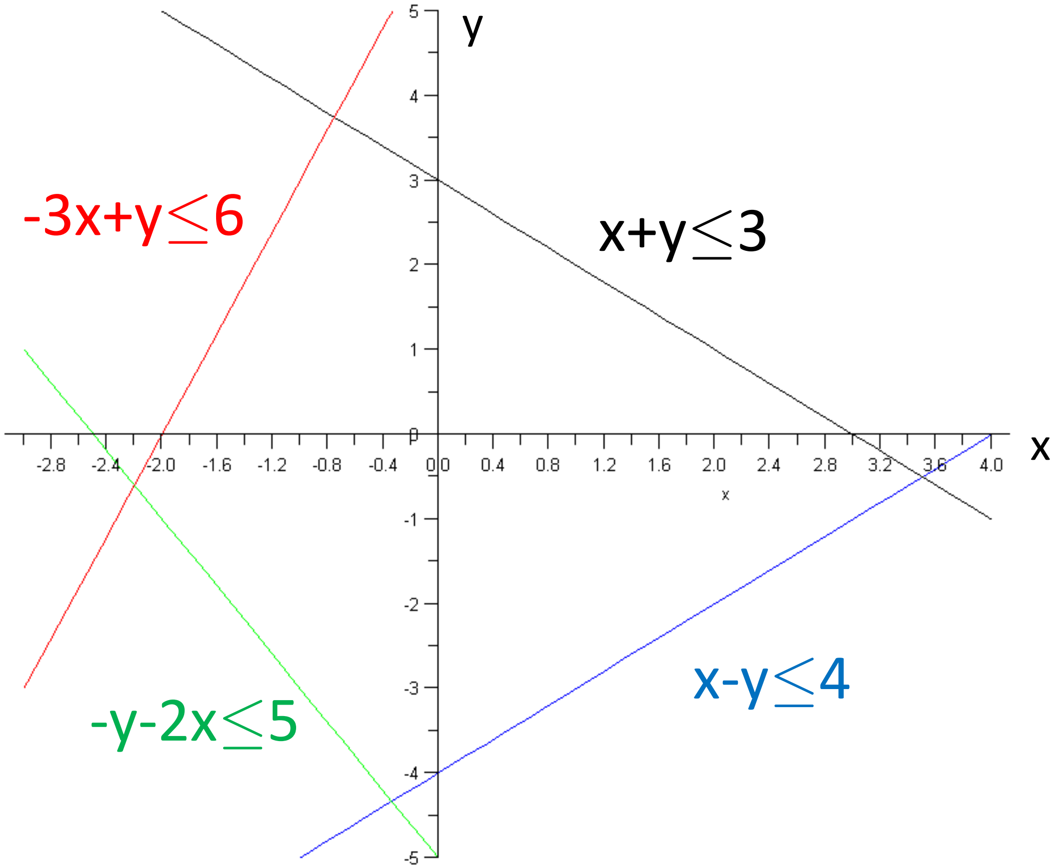
These are all "equivalent"

(each can be proved using another)

This is the simple lemma on systems of **eq**ualities

2D System of Inequalities

Consider the polyhedron
$Q = \{ (x,y) : -3x+y \leq 6,$
$x+y \leq 3,$
$-y-2x \leq 5,$
$x-y \leq 4 \}$



- **Given x, for what values of y is (x,y) feasible?**
  - Need: $y \leq 3x+6$, $y \leq -x+3$, $y \geq -2x-5$, and $y \geq x-4$

