Practical Models of Optimization for Controlled Sweeping Processes

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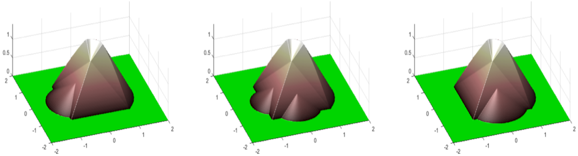
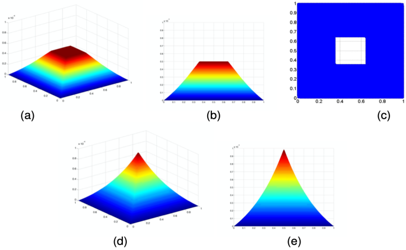
Sandpile model





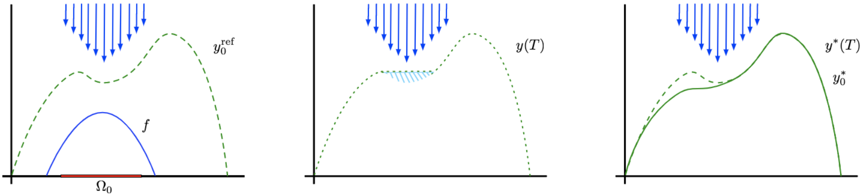


Figure: (**LEFT**) Depiction of the initial supporting structure y_0^{ref}, the density rate of poured material f and the location of the subdomain Ω_0, where the material should not accumulate. (**CENTER**) Final resulting shape at time $t = T$ for the material with a very flat angle of repose. (**RIGHT**) Optimal supporting structure y_0^*, which coincides with the final growth shape y^* at time $t = T$ given that no material is accumulating anywhere.

Optimal Control of a Quasi-Variational Sweeping Process

Let $\sigma > 0$, $\mathbf{f} : (0, T) \to \mathbb{R}^N$ non-negative and $\mathbf{a}, \mathbf{y}_0^{\mathrm{ref}} \in \mathbb{R}^N$ be given.

$$\text{minimize} \quad J(\mathbf{y}, \mathbf{y}_0) := \int_0^T \mathbf{a}^\intercal(\mathbf{y}(t) - \mathbf{y}_0)\mathrm{d}t + \frac{\sigma}{2}(\mathbf{y}_0 - \mathbf{y}_0^{\mathrm{ref}})^\intercal(\mathbf{y}_0 - \mathbf{y}_0^{\mathrm{ref}}),$$

- $\mathbf{y}$ solves $\mathrm{QVI}(\mathbf{y}_0)$

$$-\mathbf{y}'(t) \in F(\mathbf{y}(t), \mathbf{y}_0) := N_{\mathcal{K}^p(\mathbf{y}(t), \mathbf{y}_0)}(\mathbf{y}(t)) - \mathbf{f}(t). \qquad (\mathrm{QVI}(\mathbf{y}_0))$$

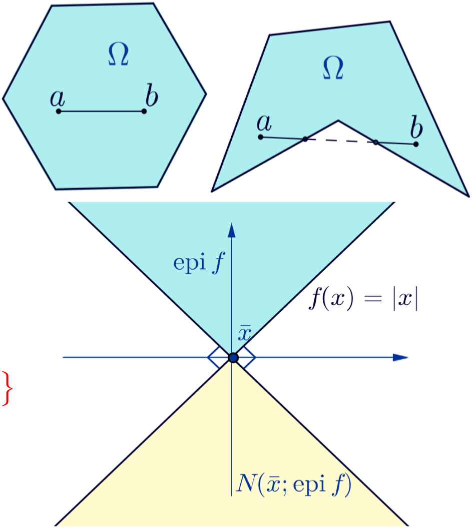
- $\mathbf{y} \in V := \{\mathbf{v} \in L^2(0, T; H_0^1(\Omega)) : \mathbf{v}' \in L^2(0, T; H^{-1}(\Omega))\}$

Note

- Convex set and nonconvex set.

- Normals to convex set.

$$N(x;C) := \left\{ v \in \mathbb{R}^n \middle| \langle v, y-x \rangle \le 0,\ y \in C \right\}$$
if $x \in C$,
$$N(x;C) := \emptyset \text{ otherwise.}$$



Optimal Control of a Quasi-Variational Sweeping Process

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- $y \in \mathcal{K}^p(y, y_0) := \{z \in H_0^1(\Omega) : |\nabla z|_p \leq M_p(y, y_0)\}$, a.e. in $(0, T)$
- The operator $M_p(w, y_0) : \Omega \to \mathbb{R}$ is given by

$$M_p(w, y_0) := \begin{cases} \alpha, & \text{if } w > y_0; \\ \max(\alpha, |\nabla y_0|_p), & \text{if } w = y_0; \end{cases}$$

where $\alpha = \tan(\theta)$.

