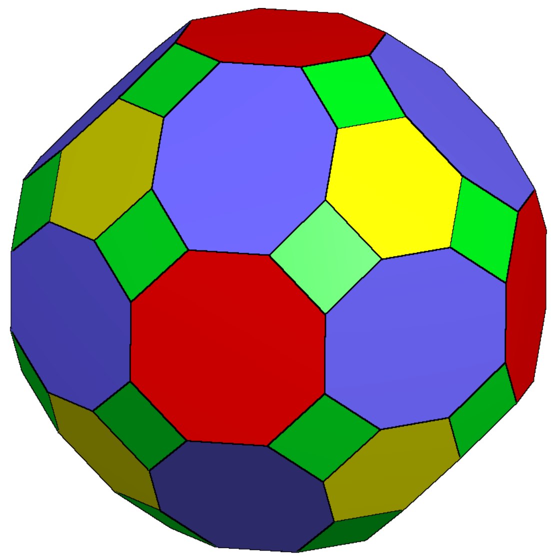
Worst-case constructions for linear optimization



Antoine Deza, McMaster

based on joint works with:

Shmuel Onn, Technion, **Sebastian Pokutta**, ZIB, **Lionel Pournin**, Paris XIII

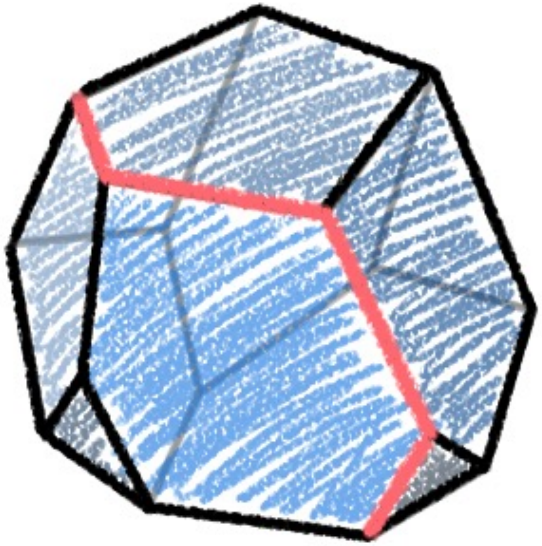
Linear optimization

Given an ***n***-dimensional vector ***b*** and an ***n*** x ***d*** matrix ***A***
find, in any, a ***d***-dimensional vector ***x*** such that :

$\boldsymbol{Ax = b}$	$\boldsymbol{Ax \leq b}$
linear algebra	**linear optimization**

"Can linear optimization be solved in ***strongly polynomial*** time?"
is listed by Smale as one of the top problems for the XXI century

Strongly polynomial *:* algorithm ***independent*** from
the ***input data length*** and polynomial in ***n*** and ***d****.*

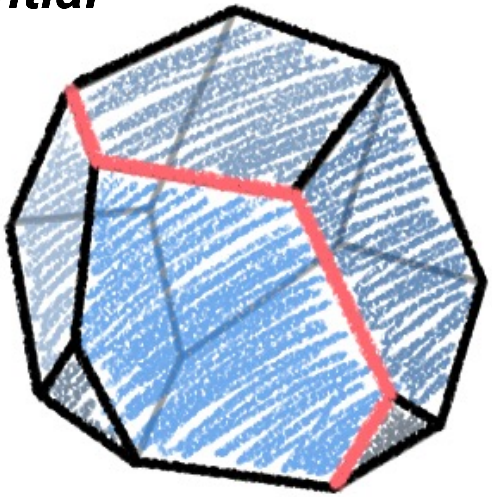


Linear optimization algorithms
simplex methods

Given an ***n***-dimensional vector ***b*** and an ***n*** x ***d*** (full row-rank) matrix ***A*** and a ***d***-dimensional cost vector ***c***, solve : max { $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$: $\boldsymbol{Ax} = \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ }

Simplex methods (Dantzig 1947): pivot-based, combinatorial, ***not proven to be polynomial***, efficient in practice

- start from a ***feasible basis***
- use a ***pivot rule***
- find an optimal solution after a ***finite number*** of iterations
- most known pivot rules are known to be ***exponential*** (worst case); ***efficient*** implementations exist

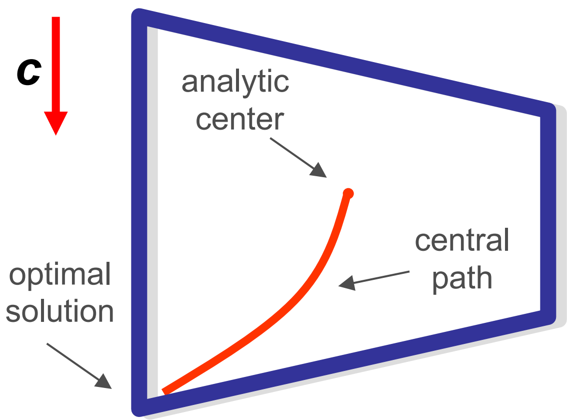


Linear optimization algorithms
(central path following) interior point methods

Given an n-dimensional vector $\boldsymbol{b}$ and an n x d (full row-rank) matrix $\boldsymbol{A}$ and a d-dimensional cost vector $\boldsymbol{c}$, solve : max { $\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{Ax} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$ }

