# C&O 355, Fall 2010
# Lecture 7 Notes

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## 1 Covering Hemispheres by Ellipsoids

Recall our notation $B = \{\ x\ :\ \|x\| \leq 1\ \}$ and $H_u = \left\{\ x\ :\ x^{\mathsf{T}} u \geq 0\ \right\}$, where $u$ is an arbitrary unit vector. The next theorem defines an ellipsoid that covers $B \cap H_u$ and analyzes its volume. For simplicity, let us assume that $n \geq 3$.

**Theorem 1.** Define

$$\begin{aligned} M &= \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}uu^{\mathsf{T}}\Big) \\ b &= \frac{u}{n+1} \\ B' &= E(M,b) = \Big\{\ x\ :\ (x-b)^{\mathsf{T}} M^{-1} (x-b) \leq 1\ \Big\}. \end{aligned}$$

Then $B'$ satisfies the following two properties.

$$B \cap H_u \ \subseteq\ B' \tag{1}$$

$$\frac{\mathrm{vol}(B')}{\mathrm{vol}(B)} \ \leq\ e^{-1/4(n+1)} \ \leq\ 1 - \frac{1}{8(n+1)} \tag{2}$$

The following two claims prove the theorem.

**Claim 2.** $B \cap H_u \subseteq B'$.

*Proof.* First note that we can derive an explicit expression for $M^{-1}$ using our Claim 1 on rank-1 updates.

$$M^{-1} \ =\ \frac{n^2-1}{n^2}\Big(I + \frac{2}{n-1}uu^{\mathsf{T}}\Big)$$

Substitute this into the definition of $E(M,b)$:

$$\begin{aligned} B' &= \left\{\ x\ :\ \Big(x - \frac{u}{n+1}\Big)^{\mathsf{T}}\Big(\frac{n^2-1}{n^2}\Big)\Big(I + \frac{2}{n-1}uu^{\mathsf{T}}\Big)\Big(x - \frac{u}{n+1}\Big) \leq 1\ \right\} \\ &= \left\{\ x\ :\ \Big(x - \frac{u}{n+1}\Big)^{\mathsf{T}}\Big(x - \frac{u}{n+1}\Big) + \frac{2}{n-1}\left(u^{\mathsf{T}}\Big(x - \frac{u}{n+1}\Big)\right)^2 \leq \frac{n^2}{n^2-1}\ \right\} \\ &= \left\{\ x\ :\ x^{\mathsf{T}}x - \frac{2x^{\mathsf{T}}u}{n+1} + \frac{1}{(n+1)^2} + \frac{2}{n-1}\Big(x^{\mathsf{T}}u - \frac{1}{n+1}\Big)^2 \leq 1 + \frac{1}{n^2-1}\ \right\} \end{aligned} \tag{3}$$

Now consider any $x \in B \cap H_u$. If we can show that $x \in B'$, then the proof is complete. By (3), it is sufficient to show that

$$x^{\mathsf{T}}x - \frac{2x^{\mathsf{T}}u}{n+1} + \frac{1}{(n+1)^2} + \frac{2}{n-1}\Big(x^{\mathsf{T}}u - \frac{1}{n+1}\Big)^2 \ \leq\ 1 + \frac{1}{n^2-1}.$$

We know that $x^\mathsf{T} x \leq 1$ (since $x \in B$), so it is sufficient to show that

$$\underbrace{-\frac{2x^\mathsf{T} u}{n+1} + \frac{1}{(n+1)^2} + \frac{2}{n-1}\Big(x^\mathsf{T} u - \frac{1}{n+1}\Big)^2}_{f(x^\mathsf{T} u)} \;\leq\; \frac{1}{n^2-1}. \tag{4}$$

The left-hand side of (4) is a function only of $x^\mathsf{T} u$, so let's call it $f(x^\mathsf{T} u)$. Every point in $B \cap H_u$ satisifies $0 \leq x^\mathsf{T} u \leq 1$, by the Cauchy-Schwarz inequality. For notational simplicity, let $y$ denote the scalar $x^\mathsf{T} u$. So, to prove (4), we must analyze the maximum value of $f(y)$ on the interval $[0,1]$. Note that $f$ is a quadratic polynomial in $y$ and it is convex (i.e., the coefficient of $y^2$ is positive), so $f$ is maximized on $[0,1]$ either at $y=0$ or $y=1$. We compute

$$f(0) \;=\; \frac{1}{(n+1)^2} + \frac{2}{n-1}\cdot\frac{1}{(n+1)^2} \;=\; \frac{1}{(n+1)^2}\Big(1+\frac{2}{n-1}\Big) \;=\; \frac{1}{(n+1)^2}\cdot\frac{n+1}{n-1} \;=\; \frac{1}{n^2-1}.$$

