CO367/CM442 Nonlinear Optimization
Lecture 1

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Structure of Class

- Course Goals
  - Formulate problems as (convex) optimization problems
  - Develop code for problems of moderate size
  - Characterize optimal solutions (optimality conditions)
- Topics
  - Convex sets, functions, optimization problems
  - Examples and applications
  - Algorithms
- Textbook: Convex Optimization - Boyd and Vandenbergh
  - http://www.stanford.edu/∼boyd/cvxbook/
- Software: CVX
  - http://www.stanford.edu/∼boyd/cvx/

Requirements

- Homeworks: 40%
    - Homework 1: due Wednesday Jan 20, 2010
- Midterm: 20%
- Final: 40%

Student Academic Discipline Policy:
Copy assignments is contrary to University policy. You must work on your assignments on your own. Late assignments are not accepted.

Lecture Outline

- Mathematical Optimization
- Least-Squares and Linear Optimization
- Nonlinear Optimization
- Convex Optimization
- CVX Demonstration

Mathematical Optimization

- Mathematical Optimization Problem

$$\begin{array}{ll} \min & f_0(\mathbf{x}) \\ \text{s.t.} & f_i(\mathbf{x}) \leq b_i, \quad i = 1, \ldots, m \end{array}$$

  - $\mathbf{x} = (x_1, \ldots, x_n)$: optimization variables
  - $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}$: objective function
  - $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \ldots, m$: constraint functions
- Optimal Solution
  - $\mathbf{x}^*$ has the *smallest* value of f_0 among all vectors that *satisfy* the constraints

Examples

- Portfolio Optimization
  - Variables: amounts invested in different assets
  - Constraints: budget, max./min. investment per asset, minimum return
  - Objective: overall risk or return variance
- Device Sizing in Electronic Circuits
  - Variables: device widths and lengths
  - Constraints: manufacturing limits, timing requirements, maximum area
  - Objective: power consumption
- Data Fitting
  - Variables: model parameters
  - Constraints: prior information, parameter limits
  - Objective: measure of misfit or prediction error

Solving Optimization Problems

- General Optimization Problem
  - Very *difficult* to solve
  - Methods involve some compromise, e.g., very long computation time, or not always finding the solution
- Exceptions: certain problem classes can be solved efficiently and reliably
  - Least-squares problems
  - Linear optimization problems
  - Convex (nonlinear) optimization problems

Least-Squares

- Problem Formulation

$$\min_{\mathbf{x}\in\mathbb{R}^n} f_0(\mathbf{x}) = ||\mathbf{A}\mathbf{x} - \mathbf{b}||_2^2 = \sum_{i=1}^{k} \left(\mathbf{a}_i^T\mathbf{x} - b_i\right)^2,$$

  where $\mathbf{A} \in \mathbb{R}^{k\times n}$, $k \geq n$, and $\text{rank}(\mathbf{A}) = n$.
- Solving Least-Squares Problems
  - Analytical solution $\mathbf{x}^* = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$
  - Computational time proportional to n^2k
- Using Least-Squares
  - Basis for regression analysis, many parameter estimation and data fitting methods
  - Flexibility in applications: weighted least-squares, least-squares with regularization

Linear Programming

- Problem Formulation

$$\begin{array}{ll} \min & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{a}_i^T\mathbf{x} \le b_i, \quad i = 1, \ldots, m. \end{array}$$

- Solving Linear Optimization Problems
  - No analytical formula for solution
  - Reliable and efficient algorithms and software
  - Computational time *in practice* proportional to $n^2 m$ (assuming $m \ge n$)
- Using Linear Programming
  - Several optimization problems can be transformed to an equivalent linear program using some standard techniques
  - Example: Chebyshev approximation problem

$$\min \max_{i=1,\ldots,k} |\mathbf{a}_i^T\mathbf{x} - b_i|.$$

Nonlinear Optimization

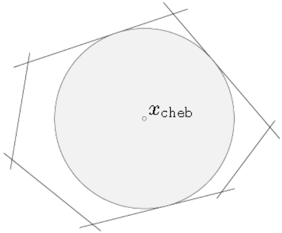
- Nonlinear Optimization
  - Either the objective or constraint functions are *not* linear (e.g. quadratic)
  - No effective methods for solving the general nonlinear optimization problem
- Convex Optimization
  - Both the objective and constraint functions are convex:

$$f_i(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \le \lambda f_i(\mathbf{x}) + (1-\lambda) f_i(\mathbf{y}), \quad \forall\, 0 \le \lambda \le 1.$$