Interior Point Methods :
path-following, ***polynomial***, efficient in practice

- start from the ***analytic center***
- follow the ***central path***
- converge to an optimal solution in $O(\sqrt{n}L)$ iterations
  (L: input data length)



$$\max \quad c^T x - \mu \sum_i \ln(b - Ax)_i$$

μ : central path parameter
$x \in \boldsymbol{P} : Ax \leq b$

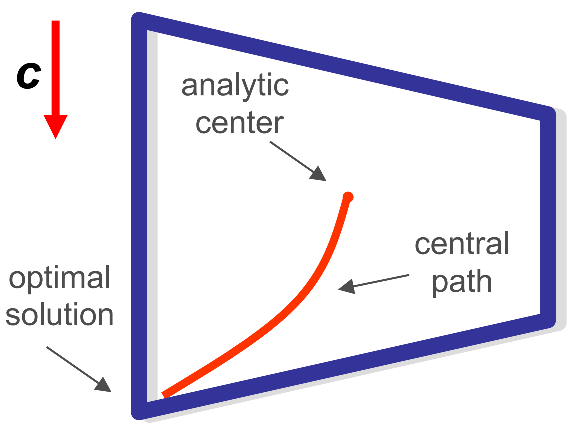
Linear optimization diameter and curvature

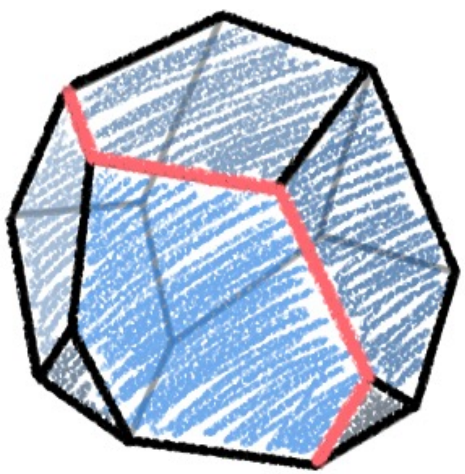
Diameter (*of a polytope*) :

lower bound for the number of iterations for *pivoting* **simplex methods**

Curvature (*of the central path associated to a polytope*) :

large curvature indicates large number of iterations for *path following* **interior point methods**





Linear optimization

Given an ***n***-dimensional vector ***b*** and an ***n*** x ***d*** matrix ***A***
find, in any, a ***d***-dimensional vector ***x*** such that :

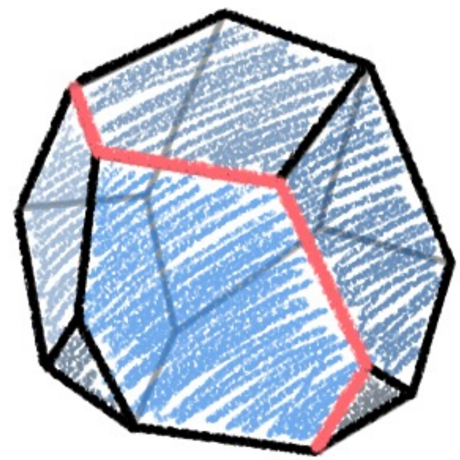
Ax* = *b ***Ax* ≤ *b***

linear algebra **linear optimization**

"Can linear optimization be solved in ***strongly polynomial*** time?"
is listed by Smale as one of the top I problems for the XXI century

- [Allamigeon, Benchimol, Gaubert, Joswig 2018]
(logarithmic barrier) ***Interior point methods***
are ***not strongly polynomial***

- [Allamigeon, Gaubert, Vandame 2022]
(self-concordant barrier) ***Interior point methods***
are ***not strongly polynomial***



(tropical counterexample to continuous Hirsch conjecture [Deza-Terlaky-Zinchenko 2008])

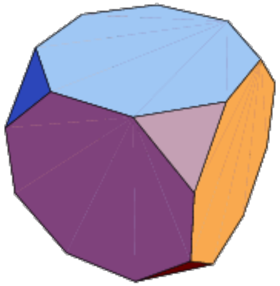
Lattice polytopes with large diameter

lattice (***d***,***k***)-polytope : convex hull of points drawn from $\{0,1,\ldots,k\}^d$

diameter δ(***P***) of polytope ***P*** : smallest number such that **any two vertices** of ***P*** can be connected by a **path with at most** δ(***P***) **edges**

δ(***d***,***k***): largest diameter over all **lattice** (***d***,***k***)-polytopes

ex. δ(3,3) = 6 and is achieved by a ***truncated cube***



- δ(***d***,***k***) : lower bound on the number of simplex pivots required in the worst case to perform linear optimization on a lattice polytope

- [Del Pia-Michini 2018] *preprocessing* and *scaling algorithm* yielding simplex paths that are ***short relative*** to δ(***d***,***k***)

Lattice polytopes with large diameter

lattice (d,k)-polytope : convex hull of points drawn from $\{0,1,\ldots,k\}^d$

diameter $\delta(P)$ of polytope P : smallest number such that **any two vertices** of P can be connected by a **path with at most** $\delta(P)$ **edges**

$\delta(d,k)$: largest diameter over all **lattice** (d,k)-polytopes

- $\delta(P)$: lower bound for the worst-case number of iterations required by *pivoting methods* (simplex) to optimize a linear function over P
- *Hirsch conjecture* : $\delta(P) \leq n - d$ (n number of inequalities) was ***disproved*** [Santos 2012]

$\delta(P) \leq (n - d)^{\log d - \ldots}$ [Kalai-Kleitman 1992, Todd 2014, Sukegawa 2019]

❖ ***no polynomial upper bound*** known for $\delta(P)$

Lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\dots,k\}^d$

upper bounds :