On the other hand,

$$\begin{aligned} f(1) &= \frac{-2}{n+1} + \frac{1}{(n+1)^2} + \frac{2}{n-1}\Big(\frac{n}{n+1}\Big)^2 \\ &= \frac{1}{(n+1)^2(n-1)}\Big(-2(n+1)(n-1) + n - 1 + 2n^2\Big) \\ &= \frac{1}{(n+1)^2(n-1)}\Big(-2(n+1)(n-1) + (n+1)(2n-1)\Big) \\ &= \frac{1}{n^2-1}. \end{aligned}$$

This proves (4), and so $B \cap H_u \subseteq B'$. □

**Claim 3.** $\mathrm{vol}(B') \leq \mathrm{vol}(B) \cdot e^{-1/4(n+1)}$.

*Proof.*

$$\begin{aligned} \left(\frac{\mathrm{vol}(B')}{\mathrm{vol}(B)}\right)^2 &= |\det M| && \text{(from Lecture 6)} \\ &= \left(\frac{n^2}{n^2-1}\right)^n \Big(1 - \frac{2}{n+1}\Big) && \text{(from Lecture 6)} \\ &= \left(1 + \frac{1}{n^2-1}\right)^n \Big(1 - \frac{2}{n+1}\Big) \\ &\leq \left(\exp\Big(\frac{1}{n^2-1}\Big)\right)^n \cdot \exp\Big(-\frac{2}{n+1}\Big) && \text{(since } 1+x \leq e^x \text{ for all } x\text{)} \\ &= \exp\left(\frac{n}{(n+1)(n-1)} - \frac{2}{n+1}\right) \\ &= \exp\left(\frac{1}{n+1}\Big(\frac{n}{n-1} - 2\Big)\right) \\ &\leq \exp\left(\frac{-1}{2(n+1)}\right) && \text{(since } n \geq 3\text{)} \end{aligned}$$

Taking square roots proves the claim. □

## 2 Covering Half-ellipsoids by Ellipsoids

Let $E = E(N, z)$ be an ellipsoid, where $N$ is a positive definite matrix and $z$ is a vector. Let $H_a = \{\ x : a^{\mathsf{T}}(x - z) \geq 0\ \}$ be a halfspace with $z$ on its boundary. We would like to find a small ellipsoid $E'$ containing the half-ellipsoid $E \cap H_a$. To solve this problem, we will use our previous result on covering hemispheres by ellipsoids. We would like to find a linear map $f$ and choose $u$ such that:

- $f(B) = E$
- $f(H_u) = H_a$. (Recall that $H_u = \{\ x : u^{\mathsf{T}} x \geq 0\ \}$.)

In Lecture 6, we showed that $E$ can be obtained by applying the affine map $f(x) = N^{1/2}x + z$ to the unit ball $B$. Now if we can judiciously choose a unit vector $u$ such that $f(H_u) = H_a$, then we'll be done. This turns out to be straightforward. Notice that

$$
\begin{aligned}
f(H_u) &= \left\{\ N^{1/2}x + z\ :\ u^{\mathsf{T}} x \geq 0\ \right\} \\
&= \left\{\ x\ :\ u^{\mathsf{T}} N^{-1/2}(x - z) \geq 0\ \right\}.
\end{aligned}
$$

So choose $u = N^{1/2}a$. Let $B'$ to be the ellipsoid covering $B \cap H_u$, as given in Theorem 1. Our solution is to define $E' = f(B')$.

**Claim 4.** $E'$ is an ellipsoid.

**Proof.** $B'$ is an ellipsoid, so it is obtained by applying an affine map to the unit ball. $E'$ is obtained by applying an affine map to $B'$. Composing these maps shows that $E'$ can also be obtained by applying an affine map to the unit ball, so $E'$ is also an ellipsoid. ■

**Claim 5.** $E \cap H_a \subseteq E'$.

**Proof.** Exercise. ■

**Claim 6.** $\mathrm{vol}(E') \leq \mathrm{vol}(E) \cdot e^{-1/4(n+1)}$.

**Proof.** Since $E' = f(B')$ and $E = f(B)$, our result on volumes under affine maps from Lecture 6 implies that

$$
\begin{aligned}
\mathrm{vol}(E') &= |\det(N^{1/2})|\,\mathrm{vol}(B') \\
\mathrm{vol}(E) &= |\det(N^{1/2})|\,\mathrm{vol}(B).
\end{aligned}
$$

Claim 3 above shows that $\mathrm{vol}(B') \leq \mathrm{vol}(B) \cdot e^{-1/4(n+1)}$. This proves the claim. ■