A Characterization of Continuous differentiability of Proximal Mappings of Composite Functions

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Based on the Joint works with N.T.V. Hang (Institute of Mathematics,Vietnam)

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Motivation

Proto-Differentiability

Strict Proto-Differentiability

Smoothness of Proximal Mappings

Recall that for a convex function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}} = [-\infty, \infty]$ and parameter value $r > 0$, the proximal mapping of f, denoted by prox_{rf}, is defined by

$$\mathrm{prox}_{rf}(x) = \operatorname*{argmin}_{w \in \mathbb{R}^n} \left\{ f(w) + \frac{1}{2r}\|w - x\|^2 \right\}, \;\; x \in \mathbb{R}^n.$$

When $f = \delta_C$, namely the indicator function of a convex set $C \subset \mathbb{R}^n$, this mapping reduces to he projection mapping of C, defined by

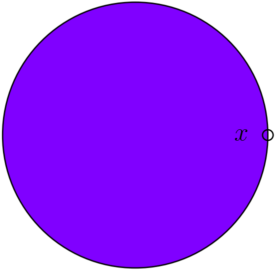
$$P_C(x) = \operatorname{argmin} \left\{ \|w - x\|^2 \,\middle|\, w \in C \right\}, \;\; x \in \mathbb{R}^n.$$

- **Question.** At what points is P_C is continuously differentiable ($\mathcal{C}^1$)? [1]

[1] "In spite of the elementary formulation of this question, a full answer is so far unknown." J.-B. Hiriart-Urruty, At what points is the projection mapping differentiable? Amer. Math. Monthly 89(7), 456–458 (1982)

What we know so far:

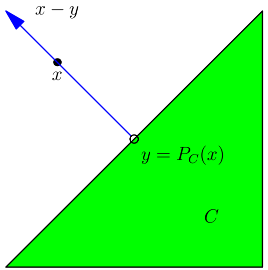
• The projection mapping P_C may fail to be differentiable in general[2]. For instance, assume that C is the unit ball and x is a vector that $\|x\| = 1$. Then P_C fails to be continuously differentiable at x.



[2] R.B. Holmes, Smoothness of certain metric projections on Hilbert space. Trans. Amer. Math. Soc. 183, 87–100 (1973)

What we know so far:

• R. Holmes [3] studied the smoothness of projection mapping onto a closed convex set in Hilbert spaces. His main result states that if $C \subset \mathbb{R}^n$ is a closed convex set, $x \in \mathbb{R}^n$, the boundary of C is a $\mathcal{C}^2$ smooth manifold around $y = P_C(x)$, then the projection mapping P_C is $\mathcal{C}^1$ in a neighborhood of the open normal ray $\{y + t(x - y) | \, t > 0\}$.



• When the projection point y is a corner point, Holmes's result fails because the boundary of C is not a $\mathcal{C}^2$ smooth manifold around y.

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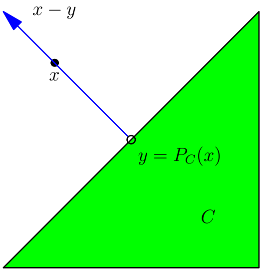
What we know so far:

Theorem. (Facchinei-Pang[4]) Assume that C is a polyhedral convex set. Then the projection mapping P_C is differentiable at x if and only if $x - y \in \operatorname{ri} N_C(y)$[5], where $y = P_C(x)$.



[4] F. Facchinei, J.-S. Pang, Finite-Dimesional Variational Inequalities and Complementarity Problems. Springer New York, New York (2003)

[5] $N_C(\bar{x}) = \left\{ v \in \mathbb{R}^n | \langle v, x - \bar{x} \rangle \leq 0 \text{ for all } x \in C \right\}$

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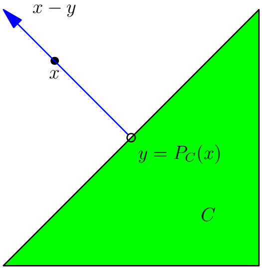

- P_C is not differentiable at y since $0 \notin \mathrm{ri}\, N_C(y)$.