$\delta(d,1) \leq d$		[Naddef 1989]
$\delta(2,k) = O(k^{2/3})$		[Balog-Bárány 1991]
$\delta(2,k) = 6(k/2\pi)^{2/3} + O(k^{1/3} \log k)$		[Thiele 1991] [Acketa-Žunić 1995]
$\delta(d,k) \leq kd$		[Kleinschmid-Onn 1992]
$\delta(d,k) \leq kd - \lceil d/2 \rceil$	for $k \geq 2$	[Del Pia-Michini 2016]
$\delta(d,k) \leq kd - \lceil 2d/3 \rceil - (k-3)$	for $k \geq 3$	[Deza-Pournin 2018]

Lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\ldots,k\}^d$

lower bounds :

$\delta(d,1) \geq d$	[Naddef 1989]
$\delta(d,2) \geq \lfloor 3d/2 \rfloor$	[Del Pia-Michini 2016]
$\delta(d,k) = \Omega(k^{2/3}\, d)$	[Del Pia-Michini 2016]
$\delta(d,k) \geq \lfloor (k+1)d/2 \rfloor$ for $k < 2d$	[Deza-Manoussakis-Onn 2018]
$\delta(d,k) = \Omega(k^{d/d+1})$ for fixed d	[Deza-Pournin-Sukegawa 2020]

- Lower bound of $\Omega(k^{d/d+1})$ obtained by counting primitive points within simplex and cross polytope blown up by an integer factor

  [Manecke-Sanyal 2020]: primitive Ehrhart theory

Lattice polytopes with large diameter

$\delta(d,k)$		*k*								
		1	2	3	4	5	6	7	8	9
d	2	**2**	**3**	**4**	**4**	**5**	**6**	**6**	**7**	**8**
	3	**3**	**4**	6	**7**	**9**	**10**			
	4	**4**	**6**	**8**						
	5	**5**	**7**	**10**						

$\delta(d,1) = d$ [Naddef 1989]

$\delta(2,k)$: close form [Thiele 1991] [Acketa-Žunić 1995]

$\delta(d,2) = \lfloor 3d/2 \rfloor$ [Del Pia-Michini 2016]

$\delta(4,3)=8$, $\delta(3,4)=7$, $\delta(3,5)=9$ [Deza-Pournin 2018], [Chadder-Deza 2017]

$\delta(5,3)=10$, $\delta(3,6)=10$ [Deza-Deza-Guan-Pournin 2019]

Lattice polytopes with large diameter

$\delta(d,k)$		k								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3	4	6	7	9	10	11+	12+	13+
	4	4	6	8	10+	12+	14+	16+	17+	18+
	5	5	7	10	12+	15+	17+	20+	22+	25+

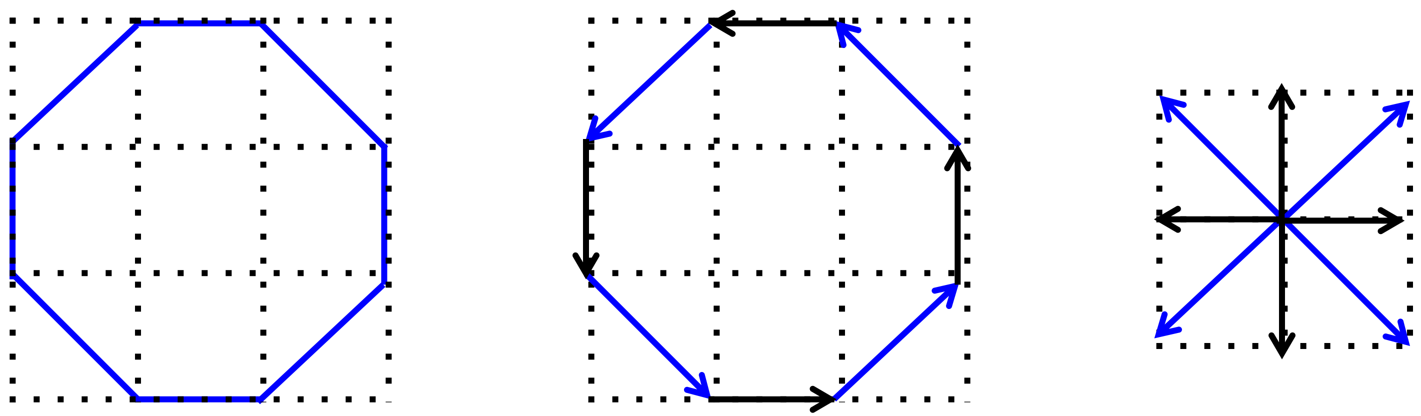
- Conjecture [Deza-Manoussakis-Onn 2018] $\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$

and $\delta(d,k)$ is achieved, up to translation, by a *Minkowski sum of primitive lattice vectors*. The conjecture holds for all known entries of $\delta(d,k)$

Lattice polygons with large diameter

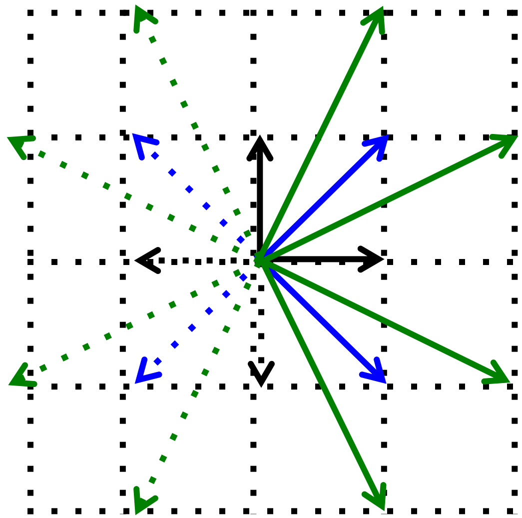
Q. What is δ(2,***k***) : largest diameter of a polygon which vertices are drawn form the ***k*** x ***k*** grid?

