JOINT POSITIVENESS OF MATRICES

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1. Introduction. For a fixed integer $n \geq 2$, denote by V either the set of all $n \times n$ hermitean matrices or, alternately, and more restrictedly the set of all $n \times n$ real symmetric matrices. In either case V is a linear space over the real numbers as skalars and it is euclidean if the inner product (x, y) of two elements x, y of V is defined to be the trace of their product xy.

The characteristic roots of a matrix $x \in V$ are real. The matrix x is positive definite if all its characteristic roots are positive. It is definite if either x or $-x$ is positive definite. Alternately, x is definite if the hermitean (or quadratic) form $\xi^* x \xi$ vanishes only if the vector $\xi = 0$.

The notion of definiteness can be extended to the joint behavior of two or more matrices; or rather to the linear subspace of V spanned by the matrices under consideration. Two alternative choices present themselves:

I. A linear subspace L of V is called positive definite if L contains a positive definite matrix.

II. A linear subspace L of V, is called jointly definite if the hermitean (quadratic) forms $\xi^* x \xi$ vanish simultaneously for all $x \in L$ only for the null vector 0.

In the first case we shall write $L \in \mathcal{P}$, and in the second case $L \in \mathcal{D}_1$.

If $p = \dim (L) = 1$ the two notions are identical. The case $p = 2$ was investigated in the real symmetric case by Greub and later by Calabi. They proved that for $n \geq 3$ the notions are identical, but that they differ for $n = 2$. In the case $p = 2$, the problem of joint definiteness is related to the simultaneous reduction of the matrices in L to diagonal form and hence to the commutativity of these matrices. This relation was observed by O. Taussky Todd and motivated the results which are to be presented.

The relationship between I and II is investigated for all values of p both in the hermitean and in the real symmetric case. To carry through the discussion in these two cases simultaneously a function f is introduced by

$$(1.1)\qquad f(r) = \begin{cases} r^2 & \text{in the hermitean case;} \\ \frac{1}{2}r(r+1) & \text{in the real symmetric case.} \end{cases}$$

With this notation we can say that in either case the dimension of the full
linear space V is equal to $f(n)$.

A stronger condition than that expressed by II is needed:

<u>Definition</u>. A linear subspace L of V is said to be jointly definite of degree r, $L \in \mathbb{D}_r$, if

$$\sum_{i=1}^{r} \xi_i^* x \xi_i = 0 \text{ for each } x \in L \text{ simultaneously,}$$

imply that each vector $\xi_i = 0$.

It is evident from the definition that

$$(1.2)\qquad \mathbb{D}_1 \supseteq \mathbb{D}_2 \supseteq \cdots \supseteq \mathbb{D}_r \supseteq \cdots \supseteq \mathbb{P}.$$

<u>Theorem I</u>. $\mathbb{D}_n = \mathbb{P}$. (See section 2 for its proof.)

Depending on the dimension p of L ~~the~~ equality between a $\mathbb{D}_r$ and $\mathbb{P}$ occurs for values of r smaller than n.

<u>Definition.</u> $L \in \mathbb{D}_r^{(p)}$, resp. $\mathbb{P}^{(p)}$ if $L \in \mathbb{D}_r$, resp. $\mathbb{P}$ and $\dim(L) = p$.

For each p, $1 \leq p \leq f(n)$ we have

(1.3) $\mathbb{D}_1^{(p)} \supset \mathbb{D}_2^{(p)} \supset \cdots \supset \mathbb{D}_n^{(p)} = \mathbb{P}^{p}$ and

<u>Theorem II.</u> For $1 \leq r \leq n-1$, $1 \leq p \leq f(n)-1$,

$$\mathbb{D}_r^{(p)} = \mathbb{P}^{(p)} \text{ iff } \cancel{\text{is equivalent to}}\ p < f(r+1) - \delta_{n,r+1}.$$

(See section 3 for the proof.)

This theorem can be stated alternately in the following form:

<u>Theorem III.</u> Let $1 \leq r \leq n-1$, $p = f(r+1) - \delta_{n,r+1} - 1$.
Given any matrices $a_1, a_2, \ldots, a_p$ in V which are jointly definite of degree r, then a linear combination of these matrices is positive definite. There exists $a_1, a_2, \ldots, a_{p+1} \in V$ which are jointly definite of degree r and yet no linear combination of these matrices is positive definite. (Except for the zero matrix, no linear combina on is ever positive semi-definite.)

The following tables exhibit this threshold value of p for various values of r and n.

Table I (Real symmetric case)

n \ r	1	2	3	4	5
2	1	-	-	-	-
3	2	4	-	-	-
4	2	5	8	-	-
5	2	5	9	13	-
6	2	5	9	14	19

The first column exhibits the results of Graeub.

Table II (Hermitean case)

n \ r	1	2	3	4	5
2	2	-	-	-	-
3	3	7	-	-	-
4	3	8	14	-	-
5	3	8	15	23	-
6	3	8	15	24	34

The first column exhibits in particular the following result valid for $n \geq 3$:

$$\xi^* a_1 \xi = \xi^* a_2 \xi = \xi^* a_3 \xi = 0 \quad \text{implies} \quad \xi = 0$$

there exists a real linear combination of the a_i's which is positive definite.