Existence of optimal solutions

Theorem

The optimal control problem $(\mathbb{P}_N)$ *admits an optimal solution.*

(Mosco convergence). Let $\mathcal{K}$ and $\mathcal{K}_n$ as $n \in \mathbb{N}$ be nonempty, closed, and convex subsets of a reflexive Banach space V. Then the sequence $\{\mathcal{K}_n\}$ is said to converge to $\mathcal{K}$ in the sense of Mosco as $n \to \infty$, which is signified by

$$\mathcal{K}_n \xrightarrow{M} \mathcal{K},$$

if the following two conditions are satisfied:

1. For each $w \in \mathcal{K}$, there exists $\{w_{n'}\}$ such that $w_{n'} \in \mathcal{K}_{n'}$ for $n' \in \mathbb{N}' \subset \mathbb{N}$ and $w_{n'} \to w$ in V.
2. If $w_n \in \mathcal{K}_n$ and $w_n \rightharpoonup w$ in V along a subsequence, then $w \in \mathcal{K}$.

Discrete approximations of feasible solutions

Given any $m \in \mathbb{N} := \{1, 2, \ldots\}$, consider the discrete mesh

$$T_M := \{0, \tau_M, \ldots, T - \tau_M, T\}, \quad \tau_M := \frac{T}{M},$$

on $[0, T]$ and

$$\mathbf{y}_j^M \in \mathcal{K}^p(\mathbf{y}_0, \mathbf{y}_j^M) \quad \Big| \quad \left(\frac{\mathbf{y}_j^M - \mathbf{y}_{j-1}^M}{\tau_M} - \mathbf{f}_j^M, \mathbf{v} - \mathbf{y}_j^M\right)_{\mathbb{R}^N} \geq 0 \quad (\mathrm{QVI}_N^M(\mathbf{y}_0))$$

for all $\mathbf{v} \in \mathcal{K}^p(\mathbf{y}_0, \mathbf{y}_j^M)$ with the discrete time $j = 1, \ldots, M$ and the rate discretization

$$\mathbf{f}_j^M = \int_{(j-1)\tau_M}^{j\tau_M} \mathbf{f}(t)\mathrm{d}t \qquad j = 1, \ldots, M. \tag{1}$$

Discrete approximations of optimal solutions

The *discretized quasi-variational sweeping process*

$$\mathbf{y}_j^M \in \mathbf{y}_{j-1}^M + \tau_M F_j^M(\mathbf{y}_j^M, \mathbf{y}_0), \quad j = 1, \ldots, M,$$

where the feasible discrete velocity mappings F_j^M are defined by

$$F_j^M(\mathbf{y}, \mathbf{y}_0) := -N_{\mathcal{K}^p(\mathbf{y}, \mathbf{y}_0)}(\mathbf{y}) + \mathbf{f}_j^M, \quad j = 1, \ldots, M,$$

The *discrete version* of the optimal control problem $(\mathbb{P}_N)$

Problem $(\mathbb{P}_N^M)$. Given $\sigma > 0$, a nonnegative mapping $\mathbf{f} : (0, T) \to \mathbb{R}^N$, and vectors $\mathbf{a}$, $\mathbf{y}_0^{\mathrm{ref}} \in \mathbb{R}^N$, consider the discrete-time optimal control problem:

$$
\begin{aligned}
&\text{minimize} && J^M(\mathbf{y}, \mathbf{y}_0) := \sum_{j=1}^{M} \tau_M \langle \mathbf{a}, \mathbf{y}_j^M - \mathbf{y}_0 \rangle + \frac{\sigma}{2} \langle \mathbf{y}_0 - \mathbf{y}_0^{\mathrm{ref}}, \mathbf{y}_0 - \mathbf{y}_0^{\mathrm{ref}} \rangle \\
&\text{over} && \mathbf{y}_0^M, \mathbf{y}_1^M, \ldots, \mathbf{y}_M^M \in \mathbb{R}^N; \\
&\text{subject to} && \mathbf{y} = \{\mathbf{y}_j^M\}_{j=1}^M \ \text{ solves } \ \mathrm{QVI}_N^M(\mathbf{y}_0), \\
& && \mathbf{y}_0 \in \mathcal{A}.
\end{aligned}
$$