A polygon can be associated to a set of vectors (*edges*) *summing up to zero*, and *without a pair of positively multiple vectors*



δ(2,3) = 4 is achieved by the 8 vectors : (±1,0), (0,±1), (±1,±1)

Primitive polygons



$||x||_1 \leq \boldsymbol{p}$

$H_1(2,\boldsymbol{p})$: Minkowski sum generated by $\{x \in \mathbb{Z}^2 : ||x||_1 \leq \boldsymbol{p},\ \mathrm{gcd}(x)=1,\ x \succeq 0\}$

$H_1(2,\boldsymbol{p})$ has diameter $\delta(2,\boldsymbol{k}) = 2\sum_{i=1}^{p} \varphi(i)$ for $\boldsymbol{k} = \sum_{i=1}^{p} i\varphi(i)$

Ex. $H_1(2,2)$ generated by (1,0), (0,1), (1,1), (1,-1) (fits, *up to translation*, in 3x3 grid)

$\varphi(\boldsymbol{p})$: ***Euler totient function*** counting positive integers less or equal to $\boldsymbol{p}$ relatively prime with $\boldsymbol{p}$

$\varphi(1) = \varphi(2) = 1$, $\varphi(3) = \varphi(4) = 2, \ldots$ $x \succeq 0$: first nonzero coordinate of x is nonnegative

Primitive zonotopes

$H_{q}(d,p)$: Minkowski $(x \in \mathbb{Z}^{d} : ||x||_{q} \leq p, \gcd(x)=1, x \succeq 0)$

$x \succeq 0$: first nonzero coordinate of x is nonnegative

Given a set G of m vectors (generators),
Minkowski (G) : convex hull of all the 2^{m} ***subsums*** of the m vectors in G

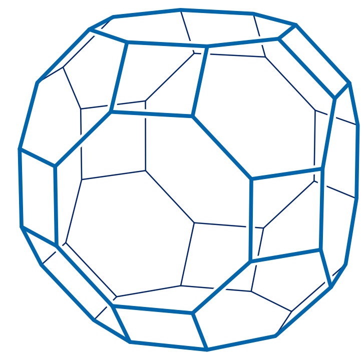
- ***Primitive zonotopes***: Minkowski sum generated by ***short integer*** vectors which are ***pairwise linearly independent***

- *Note*: convex hull of all the ***signed*** subsums of the vectors of $H_{q}(d,p)$ *is a generalization of the permutahedron of type* B_{d}

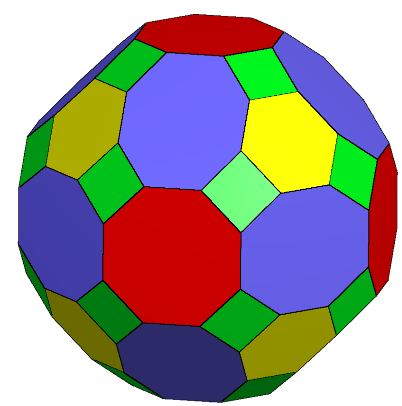
Primitive zonotopes

$H_q(d,p)$: Minkowski $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$x \succeq 0$: first nonzero coordinate of x is nonnegative



- $H_q(d, 1)$: $[0, 1]^d$ cube for $q \neq \infty$
- $H_1(d,2)$: permutahedron of type B_d *(up to a homothety)*
- $H_1(3,2)$: great rhombicuboctahedron
- $H_\infty(3,1)$: truncated small rhombicuboctahedron



Primitive zonotopes

❖ lattice polytopes with *large diameter*

$H_q(d,p)$: Minkowski ($x \in \mathbb{Z}^d$: $||x||_q \leq p$, gcd(x)=1, $x \succeq 0$)

$x \succeq 0$: first nonzero coordinate of x is nonnegative

➢ For $k < 2d$, Minkowski sum of a subset of the generators of $H_1(d,2)$ is, up to translation, a lattice (d,k)-polytope with diameter $\lfloor (k+1)d/2 \rfloor$

Positive primitive zonotopes

$H_q(d,p)$: Minkowski $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$x \succ 0$: first nonzero coordinate of x is nonnegative

$H_q(d,p)^+$: Minkowski $(x \in \mathbb{Z}_+^d : ||x||_q \leq p, \gcd(x)=1)$

- $H_1(d,2)^+$: Minkowski sum permutahedron + unit cube *(graphical zonotope)*
- $H_\infty(d,1)^+$ *:* ***white whale*** *(hypergraphical zonotope)*

$$a(d) = |H_\infty(d,1)^+|$$

number $a(d)$ of generalized retarded functions in quantum field theory is equal to the number of vertices *of* $H_\infty(d,1)^+$

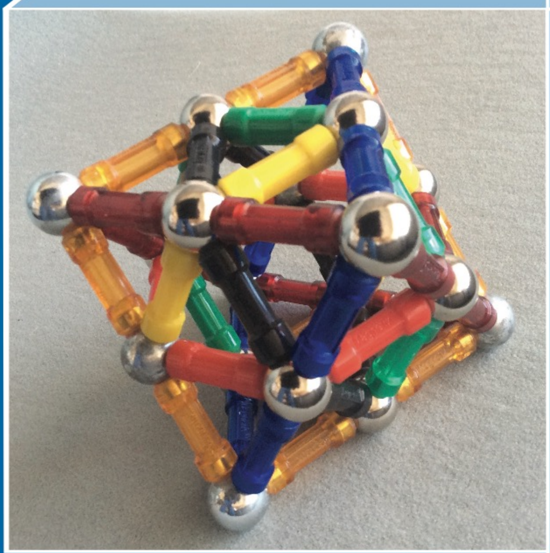
Berlin Mathematical School

BMS Friday Colloquium

Friday 27 April 2018 at 14:15

Tea & Cookies starting at 13:00

BMS Loft, Urania, An der Urania 17, 10787 Berlin



Louis J. Billera

(Cornell University)

In pursuit of a white whale: On the real linear algebra of vectors of zeros and ones

We are interested in the real linear relations (the real *matroid*) on the set of all 0-1 n-vectors. This fundamental combinatorial object is behind questions arising over the past 50 years in a variety of fields, from economics, circuit theory and integer programming to quantum physics, and has connections to an 1893 problem of Hadamard. Yet there has been little real progress on some of the most basic questions.