2. Equivalent definition of $\mathbb{D}_r$ and $\mathbb{P}$. A matrix a is positive semi-definite of rank $\leq r$ if and only if

$$a = \sum_{i=1}^{r} \xi_i \xi_i^* .$$

The condition $\Sigma \xi_i^* x \xi_i = 0$ is equivalent to $\operatorname{Tr} x a = 0$, i.e. x and a are perpendicular to each other in the euclidean space V. Thus

(2.1) $L \in \mathbb{D}_r$ if and only if $K = L^{\perp}$ contains no positive semi-definite matrix of rank $\leq r$ except the zero matrix.

Next we show that

(2.2) $L \in \mathbb{P}$ if and only if $K = L^{\perp}$ contains no positive semi-definite matrix except the zero matrix.

If $L \in \mathbb{P}$ then $L \in \mathbb{D}_n$ and K contains no non zero positive semi-definite matrix. Conversely, given such a K, there exists a K' of dimension $f(n)-1$, $K' \supset K$, and such that K' contains no non zero positive semi-definite matrix. (The set of positive semi-definite matrices is a closed convex cone in V.) Let L' be the linear space perpendicular to K'. Then $L' \subset L$ and L' is one-dimensional. Let $a \in L'$, $a \neq 0$. If a is not definit there exists a vector ξ such that $\xi \neq 0$, and $\xi^* a \xi = 0$. The matrix $\xi \xi^* \in K'$ and would be $\neq 0$ and positive semi-definite.

The proof of Theorem I is a comparison of (2.2) and (2.1) for the case $r = n$.

3. <u>The proof of Theorem II</u>. By means of the alternate definitions given in section 2, and replacing $r+1$ by r, Theorem II will be proved if we can show:

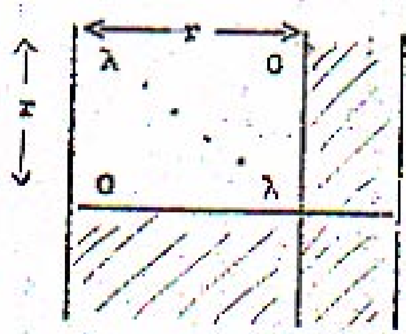
<u>Theorem IV</u>. Given r, k such that $2 \leq r \leq n$, $1 \leq k \leq f(n) - f(r) + \delta_{r,n}$, there exists a linear subspace of V of dimension k which contains a positive semi-definite matrix of rank r, but none of smaller rank except zero.

<u>Theorem V</u>. Let K be a k-dimensional linear subspace of V which contains a positive semi definite matrix $\neq 0$. If $k \geq f(n) - f(r) + \delta_{r,n} + 1$, $2 \leq r \leq n$, then K contains a positive semi definite matrix $\neq 0$ of rank $< r$.

If $r = n$, both theorems are obvious and hence we shall assume that $2 \leq r \leq n-1$.

4. <u>An example</u>. Let r be an integer, $2 \leq r \leq n-1$. Let K be the linear space of all matrices x in V which are of the form

$x =$



where the elements in the shaded areas are arbitrary, subject to the condition that $x \in V$ and to the condition that the trace of x

be equal to $n\lambda$. The dimension of K is equal to $f(n) - f(r)$. For any positive semi-definite matrix $x \in K$, $x \neq 0$, the trace of x is positive ~~semi-definite matrix $x \in K$, $x \neq 0$, the trace of x is positive~~ and hence λ is positive. This implies that the rank $\rho(x)$ is at least equal to r. Finally, the linear space K contains the semi-definite matrix

$$\begin{array}{c} \\ r \updownarrow \\ \\ \end{array} \left(\begin{array}{ccc|c} \overbrace{1 \quad \cdot \quad \cdot}^{r} & & & \\ & \cdot & & \\ & & 1 & \mu \\ \hline & & \mu & \mu^2 \end{array}\right)$$

where $\mu = \sqrt{n-r}$. This matrix is of rank r, and is positive semi-definite. This example proves Theorem IV.

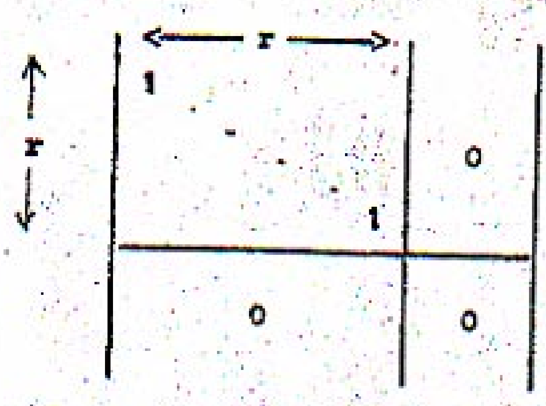
5. Two lemmas.