2D System of Inequalities

Consider the polyhedron

$Q = \{ (x,y) : -3x+y \leq 6,$

$x+y \leq 3,$

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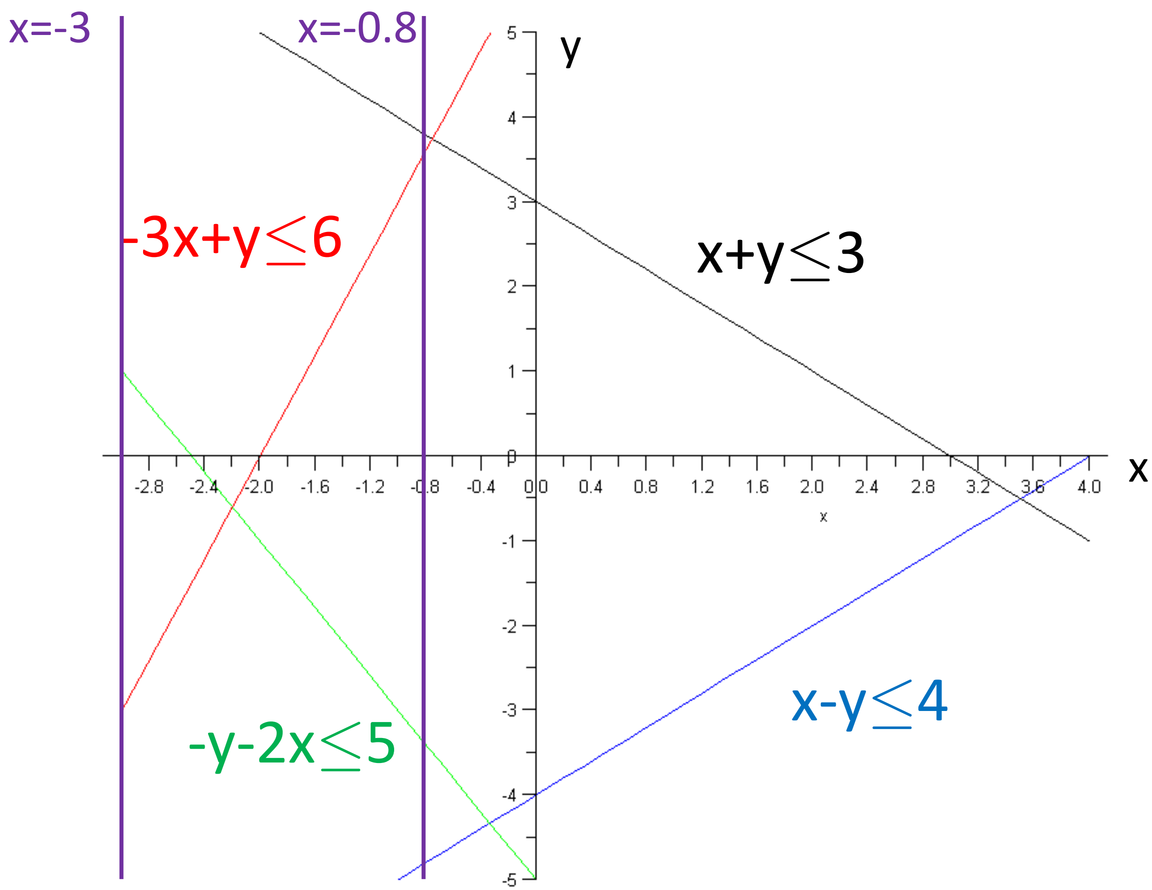

- **Given x, for what values of y is (x,y) feasible?**
  - i.e., $y \leq \min\{3x+6, -x+3\}$ and $y \geq \max\{-2x-5, x-4\}$
  - For x=-0.8, (x,y) feasible if $y \leq \min\{\mathbf{3.6}, 3.8\}$ and $y \geq \max\{\mathbf{-3.4}, -4.8\}$
  - For x=-3, (x,y) feasible if $y \leq \min\{\mathbf{-3}, 6\}$ and $y \geq \max\{\mathbf{1}, -7\}$ **Impossible!**

2D System of Inequalities

- Consider the set
  $Q = \{ (x,y) : -3x+y \leq 6,\ x+y \leq 3,\ -y-2x \leq 5,\ x-y \leq 4 \}$
- Given x, for what values of y is (x,y) feasible?
  - i.e., $y \leq \min\{3x+6, -x+3\}$ and $y \geq \max\{ -2x-5, x-4 \}$
  - Such a y exists $\Leftrightarrow$ $\max\{-2x-5, x-4\} \leq \min\{3x+6, -x+3\}$
    $\Leftrightarrow$ the following inequalities are solvable

Every "lower" constraint is $\leq$ every "upper" constraint

$$\begin{matrix} -2x-5 \leq 3x+6 \\ x-4 \leq 3x+6 \\ -2x-5 \leq -x+3 \\ x-4 \leq -x+3 \end{matrix} \equiv Q' = \left\{ x : \begin{matrix} -5x \leq 11 \\ -2x \leq 10 \\ -x \leq 8 \\ 2x \leq 7 \end{matrix} \right\} = \left\{ x : \begin{matrix} x \geq -11/5 \\ x \geq -5 \\ x \geq -8 \\ x \leq 7/2 \end{matrix} \right\}$$

- **Conclusion:** Q is non-empty $\Leftrightarrow$ Q' is non-empty.
- This is easy to decide because Q' involves only 1 variable!