Some applications seek the number of regions in $\mathbf{R}^n$ that are determined by the $2^n - 1$ linear hyperplanes having 0–1 normals. This number, asymptotically 2^{n^2}, can be obtained exactly from the characteristic polynomial of the geometric lattice of all real subspaces spanned by these 0–1 vectors. These polynomials are known only through $n = 7$, while the number of regions is known through $n = 8$.

Discrete optimization and theoretical physics

- Ising model (spin glasses)
  maxcut, cut and metric polytopes [Deza-Laurent 1997]

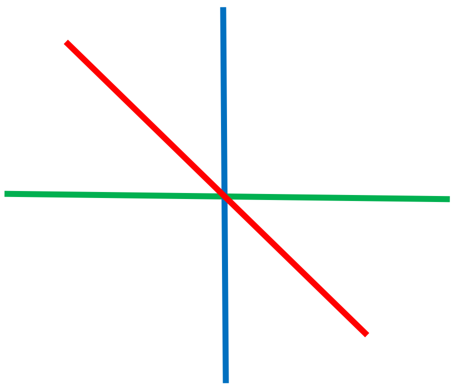
- a(d) : number of generalized retarded functions in quantum field theory
  (number of real-time Green functions) [Evans 1994]

a(d) = number of regions of the arrangement formed by the $2^d -1$ hyperplanes with {0,1}-valued normals in dimension d

$d = 2$ $2^d -1 = 3$ hyperplanes

(0,1)
(1,0)
(1,1)

- a(**2**)=6



6 regions

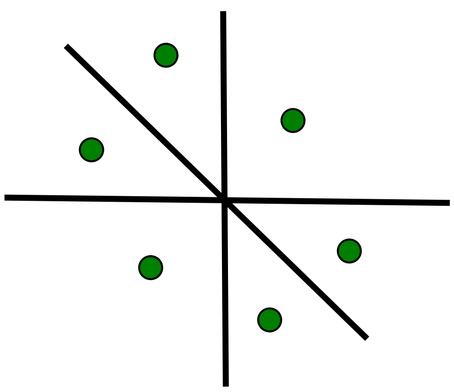
Discrete optimization and theoretical physics

- Ising model (spin glasses)
  maxcut, cut and metric polytopes [Deza-Laurent 1997]

- $a(d)$: number of generalized retarded functions in quantum field theory
  (number of real-time Green functions) [Evans 1994]

$a(d)$ = number of regions of the arrangement formed by the $2^d - 1$ hyperplanes with {0,1}-valued normals in dimension d

- is $a(d) \geq d$! [question by Evans]
- $a(d)$ determined till $d = 9$
- how to estimate $a(d)$?



$a(d)$ regions <=> $a(d)$ vertices

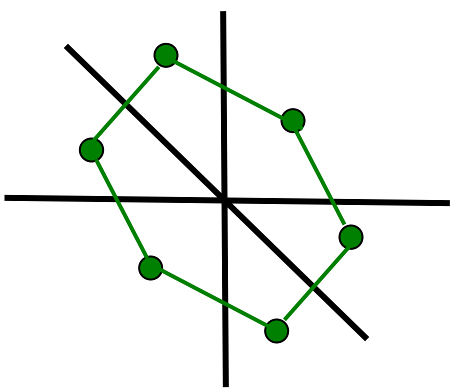
Discrete optimization and theoretical physics

- Ising model (spin glasses)
  maxcut, cut and metric polytopes [Deza-Laurent 1997]

- a(***d***) : number of generalized retarded functions in quantum field theory
  (number of real-time Green functions) [Evans 1994]

a(***d***) = number of regions of the arrangement formed by the 2^{d} -1 hyperplanes with {0,1}-valued normals in dimension ***d***

- is a(***d***) ≥ ***d*** ! [question by Evans]
- a(***d***) determined till ***d*** = 9
- how to estimate a(***d***) ?
- a(***d***) vertices of the *white whale*



a(**2**) = 6

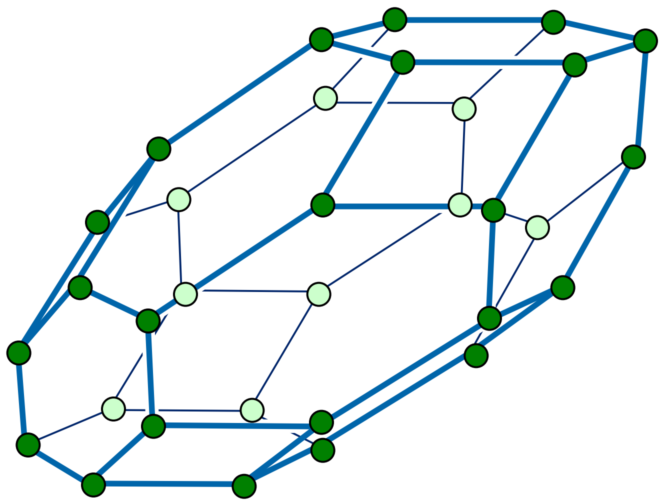
Discrete optimization and theoretical physics