  - Least-squares and linear programs are convex problems
  - Efficient methods for solving (nonlinear) convex problems
  - Several problems can be solved by convex optimization
- Nonconvex Optimization
  - Local optimization methods: usually iterative methods
  - Global optimization methods: exponential complexity
  - Based on solving convex subproblems

Example

- Chebyshev Center of a Polyhedron
  - Polyhedron $\mathcal{P} = \{\mathbf{x} \mid \mathbf{a}_i^T \mathbf{x} \leq b_i,\ i = 1, \ldots, m\}$
  - Ball $\mathcal{B}(\mathbf{x}_c, r) = \{\mathbf{x}_c + \mathbf{u} \mid \|\mathbf{u}\|_2 \leq r\}$
  - Find the largest *inscribed* ball $\mathcal{B}(\mathbf{x}_c, r)$ in the polyhedron $\mathcal{P}$



- Problem Formulation

$$\begin{aligned} \max_{\mathbf{x}_c, r} \quad & r \\ \text{s.t.} \quad & \mathcal{B}(\mathbf{x}_c, r) \subset \mathcal{P} \end{aligned}$$

Reformulation

- Constraint Reformulation

$$\begin{aligned}
\mathcal{B}(\mathbf{x}_c, r) \subset \mathcal{P} &\Leftrightarrow \mathbf{x}_c + \mathbf{u} \in \mathcal{P},\ \forall\, \mathbf{u} \in \mathcal{B}(\mathbf{0}, r) \\
&\Leftrightarrow \mathbf{a}_i^T(\mathbf{x}_c + \mathbf{u}) \leq b_i,\ \forall\, \mathbf{u} : ||\mathbf{u}||_2 \leq r,\ i = 1, \ldots, m \\
&\Leftrightarrow \sup_{\mathbf{u}:||\mathbf{u}||_2 \leq r} \mathbf{a}_i^T \mathbf{u} \leq b_i - \mathbf{a}_i^T \mathbf{x}_c,\ i = 1, \ldots, m
\end{aligned}$$

- Cauchy-Schwarz Inequality

$$\mathbf{a}_i^T \mathbf{u} \leq ||\mathbf{a}_i||_2\, ||\mathbf{u}||_2 \leq r\, ||\mathbf{a}_i||_2 \quad \forall\, i = 1, \ldots, m$$

- Problem Reformulation

$$\begin{aligned}
\max_{\mathbf{x}_c, r} \quad & r \\
\text{s.t.} \quad & \mathbf{a}_i^T \mathbf{x}_c + r\, ||\mathbf{a}_i||_2 \leq b_i, \quad i = 1, \ldots, m.
\end{aligned}$$

CVX Model

- http://www.stanford.edu/~boyd/cvx/
- Run `cvx_setup` in Matlab
- Code `cvx` model file
- Example: Chebyshev center of a polyhedron
  - Inputs: $\mathbf{A}, \mathbf{b}$, n, and m
  - `cvx` model:

```
cvx_begin
  variable r;
  variable x_c(n);
  maximize r
  subject to
  for i=1:1:m
    A(i,:)*x_c + r*norm(A(i,:),2) <= b(i);
  end
cvx_end
```

CVX Run

- Example in 2D by Joëlle Skaf
- Inputs $n = 2$, $m = 4$, and

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & -1 & -1 \\ 1 & -1 & 2 & -2 \end{pmatrix}^T, \mathbf{b} = (1;1;1;1)$$

- Optimal solutions $\mathbf{x}_c^* = (0;0)$ and $r^* = 0.4472$