The *discrete version* of the optimal control problem $(\mathbb{P}_N)$

The *dynamics constraints* can be written in the quasi-variational sweeping form

$$\dot{\mathbf{y}}(t_j^M) \in F_j^M(\mathbf{y}(t_j^M), \mathbf{y}_0) \ \text{ for all } \ t_j^M \in (0, T),$$

the *control constraint* $\mathbf{y}_0 \in \mathcal{A}$ is expressed in terms of the set

$$\mathcal{A} := \left\{ \mathbf{z} \in \mathbb{R}^N \;\middle|\; \mathbf{y}_0^{\mathrm{ref}} + \lambda_0 \le \mathbf{z} \le \mathbf{y}_0^{\mathrm{ref}} + \lambda_1 \right\},$$

where $\lambda_0, \lambda_1 \in \mathbb{R}^N$ with $0 \le \lambda_0 \le \lambda_1$, and the *hidden state constraints* are given by

$$-\left(M_\infty(\mathbf{y}(t_j^M), \mathbf{y}_0)\right)_i \le \left(\mathbf{D}_k \mathbf{y}(t_j^M)\right)_i \le \left(M_\infty(\mathbf{y}(t_j^M), \mathbf{y}_0)\right)_i$$

with $i = 1, \ldots, N$, $k = 1, 2$, and $j = 1, \ldots, M$, where the mapping M_p is defined above.

Tools of Variational Analysis and Generalized Differentiation

- The (Painlevé-Kuratowski) *outer limit* of $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $\bar{x}$

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \left\{ y \in \mathbb{R}^m \;\middle|\; \exists\, x_k \to \bar{x},\ y_k \to y:\ y_k \in F(x_k) \right\}$$

- The (Mordukhovich) *basic/limiting normal cone*

$$N_\Omega(\bar{x}) := \operatorname*{Lim\,sup}_{x\to\bar{x}} \left[\operatorname{cone}\big(x - \Pi_\Omega(x)\big) \right]$$

- The *coderivative* of F at $(\bar{x}, \bar{y})$

$$D^*F(\bar{x},\bar{y})(u) := \left\{ v \in \mathbb{R}^n \;\middle|\; (v,-u) \in N\big((\bar{x},\bar{y}); \operatorname{gph} F\big) \right\}, \quad u \in \mathbb{R}^m$$

Tools of Variational Analysis and Generalized Differentiation

- Let $\phi\colon \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty]$

  $$\operatorname{dom}\phi := \{x \in \mathbb{R}^n \mid \varphi(x) < \infty\},\ \operatorname{epi}\phi := \{(x,\alpha) \in \mathbb{R}^{n+1} \mid \alpha \geq \phi(x)\}$$

- The (*first-order*) *subdifferential*

  $$\partial\phi(\bar{x}) := \{v \in \mathbb{R}^m \mid (v,-1) \in N((\bar{x},\phi(\bar{x})); \operatorname{epi}\phi)\}$$

- The *second-order subdifferential*

  $$\partial^2\phi(\bar{x},\bar{v})(u) := (D^*\partial\phi)(\bar{x},\bar{v})(u), \quad u \in \mathbb{R}^n$$

Second-order computation for the discretized QVI sweeping process

$$D^*F(\mathbf{y},\mathbf{y}_0,w)(y) =$$

$$\bigcup_{\lambda\geq 0,-\nabla g(\mathbf{y},\mathbf{y}_0)\lambda=w+\mathbf{f}} \left\{ \left(-\sum_{k=1}^{2}\sum_{l=1}^{2N} \lambda_k^l \langle \nabla_{\mathbf{y}}^2 g_k^l(\mathbf{y},\mathbf{y}_0), y\rangle - \nabla_{\mathbf{y}} g(\mathbf{y},\mathbf{y}_0)^*\gamma, 0 \right) \right\}$$

$$\mathrm{dom}\, D^*N_{\widetilde{\mathcal{K}}^\infty(\mathbf{y},\mathbf{y}_0)}(\mathbf{y}, w+\mathbf{f}) = \Big\{ y \;\Big|\; \exists\, \lambda \geq 0 \ \text{ such that}$$
$$-\nabla g(\mathbf{y},\mathbf{y}_0)\lambda = w+\mathbf{f}, \lambda_k^l \langle \nabla g_k^l(\mathbf{y},\mathbf{y}_0), y\rangle = 0 \ \text{ for } \ l=1,\ldots,2N,\ k=1,2 \Big\}$$