- Ising model (spin glasses)
  maxcut, cut and metric polytopes [Deza-Laurent 1997]

- a(*d*) : number of generalized retarded functions in quantum field theory
  (number of real-time Green functions) [Evans 1994]

a(*d*) = number of regions of the arrangement formed by the 2^d -1 hyperplanes with {0,1}-valued normals in dimension *d*

- is a(*d*) ≥ *d* ! [question by Evans]
- a(*d*) determined till *d* = 9
- how to estimate a(*d*) ?
- a(*d*) vertices of the *white whale*



a(**3**) = 32

Vertices of primitive zonotopes

Sloane OEI sequences

$H_\infty(\boldsymbol{d},1)^{+}$ vertices : A034997 = number of generalized retarded functions in quantum Field theory (determined till $\boldsymbol{d} = 9$)

$H_\infty(\boldsymbol{d},1)$ vertices : A009997 = number of regions of hyperplane arrangements with {-1,0,1}-valued normals in dimension $\boldsymbol{d}$ (determined till $\boldsymbol{d} = 7$)

Estimating the number of vertices of $H_\infty(\boldsymbol{d},1)^{+}$ (*white whale*)

$\boldsymbol{d}(\boldsymbol{d}-1)/2 \leq \log_2 | H_\infty(\boldsymbol{d},1)^{+} | \leq \boldsymbol{d}^{2}$ [Billera et al 2012]

$\boldsymbol{d}(\boldsymbol{d}-1)/2 \leq \log_2 | H_\infty(\boldsymbol{d},1)^{+} | \leq \boldsymbol{d}(\boldsymbol{d}-3)$ [Deza-Pournin-Rakotonarivo 2021]

$\boldsymbol{d}^{2}(1-\varepsilon_{\boldsymbol{d}}) \leq \log_2 | H_\infty(\boldsymbol{d},1)^{+} | \leq \boldsymbol{d}(\boldsymbol{d}-3)$ [Gutekunst, Mészáros, Petersen 2021]

(root resonance arrangement, maximal unbalanced families...)

Sizing the White Whale

d	*a*(***d***)	
2	6	[**Evans 1995**]
3	32	[**Evans 1995**]
4	370	[**Evans 1995**]
5	11 292	[**Evans 1995, van Eijck 1995**]
6	1 066 044	[**Evans 1995, van Eijck 1995**]
7	347 326 352	[**van Eijck 1995**, **Kamiya, Takemura, Terao 2011**]
8	419 172 756 930	[**Evans 2011**]
9	1 955 230 985 997 140	[**Brysiewicz, Eble, Kühne 2021**]
		[**Chroman-Singhal 2021**]
		[**Deza-Hao-Pournin 2021**]

Generating and **counting** the vertices of the White Whale

- [Deza-Hao-Pournin 2021] : Generating all the edges of White While till ***d*** = 9, and exhibiting a family of White While vertices of degree roughly 2^{d}

Convex matroid optimization

The optimal solution of max { $\mathbf{f}(\mathbf{W}x) : x \in \mathbf{S}$} is attained at a vertex of the projection integer polytope in $\mathbb{R}^d$: conv(**WS**) = **W**conv(**S**)

S : set of feasible point in $\mathbb{Z}^n$ (in the talk $\mathbf{S} \in \{0,1\}^n$)
W : integer d x n matrix (**W** is {0,1,…, p}-valued)
f : convex function from $\mathbb{R}^d$ to $\mathbb{R}$

Q. What is the maximum number **v**(d,n) of vertices of conv(**WS**) when $\mathbf{S} \in \{0,1\}^n$ and **W** is a {0,1}-valued d x n matrix ?

obviously $v(d,n) \le |\mathbf{WS}| = O(n^d)$
in particular $v(2,n) = O(n^2)$, and $v(2,n) = \Omega(n^{0.5})$

- [Hunkenschröder, Pokutta, Weismantel 2022] : min { $\mathbf{g}(\mathbf{W}x) : x \in \{0,1\}^n$

  Machine Learning setting with **W** unknow, but $||\mathbf{W}||_\infty$ and the number of rows $m \ll n$ are revealed, some conditions on **g** such as having Lipschitz continuous gradients

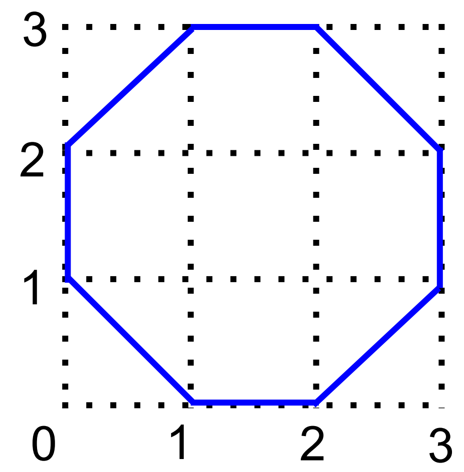
Convex matroid optimization

[Melamed-Onn 2014] Given matroid **S** of order ***n*** and {0,1,…,***p***}-valued ***d*** x ***n*** matrix **W**, the maximum number **m**(***d***,***p***) of vertices of conv(**WS**) is independent of ***n*** and **S**

Ex: maximum number **m**(2,1) of vertices of a planar projection conv(**WS**) of matroid **S** by a binary matrix **W** is attained by the following matrix and uniform matroid of rank 3 and order 8:

$$\mathbf{W} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{S} = U(3,8) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$



conv(**WS**)

Convex matroid optimization

The optimal solution of max { **f**(**W**x) : $x \in$ **S**} is attained at a vertex of the projection integer polytope in $\mathbb{R}^{d}$: conv(**WS**) = **W**conv(**S**)