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What we know so far:

- The projection mapping P_C is always directionally differentiable if we assume a second-order regularity on C such as parabolic regularity [6]. Recall that a function $g : \mathbb{R}^n \to \mathbb{R}^m$ is directionally differentiable at $\bar{x}$ if the following limit exists for any $w \in \mathbb{R}^n$:

$$\lim_{t \downarrow 0} \frac{g(\bar{x} + tw) - g(\bar{x})}{t}.$$

[6] A. Mohammadi, B.S. Mordukhovich and M.E. Sarabi, Parabolic regularity via geometric variational analysis. Trans. Amer. Soc. 374(3), 1711–1763 (2021)

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• We likely need a second-order regularity to ensure continuous differentiability of the projection mapping onto a closed convex (prox-regular) set.

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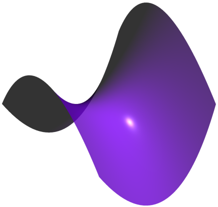
What we know so far:

Assume that $C \subset \mathbb{R}^n$ is a $\mathcal{C}^2$ smooth manifold around a point $\bar{x} \in C$, meaning that there exists a neighborhood O of $\bar{x}$ on which C has the representation

$$C \cap O = \{x \in O | \, \Phi(x) = 0\},$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ is a $\mathcal{C}^2$ function with $\nabla\Phi(\bar{x})$ having full rank.



[7] A.S. Lewis and J. Malick, Alternating projections on manifolds, Math. Oper. Res., 33 (2008) 216–234.

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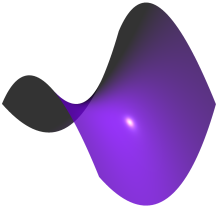

- It is well-known that the projection mapping P_C is locally single-valued and Lipschitz continuous and directionally differentiable.
- Lewis and Malick [7] showed that P_C is $\mathcal{C}^1$ around $\bar{x}$.

[7] A.S. Lewis and J. Malick, Alternating projections on manifolds, Math. Oper. Res., 33 (2008) 216–234.

• Given $C \subset \mathbb{R}^n$ and $\bar{x} \in C$, recall that the tangent cone and the adjacent cone to C at $\bar{x}$ are defined, respectively, by

$$T_C(\bar{x}) = \limsup_{t \downarrow 0} \frac{C - \bar{x}}{t}{}^{8} \quad \text{and} \quad A_C(\bar{x}) = \liminf_{t \downarrow 0} \frac{C - \bar{x}}{t}{}^{9},$$

where both limits are understood in the sense of Painlevé-Kuratowski.

[8] $T_C(\bar{x}) = \{w \in \mathbb{R}^n \mid \exists\, t_k \downarrow 0, \ \ w_k \to w \ \text{ as } \ k \to \infty \ \text{ with } \ \bar{x} + t_k w_k \in C\}$
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- • Clearly we always have $A_C(\bar{x}) \subset T_C(\bar{x})$.
- • **Definition**. Suppose that $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a convex function and $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$. We say ∂f is proto-differentiable at $\bar{x}$ for $\bar{v}$ if

$$A_{\operatorname{gph} \partial f}(\bar{x}, \bar{v}) = T_{\operatorname{gph} \partial f}(\bar{x}, \bar{v}),$$

where

$$\operatorname{gph} \partial f = \big\{(x, v) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, v \in \partial f(x)\big\}.$$

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Theorem.(Rockafellar[10] (1990)). Suppose that $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper convex function and $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$. Then the following properties are equivalent:

- ∂f is proto-differentiable at $\bar{x}$ for $\bar{v}$;
- prox_f is directionally differentiable at $\bar{x} + \bar{v}$.

The proof is based on the identity

$$\operatorname{prox}_f = (I + \partial f)^{-1},$$

which holds for any convex functions.