Fourier-Motzkin Elimination

Joseph Fourier

Theodore Motzkin

- **Generalization:** given a set $Q = \{ (x_1,\ldots,x_n) : Ax \leq b \}$, we want to find set $Q' = \{ (x'_1,\ldots,x'_{n-1}) : A'x' \leq b' \}$ satisfying

$$(x_1,\ldots,x_{n-1}) \in Q' \quad \Leftrightarrow \quad \exists x_n \text{ s.t. } (x_1,\ldots,x_{n-1},x_n) \in Q$$

- Q' is called a **projection** of Q (onto the first n-1 coordinates)
- Fourier-Motzkin Elimination is a procedure for producing Q' from Q
- **Consequences:**
  - An (inefficient!) algorithm for solving systems of inequalities, and hence for solving LPs too
  - A way of proving Farkas' Lemma by induction

- **Lemma:** Let $Q = \{ (x_1,\ldots,x_n) : Ax \leq b \}$. We can construct $Q' = \{ (x'_1,\ldots,x'_{n-1}) : A'x' \leq b' \}$ satisfying
  (1) $(x_1,\ldots,x_{n-1}) \in Q' \quad \Leftrightarrow \quad \exists x_n$ s.t. $(x_1,\ldots,x_{n-1},x_n) \in Q$
  (2) Every inequality defining Q' is a non-negative linear combination of the inequalities defining Q.
- **Proof:** Put inequalities of Q in three groups: (a_i = i^{th} row of A)
  $Z=\{ i : a_{i,n}=0 \}$ $P=\{ j : a_{j,n}>0 \}$ $N=\{ k : a_{k,n}<0 \}$
- WLOG, $a_{j,n}=1 \ \forall j \in P$ and $a_{k,n}=-1 \ \forall k \in N$
- For any $x \in \mathbb{R}^n$, let $x' \in \mathbb{R}^{n-1}$ be vector obtained by deleting coordinate x_n
- The constraints defining Q' are:
  - $a_i' x' \leq b_i \quad \forall i \in Z$
  - $a_j' x' + a_k' x' \leq b_j + b_k \quad \forall j \in P, \forall k \in N$
- This proves (2).

This is sum of j^{th} and k^{th} constraints of Q, because n^{th} coordinate of $a_j + a_k$ is zero!

- In fact, (2) implies the "$\Leftarrow$ direction" of (1):
  For every $x \in Q$, x' satisfies all inequalities defining Q'.
- Why? Because every constraint of Q' is a non-negative lin. comb. of constraints from Q, with n^{th} coordinate equal to 0.

- **Lemma:** Let $Q = \{ (x_1,\dots,x_n) : Ax \leq b \}$. We can construct $Q' = \{ (x'_1,\dots,x'_{n-1}) : A'x' \leq b' \}$ satisfying
  (1) $(x_1,\dots,x_{n-1}) \in Q' \quad \Leftrightarrow \quad \exists x_n$ s.t. $(x_1,\dots,x_{n-1},x_n) \in Q$
  (2) Every inequality defining Q' is a non-negative linear combination of the inequalities defining Q.
- **Proof:** Put inequalities of Q in three groups:
  $Z=\{ i : a_{i,n}=0 \}$ $\qquad P=\{ j : a_{j,n}=1 \}$ $\qquad N=\{ k : a_{k,n}=-1 \}$
- The constraints defining Q' are:
  - $a_i'x' \leq b_i \quad \forall i \in Z$
  - $a_j'x' + a_k'x' \leq b_j + b_k \quad \forall j \in P, \forall k \in N$
- It remains to prove the "$\Rightarrow$ direction" of (1).
- Note that: $\qquad a_k'x' - b_k \leq b_j - a_j'x' \quad \forall j \in P, \forall k \in N.$

  $$\Rightarrow \underbrace{\max_{k \in N} \{ a_k'x' - b_k \}} \leq \min_{j \in P} \{ b_j - a_j'x' \}$$

  Let x_n be this value, and let $x = (x'_1,\dots,x'_{n-1},x_n)$.