S : set of feasible point in $\mathbb{Z}^{n}$ (in the talk **S** $\in \{0,1\}^{n}$)
W : integer d x n matrix (**W** is mostly {0,1,…, p}-valued)
f : convex function from $\mathbb{R}^{d}$ to $\mathbb{R}$

Q. What is the maximum number **v**(d,n) of vertices of conv(**WS**) when **S** $\in \{0,1\}^{n}$ and **W** is a {0,1}-valued d x n matrix ?

$$v(d,n) \leq |\mathbf{WS}| = O(n^{d})$$
$$v(2,n) = O(n^{2}), \quad \text{and } v(2,n) = \Omega(n^{0.5})$$

[Melamed-Onn 2014] Given matroid **S** of order n and {0,1,…,p}-valued d x n matrix **W**, the maximum number **m**(d,p) of vertices of conv(**WS**) is independent of n and **S**

Convex matroid optimization

[Melamed-Onn 2014] Given matroid **S** of order n and $\{0,1,\ldots,p\}$-valued d x n matrix **W**, the maximum number $\mathbf{m}(d,p)$ of vertices of conv(**WS**) is independent of n and **S**

[Deza-Manoussakis-Onn 2018] Given matroid **S** of order n, $\{0,1,\ldots,p\}$-valued d x n matrix **W**, maximum number $\mathbf{m}(d,p)$ of vertices of conv(**WS**) is equal to the number of vertices of $H_\infty(d,p)$

$$\mathbf{m}(d,p) = | H_\infty(d,p) |$$

[Melamed-Onn 2014]

$$d\,2^d \leq \mathbf{m}(d,1) \leq 2\sum_{i=0}^{d-1}\binom{(3^d-3)/2}{i}$$

$24 \leq \mathbf{m}(3,1) \leq 158$

$64 \leq \mathbf{m}(4,1) \leq 19840$

$\mathbf{m}(2,1) = 8$

[Deza-Pournin-Rakotonarivo 2021]

$$3^{d(d-1)/2} \leq \mathbf{m}(d,1) \leq 3^{d(d-2)}$$

$\mathbf{m}(3,1) = \mathbf{96}$

$\mathbf{m}(4,1) = \mathbf{5376}$

$$\mathbf{m}(2,p) = 8\sum_{i=1}^{p}\varphi(i)$$

Geometric scaling

(IP) integer optimization max $\{\boldsymbol{c}^T \boldsymbol{x} : \boldsymbol{x} \in \boldsymbol{P} \cap \{0,1\}^{\boldsymbol{d}}\}$

[Schultz-Weismantel-Ziegler 1995] optimization and augmentation are equivalent (***bit scaling***)

[Schulz-Weismantel 2002] ***geometric scaling*** solves (IP) by $O(\boldsymbol{d} \log \boldsymbol{d} \, ||\boldsymbol{c}||_\infty)$ augmentation oracle calls

[Le Bodic-Pavelka-Pfetsch-Pokutta 2018] ***geometric scaling*** solves (IP) by $O(\boldsymbol{d} \log ||\boldsymbol{c}||_\infty)$ augmentation oracle calls

[Deza-Pournin-Pokutta 2022] ***geometric scaling*** may require $\boldsymbol{d} + \log ||\boldsymbol{c}||_\infty + 1$ iterations over a simplex

[Le Bodic-Pavelka-Pfetsch-Pokutta 2018] tight upper and lower bound for ***bit scaling***

Maximum ratio augmentation based geometric scaling

(IP) integer optimization max $\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{x} \in \boldsymbol{P} \cap \{0,1\}^{d}\}$

Input: $\boldsymbol{P}$, $\boldsymbol{c} \in \mathbb{Z}^{d}$, vertex $\boldsymbol{x^o} \in \boldsymbol{P}$, $\mu_o \geq ||\boldsymbol{c}||_\infty$
Output: vertex $\boldsymbol{x}^*$ maximizing $\boldsymbol{c}^T\boldsymbol{x}$

1. $\mu \leftarrow \mu_o$, $\boldsymbol{x}^* \leftarrow \boldsymbol{x^o}$
2. repeat
3. compute vertex $\boldsymbol{x}^+ \in \boldsymbol{P}$ **maximizing** $\boldsymbol{c}^T(\boldsymbol{x}^+ - \boldsymbol{x}^*) / || \boldsymbol{x}^+ - \boldsymbol{x}^* ||_1$
4. if $\boldsymbol{x}^+ = \boldsymbol{x}^*$ or $\boldsymbol{c}^T(\boldsymbol{x}^+ - \boldsymbol{x}^*) < \mu || \boldsymbol{x}^+ - \boldsymbol{x}^* ||_1$ then $\mu \leftarrow \mu/2$ (halving step)
5. else $\boldsymbol{x}^* \leftarrow \boldsymbol{x}^+$ (augmenting step)
6. end
7. until $\mu < 1/d$
8. return $\boldsymbol{x}^*$

$\boldsymbol{P}$ = convex hull $(v^0, v^1, \ldots, v^d)$ where $v^i = (0,\ldots0,1\ldots1)$ with i ones
$\boldsymbol{c} = (1,2,3,\ldots, d)$, $x^0 = v^0$

- requires d augmenting steps and $\log ||\boldsymbol{c}||_\infty + 1$ halving steps

Feasibility test based geometric scaling

(IP) integer optimization max $\{c^T x : x \in P \cap \{0,1\}^d\}$

Input: P, $c \in \mathbb{Z}^d$, vertex $x^o \in P$, $\mu_o \geq ||c||_\infty$
Output: vertex x^* maximizing $c^T x$