[10] R.T. Rockafellar, Generalized second derivatives of convex functions and saddle functions. Trans. Amer. Math. Soc. 322(1), 51–77 (1990)

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$$(w, q) \in T_{\operatorname{gph} \partial f}(\bar{x}, \bar{v}) \iff (w + q, w) \in T_{\operatorname{gph} \operatorname{prox}_f}(\bar{x} + \bar{v}, \bar{y})$$

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which holds for any convex functions. Proto-differentiability holds for many important sets and functions including

- polyhedral convex sets, the second-order cone, the cone of positive semidefinite symmetric matrices;
- polyhedral functions; convex piecewise linear-quadratic functions, spectral functions.

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Motivation

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Smoothness of Proximal Mappings

• Given $C \subset {\rm I\!R}^n$ and $\bar{x} \in C$, recall that the paratingent cone and the regular (Clarke) tangent cone to C at $\bar{x}$ are defined, respectively, by

$$\widehat{T}_C(\bar{x}) = \limsup_{x \to \bar{x} t \downarrow 0} \frac{C - \bar{x}}{t} \quad \text{and} \quad \widetilde{T}_C(\bar{x}) = \liminf_{x \to \bar{x}, t \downarrow 0} \frac{C - x}{t},$$

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- **Definition**. Suppose that $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a convex function and $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$. We say ∂f is strictly proto-differentiable [11] at $\bar{x}$ for $\bar{v}$ if

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Theorem.(Poliquin-Rockafellar[12] (1996)). Suppose that $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper convex function and $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$. Then the following properties are equivalent:

- ∂f is strictly proto-differentiable at x for v for any $(x, v) \in \operatorname{gph} \partial f$ sufficiently close to $(\bar{x}, \bar{v})$;
- for any $r > 0$, prox_{rf} is continuously differentiable in a neighborhood of $\bar{x} + r\bar{v}$.

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Poliquin-Rockafellar showed that this result holds for prox-regular functions at $\bar{x}$ for $\bar{v} = 0$ provided that $\bar{x} \in$ argmin f. It is, however, possible to show that the latter condition can be dropped using the stability properties of generalized equations.[13].

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Question. When does strict proto-differentiability hold?

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Recall that $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is called polyhedral if $\operatorname{epi} f$ is a polyhedral convex set. Important examples of polyhedral functions include

- the indicator function of a polyhedral convex set;
- $f(x) = \max\{\langle a_i, x\rangle + \alpha_i |\ i = 1, \ldots, m\}$ with $a_i \in \mathbb{R}^n$ and $\alpha_i \in \mathbb{R}$.

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Theorem.(Hang-S[14] (2022)). Suppose that $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a polyhedral function and $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$. Then the following properties are equivalent:

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Theorem.(Hang-S[15] (2022)). Suppose that $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a polyhedral function and $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$. Then the following properties are equivalent:

- for any $r > 0$, prox_{rf} is continuously differentiable in a neighborhood of $\bar{x} + r\bar{v}$;
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Theorem.(Hang-S[15] (2022)). Suppose that $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a polyhedral function and $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$. Then the following properties are equivalent:

- for any $r > 0$, prox_{rf} is continuously differentiable in a neighborhood of $\bar{x} + r\bar{v}$;
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Corollary. (Hang-S (2022)). Assume that $C \subset \mathbb{R}^n$ is a polyhedral convex set and $x \in \mathbb{R}^n$. Then P_C is continuously differentiable in a neighborhood of x if and only if $x - y \in \operatorname{ri} N_C(z)$, where $y = P_C(x)$.

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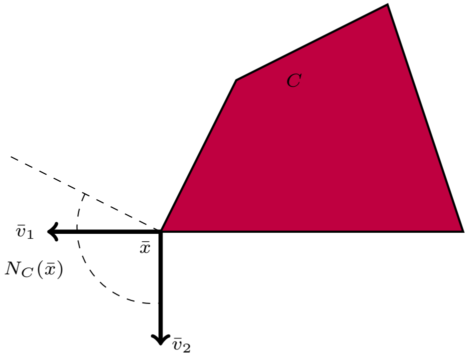
For the polyhedral set C, P_C is continuously differentiable at $\bar{x} + \bar{v}_1$ but is not continuously differentiable at $\bar{x} + \bar{v}_2$.