  Then:
  $$\left.\begin{array}{l} a_k x - b_k = a_k'x' - x_n - b_k \leq 0 \quad \forall k \in N \\ b_j - a_j x = b_j - a_j'x' - x_n \geq 0 \quad \forall j \in P \\ a_i x = a_i'x' \leq b_i \quad \forall i \in Z \end{array}\right\} \Rightarrow x \in Q$$

By definition of x, and since $a_{k,n} = -1$

By definition of x_n, $a_k'x' - b_k \leq x_n$

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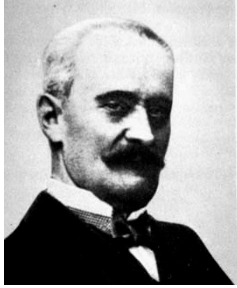
Variants of Farkas' Lemma



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has **no** solution $x \in \mathbb{R}^{\mathbf{n}}$ iff	$\exists y \geq 0,\ A^T y \mathbf{=0},\ b^T y < 0$	$\exists y \in \mathbb{R}^{\mathbf{n}},\ A^T y \mathbf{=0},\ b^T y < 0$

We'll prove this one

- **Lemma:** Exactly one of the following holds:
  - There exists $x \in \mathbb{R}^n$ satisfying $Ax \leq b$
  - There exists $y \geq 0$ satisfying $y^\mathsf{T} A = 0$ and $y^\mathsf{T} b < 0$
- **Proof:** Suppose x exists. Need to show y cannot exist.

  Suppose y also exists. Then:

$$0 = 0x = y^\mathsf{T} A x \leq y^\mathsf{T} b < 0$$

  Contradiction! y cannot exist.

- **Lemma:** Exactly one of the following holds:
  - There exists $x \in \mathbb{R}^n$ satisfying $Ax \leq b$
  - There exists $y \geq 0$ satisfying $y^T A = 0$ and $y^T b < 0$
- **Proof:** Suppose no solution x exists.

  We use induction. Trivial for $n=0$, so let $n \geq 1$.

  We use Fourier-Motzkin Elimination.

  Get an **equivalent** system $A'x' \leq b'$ where

$$(A' \mid 0) = MA \qquad b' = Mb$$

  for some **non-negative matrix** M.

Lemma: Let $Q = \{ (x_1, \ldots, x_n) : Ax \leq b \}$. We can construct $Q' = \{ (x_1, \ldots, x_{n-1}) : A'x' \leq b' \}$ satisfying

1) Q is non-empty $\Leftrightarrow$ Q' is non-empty
2) Every inequality defining Q' is a **non-negative linear combination** of the inequalities defining Q.

(This statement is slightly simpler than our previous lemma)

- **Lemma:** Exactly one of the following holds:
  - There exists $x \in \mathbb{R}^n$ satisfying $Ax \leq b$
  - There exists $y \geq 0$ satisfying $y^T A = 0$ and $y^T b < 0$
- **Proof:**

Get an equivalent system $A'x' \leq b'$ where

$$(A' | 0) = MA \qquad b' = Mb$$

for some non-negative matrix M.

We assume $Ax \leq b$ has no solution, so $A'x' \leq b'$ has no solution.

By induction, $\exists y' \geq 0$ s.t. $y'^T A' = 0$ and $y'^T b' < 0$.

Define $y = M^T y'$.

Then: $y \geq 0$, because $y' \geq 0$ and M non-negative

$$y^T A = y'^T MA = y'^T (A' | 0) = 0$$

$$y^T b = y'^T Mb = y'^T b' < 0$$

■

Farkas' Lemma $\Rightarrow$ Strong Duality

Primal LP: $\begin{array}{ll} \max & c^{\mathsf{T}} x \\ \text{s.t.} & Ax \leq b \end{array}$

Dual LP: $\begin{array}{ll} \min & b^{\mathsf{T}} y \\ \text{s.t.} & A^{\mathsf{T}} y = c \\ & y \geq 0 \end{array}$

- By Asst 1 Q6, if x is feasible then either:
  (1) x is not an optimal solution
  (2) c is a non-negative lin. comb of the constraints tight at x
- More formally, (2) says: there exists $y \geq 0$ s.t.
  $y^{\mathsf{T}} A = c^{\mathsf{T}}$ and $y_i > 0$ only when $a_i^{\mathsf{T}} x = b_i$
  (a_i = i^{th} row of A)
- Suppose x is optimal for Primal. (i.e., (1) doesn't hold)
- Then such a y exists. It is clearly feasible for Dual, and:

$$c^{\mathsf{T}} x = y^T A x = \sum_{i : y_i > 0} y_i (Ax)_i = \sum_{i : y_i > 0} y_i b_i = y^{\mathsf{T}} b$$

- So x and y are both optimal ■