Explicit necessary conditions for discretized QVI sweeping control problems

Theorem

Let $\bar{z}^M = (\bar{\mathbf{y}}^M, \bar{\mathbf{y}}_0)$ be an optimal control to the smoothed problem $(\mathbb{P}_N^M)$. Then there exist dual elements $(\lambda^M, \alpha^{kM}, p^M)$ and $\psi \in N_{\mathcal{A}}(\bar{\mathbf{y}}_0)$ together with vectors $\eta_j^{kM} = \left(\eta_{1j}^{kM}, \ldots, \eta_{2Nj}^{kM}\right) \in \mathbb{R}_+^{2N}$ as $j = 1, \ldots, M$, $k = 1, 2$ and $\gamma_j^{kM} = \left(\gamma_{1j}^{kM}, \ldots, \gamma_{2Nj}^{kM}\right) \in \mathbb{R}^{2N}$ as $j = 1, \ldots, M-1$ and $k = 1, 2$ such that the following relationships hold:

- NONTRIVIALITY CONDITION

$$\lambda^M + \|\eta_M^{kM}\| + \sum_{j=1}^{M} \|p_j^M\| \neq 0.$$

- DYNAMIC RELATIONSHIPS for all $j = 1, \ldots, M-1$:

$$\frac{\bar{\mathbf{y}}_{j+1}^M - \bar{\mathbf{y}}_j^M}{\tau_M} + \mathbf{f}_j^M = -\sum_{k=1}^{2} \sum_{l \in I(\bar{\mathbf{y}}_j^M)} \eta_{lj}^{kM} \nabla_{\bar{\mathbf{y}}_j^M} g_k^l(\bar{\mathbf{y}}_j^M, \bar{\mathbf{y}}_0),$$

$$\frac{p_{j+1}^M - p_j^M}{\tau_M} - \frac{\lambda^M T\mathbf{a}^\mathsf{T}}{\tau_M} = -\sum_{k=1}^{2} \sum_{l=1}^{2N} \eta_{lj}^{kM} \left\langle \nabla^2_{\bar{\mathbf{y}}_j^M} g_k^l(\bar{\mathbf{y}}_j^M, \bar{\mathbf{y}}_0), -p_{j+1}^M \right\rangle - \sum_{k=1}^{2} \sum_{l=1}^{2N} \gamma_{lj}^{kM} \nabla_{\bar{\mathbf{y}}_j^M} g_k^l(\bar{\mathbf{y}}_j^M, \bar{\mathbf{y}}_0),$$

$$-\frac{1}{\tau_M}\lambda^M \left(T\mathbf{a}^\mathsf{T} + \sigma\bar{\mathbf{y}}_0\right) + \frac{1}{\tau_M} \sum_{k=1}^{2} \sum_{l=1}^{2N} \eta_{lM}^{kM} \nabla_{\bar{\mathbf{y}}_0} g_k^l(\bar{\mathbf{y}}_M^M, \bar{\mathbf{y}}_0) - \frac{1}{\tau_M}\psi = 0.$$

Explicit necessary conditions for discretized QVI sweeping control problems

- TRANSVERSALITY CONDITION

$$p_M^M = -\lambda^M T\mathbf{a} + \sum_{k=1}^{2}\sum_{l=1}^{2N} \eta_{lM}^{kM} \nabla_{\bar{\mathbf{y}}_M^M} g_k^l(\bar{\mathbf{y}}_M^M, \bar{\mathbf{y}}_0).$$

- COMPLEMENTARITY SLACKNESS CONDITIONS

$$g_k^l(\bar{\mathbf{y}}_j^M, \bar{\mathbf{y}}_0) > 0 \Longrightarrow \eta_{lj}^{kM} = 0,$$

$$\left[g_k^l(\bar{\mathbf{y}}_j^M, \bar{\mathbf{y}}_0) > 0 \text{ or } \eta_{lj}^{kM} = 0,\ \langle \nabla g_k^l(\bar{\mathbf{y}}_j^M, \bar{\mathbf{y}}_0), -p_{j+1}^M \rangle > 0\right] \Longrightarrow \gamma_{lj}^{kM} = 0,$$

$$\left[g_k^l(\bar{\mathbf{y}}_j^M, \bar{\mathbf{y}}_0) = 0,\ \eta_{lj}^{kM} = 0,\ \text{ and } \langle \nabla g_k^l(\bar{\mathbf{y}}_j^M, \bar{\mathbf{y}}_0), -p_{j+1}^M \rangle < 0\right] \Longrightarrow \gamma_{lj}^{kM} \geq 0$$

for $j = 1, \ldots, M-1$, $l = 1, \ldots, 2N$, and $k = 1, 2$.