• Similar results[16] were established recently for the composite function

$$f \circ \Phi,$$

where f is a polyhedral function and Φ is a $\mathcal{C}^2$ function, and the constraint qualification

$$\text{par}\{\partial f(\Phi(\bar{x}))\}^{17} \cap \ker \nabla\Phi(\bar{x})^* = \{0\}$$

is satisfied at $\bar{x} \in \mathbb{R}^n$ with $\Phi(\bar{x}) \in \operatorname{dom} f$.

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is satisfied at $\bar{x} \in \mathbb{R}^n$ with $\Phi(\bar{x}) \in \operatorname{dom} f$.

- The condition above boils down to the classical linear independent constraint qualification when $f = \delta_{\mathbb{R}^m_- \times \{0\}^{n-m}}$ with $0 \le m \le n$.
- This composite function is prox-regular and thus its proximal mapping is locally single-valued and Lipschitz continuous.[18].

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Theorem.(Hang-S (2022)). Given the composite function $g = f \circ \Phi$ with $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial g$, the following properties are equivalent:

- ∂g[19] is strictly proto-differentiable at x for v for any $(x, v) \in \operatorname{gph} \partial g$ sufficiently close to $(\bar{x}, \bar{v})$;
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Theorem(Hang-S (2022)). For the composite function $g = f \circ \Phi$ with $(\bar{x}, \bar{v}) \in \mathrm{gph}\, \partial g$, the following properties are equivalent:

- $\bar{v} \in \mathrm{ri}\, \partial g(\bar{x})$;
- for any $r > 0$ sufficiently small, the proximal mapping prox_{rg} is continuously differentiable in a neighborhood of $\bar{x} + r\bar{v}$.

[19] the limiting subdifferential of g

Assume that $C \subset \mathbb{R}^n$ is fully amenable around a point $\bar{x} \in C$, meaning that there exists a neighborhood O of $\bar{x}$ on which C has the representation

$$C \cap O = \{x \in O | \, \Phi(x) \in \Theta\},$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ is a $\mathcal{C}^2$ function and $\Theta \subset \mathbb{R}^m$ is a polyhedral convex set, and the condition

$$\text{span}\{N_C(\Phi(\bar{x}))\}^{20} \cap \ker \nabla\Phi(\bar{x})^* = \{0\}$$

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Theorem(Hang-S (2022)). For a fully amenable set C with $(\bar{x}, \bar{v}) \in \text{gph}\, N_C$, the following properties are equivalent:

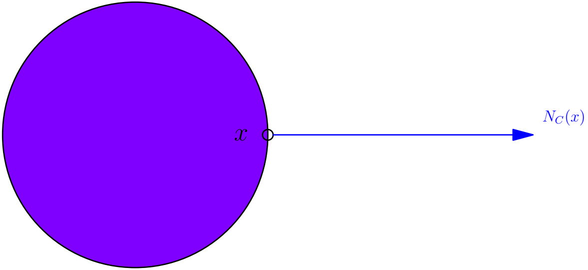
- $\bar{v} \in \text{ri}\, N_C(\bar{x})$;
- for any $r > 0$ sufficiently small, the projection mapping P_C is continuously differentiable in a neighborhood of $\bar{x} + r\bar{v}$.

[20] the linear subspace $N_C(\Phi(\bar{x}))$.

Example. Assume that C is the unit ball in $\mathbb{R}^n$. Then C is full amenable at every point $x \in C$ since

$$C = \left\{ x \in \mathbb{R}^n \middle| \, \Phi(x) \leq 0 \right\} \quad \text{with} \;\; \Phi(x) = \|x\|^2 - 1.$$

If $\|x\| = 1$, then we have $0 \notin \operatorname{ri} N_C(x)$ and thus P_C can't be continuously differentiable around x.



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Thank you for you attention!