Lemma 1. Let $2 \leq r \leq n-1$, $k \geq f(n) - f(r) + 1$ and K a k-dimensional linear subspace of V, which contains a positive semi-definite matrix a of rank r. Then K contains either

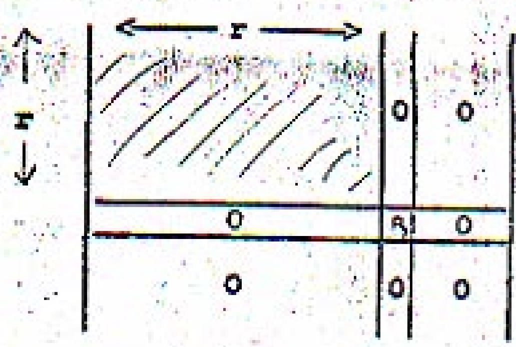
(1) an indefinite matrix of rank $\leq r$, or

(2) a positive semi-definite matrix $\neq 0$, of rank $< r$.

Proof. We may as well assume that the matrix a is the matrix

$a =$ 

The linear space L of all matrices in V of the form

$b =$ 

has dimension $f(r) + 1$. (The shaded area is arbitrary.) Since $k \geq f(n) - f(r) + 1$ the intersection $K \cap L$ is at least two-dimensional. There exists thus a matrix b in $K \cap L$ which is linearly independent from the matrix a. Let $\beta_1, \beta_2, \dots, \beta_r, \beta, 0, \dots, 0$ be its characteristic values, where the β_i's are so labeled that $\beta_1 \geq \beta_2 \geq \dots \geq \beta_r$; $\beta \geq 0$. The matrix $c = b - \beta_1 a$ has the characteristic roots

$$0, \beta_2 - \beta_1, \dots, \beta_r - \beta_1, \beta, 0, \dots, 0 .$$

Either c, or $-c$, satisfies the requirements of Lemma [illegible].

Lemma 2. Let $1 \leq r \leq n$, $k \geq f(n) - f(r) + 1$ and K a k-dimensional linear subspace of V. Let K_1 be the subset of K consisting of all $x \in K$, of norm $= 1$, and of rank $\leq r$. Then K_1 is projectively connected, i.e. if $A = -A$, $B = -B$ are closed sets, $A \cup B = K_1$, $A \cap B = \emptyset$ then either A or B is void.

Proof. Let $A = -A$ be a closed subset of the closed set K_1. Denote by $E(A)$ the set of all idempotent matrices e in V of rank $(n-r)$ for which there exists an $a \in A$ such that $ae = 0$. The set $E(A)$ is evidently closed.

Similarly, if $B = -B$ is a closed subset of K_1, then $E(B)$ is closed. From the definition follows $E(A) \cup E(B) = E(K_1)$.

Let e be any idempotent matrix in V of rank $(n-r)$. The linear space $\{x \mid x \in V, xe = 0\}$ has dimension $f(r)$. Since $k \geq f(n) - f(r) + 1$ the intersection of this space with K is at least one-dimensional. Furthermore, the rank of x is $\leq r$ if $xe = 0$. This shows that $E(K_1)$ is the set of all idempotent matrices in V of rank $(n-r)$. This set is connected and hence either $E(A)$ is void, or $E(B)$ is void, or $E(A) \cap E(B)$ is non void.

If $E(A)$ is void, then A is void since to each x of rank $\leq r$, there exists an idempotent e of rank $n-r$ such that $xe = 0$. If $E(B)$ is void, then similarly B must be void.

If neither A, nor B are void then $E(A) \cap E(B)$ is non void. This is impossible if $A \cap B = \emptyset$. Indeed, let $e \in E(A) \cap E(B)$, and let $a \in A$, $b \in B$ be such that $ae = be = 0$. Since $A = -A$, $A \cap B = \emptyset$, $\|a\| = \|b\| = 1$, the matrices a, b must be linearly independent. Consider the circle C of radius one which passes through a and b. For any $x \in C$, $xe = 0$, hence $\rho(x) \leq r$ and obviously $x \in K_1$. Thus $C \subset A \cup B$ and the intersection $A \cap B$ could not be void.

$\geq f(n) - f(r+p) - 1$

6. Proof of Theorem V. It is a simple consequence of ~~the~~ lemmas 1 and 2. Let $2 \leq r \leq n-1$, and let K be a k-dimensional linear subspace of V which contains a positive semi-definite matrix a of rank r. Assume, furthermore, that $k \geq f(n) - f(r) + 1$. Apply lemma 2 to the set A of all matrices in K, of norm 1, semi-definite of rank $\leq r$ and B the closure of the set of all matrices in K, of norm 1, and indefinite. By assumption the set A is non void.

If B is non void, the intersection is $A \cap B$ is non void. Any $x \in A \cap B$ is semi-definite, $\neq 0$, of rank $< r$.

If B is void, lemma 1 shows that there exists in K a positive semi-definite matrix, $\neq 0$, of rank $< r$. This completes the proof of Theorem V.

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