1. $\mu \leftarrow \mu_o$, $x^* \leftarrow x^o$
2. repeat
3. compute **a vertex** $x^+ \in P$ such that $c^T(x^+ - x^*) > \mu \, || x^+ - x^* ||_1$
4. if there is no such vertex then $\mu \leftarrow \mu/2$ (halving step)
5. else $x^* \leftarrow x^+$ (augmenting step)
6. end
7. until $\mu < 1/d$
8. return x^*

P = convex hull $(v^0, v^1, \ldots, v^d)$ where $v^i = (0,\ldots 0,1\ldots 1)$ with i ones
$c = (2,4,8,\ldots,2^d)$, $x^0 = v^0$

- requires $d/3$ augmenting steps and $\log ||c||_\infty + 1$ halving steps

Feasibility test based geometric scaling

(IP) integer optimization max $\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in \mathbf{P} \cap \{0,1\}^d\}$

Input: $\mathbf{P}$, $\mathbf{c} \in \mathbb{Z}^d$, vertex $\mathbf{x}^o \in \mathbf{P}$, $\mu_o \geq ||\mathbf{c}||_\infty$
Output: vertex $\mathbf{x}^*$ maximizing $\mathbf{c}^T\mathbf{x}$

1. $\mu \leftarrow \mu_o$, $\mathbf{x}^* \leftarrow \mathbf{x}^o$
2. repeat
3. compute **a vertex** $\mathbf{x}^+ \in \mathbf{P}$ such that $\mathbf{c}^T(\mathbf{x}^+ - \mathbf{x}^*) > \mu\, ||\mathbf{x}^+ - \mathbf{x}^*||_1$
4. if there is no such vertex then $\mu \leftarrow 3\mu/4$ (halving step)
5. else $\mathbf{x}^* \leftarrow \mathbf{x}^+$ (augmenting step)
6. end
7. until $\mu < 1/d$
8. return $\mathbf{x}^*$

$\mathbf{P}$ = convex hull $(v^0, v^1, \ldots, v^d)$ where $v^i = (0,\ldots 0,1\ldots 1)$ with i ones
$\mathbf{c} = (2,4,8,\ldots,2^d)$, $x^0 = v^0$

- ➢ requires d augmenting steps and log $||\mathbf{c}||_\infty + 1$ halving steps

Primitive zonotopes

(degree sequences)

$\boldsymbol{D_d}$: convex hull of the degree sequences of all hypergraphs on $\boldsymbol{d}$ nodes

$$\boldsymbol{D_d} = H_\infty(\boldsymbol{d},1)^+$$

$\boldsymbol{D_d}(\boldsymbol{k})$: convex hull of the degree sequences of all $\boldsymbol{k}$-uniform hypergraphs on $\boldsymbol{d}$ nodes

Q: check whether $x \in \boldsymbol{D_d}(\boldsymbol{k}) \cap \mathbb{Z}^{\boldsymbol{d}}$ is the degree sequence of a $\boldsymbol{k}$-uniform hypergraph. Necessary condition: sum of the coordinates of x is multiple of $\boldsymbol{k}$.

[Erdős-Gallai 1960]: for $\boldsymbol{k} = 2$ (graphs) necessary condition is sufficient

[Liu 2013] exhibited counterexamples (holes) for $\boldsymbol{k} = 3$ (Klivans-Reiner **Q**.)

- Answer to Colbourn-Kocay-Stinson **Q.** (1986)
  Deciding whether a given integer sequence is the degree sequence of a **3**-uniform hypergraph is NP-complete [Deza-Levin-Meesum-Onn 2018]

(reduction to 3-partition problem)

Primitive zonotopes, convex matroid optimization, and degree sequences of hypergraphs

$\delta(d,k)$: *largest diameter* over all lattice (d,k)-polytopes

- Conjecture : $\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$ and $\delta(d,k)$ is achieved, up to translation, by a *Minkowski sum* of primitive lattice vectors (holds for all known $\delta(d,k)$)

$$\Rightarrow \delta(d,k) = \lfloor (k+1)d/2 \rfloor \text{ for } k < 2d$$

- $\mathbf{m}(d,p) = | H_\infty(d,p) |$ (*convex matroid optimization complexity*)

- tightening of the *bounds* for $\mathbf{m}(d,1) = | H_\infty(d,1) |$

- tightening of the *bounds* for $a(d) = | H_\infty(d,1)^+ |$ (*white whale*)

- Answer to [Colbourn-Kocay-Stinson 1986] question:
  Deciding whether a given integer sequence is the *degree sequence* of a 3-hypergraph is *NP-complete* [Deza-Levin-Meesum-Onn 2018]

- ✓ Deza, Pournin: *Primitive point packing.*
  Mathematika (2022)

- ✓ Deza, Pokutta, Pournin: *The complexity of geometric scaling*.
  arXiv (2022)

- ✓ Deza, Hao, Pournin: *Sizing the White Whale.*
  arXiv (2022)

- ✓ Deza, Pournin, Rakotonarivo: *Vertices of primitive zonotopes.*
  Contemporary Mathematics (2021)

- ✓ Deza, Pournin, Sukegawa: *The diameter of lattice zonotopes*.
  Proceedings of the American Mathematical Society (2020)

- ✓ Deza, Deza, Guan, Pournin: *Distances between vertices of lattice polytopes*.
  Optimization Letters (2019)

- ✓ Deza, Levin, Meesum, Onn: *Optimization over degree sequences*.
  SIAM Journal on Discrete Mathematics (2018)

- ✓ Deza, Manoussakis, Onn: *Primitive zonotopes.*
  Discrete and Computational Geometry (2018)