The necessary optimality conditions for the nonsmooth perturbed sweeping process

Furthermore, we have the implications

$$g_k^l(\bar{\mathbf{y}}_j^M, \bar{\mathbf{y}}_0) > 0 \Longrightarrow \gamma_{lj}^{kM} = 0 \text{ for } j = 1, \ldots, M-1,$$

$$g_k^l(\bar{\mathbf{y}}_M^M, \bar{\mathbf{y}}_0) > 0 \Longrightarrow \eta_{lM}^{kM} = 0 \text{ for } l = 1, \ldots, 2N \text{ and } k = 1, 2,$$

$$\eta_{lj}^{kM} > 0 \Longrightarrow \langle \nabla g_k^l(\bar{\mathbf{y}}_j^M, \bar{\mathbf{y}}_0), -p_{j+1}^M \rangle = 0.$$

Mobile robot model

- This model was introduced by Ramdane Hedjar and Messaoud Bounkhel.
- Purpose and model:
  - The configuration vector $x = (x^1, x^2, \dots, x^n) \in \mathbb{R}^{2n}$.
  - $\bar{x}^i(t) = \left(\|\bar{x}^i(t)\| \cos\theta_i(t), \|\bar{x}^i(t)\| \sin\theta_i(t)\right)$.
  - The admissible configuration set $Q_0 = \left\{x = \left(x^1, \dots, x^n\right) \in \mathbb{R}^{2n},\ D_{ij}(x) \geq 0,\ \forall i, j \in \{1, \dots, n\}\right\}$, where $D_{ij}(x) = \|x^i - x^j\| - 2R$.

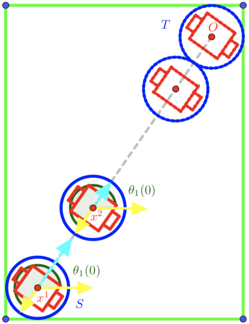

Figure: Mobile robot model.

Mobile robot model

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- We define next:
  - The set of admissible velocities $V_h(x) = \{v \in \mathbb{R}^{2n} : D_{ij}(x) + h \bigtriangledown D_{ij}(x) \cdot v \geq 0, \forall i < j,\ i,j \in \{1,\dots,n\}\}$.
  - $S(x)$ be the mobile robot's desired velocity, with $S(x) = (S_0(x^1),\dots,S_0(x^n))$, where $S_0(x) = -s\nabla D(x)$.

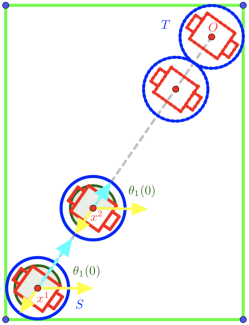

Figure: Mobile robot model.

Mobile robot model

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- Define:
  - We will involve the control function $u(t) = (u^1(t), \dots, u^d(t)) \in U, \quad \text{a.e. } t \in [0, T]$.

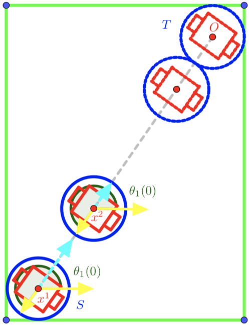


Figure: Mobile robot model.

Mobile robot model

- The *controlled motion dynamics* is presented as follows:

$$\begin{cases} -\dot{x}(t) \in N\big(x(t); C\big) - g\big(x(t), u(t)\big) & \text{for a.e.} \quad t \in [0, T], \\ x(0) = x_0 \in C, \; u(t) \in U \; \text{ a.e. on } [0, T] \end{cases}$$

- The state constraints form

$$x(t) \in C \text{ for all } t \in [0, T],$$

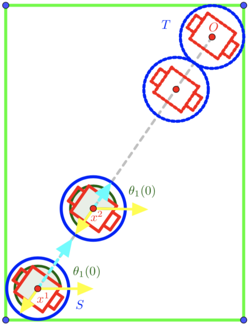
where $C := \big\{x \in \mathbb{R}^{2n} \mid \langle x_*^i, x\rangle \le c_i, \; i = 1, \ldots, s = n-1\big\}$, with
$x_*^i := e_{i1} + e_{i2} - e_{(i+1)1} - e_{(i+1)2}, \quad c_i = -2R, \quad i = 1, \ldots, s,$

- The *cost functional*

$$\text{minimize} \quad J[x, u] := \frac{1}{2}\|x(T)\|^2,$$

Solving the mobile robot problem with two robots[1]

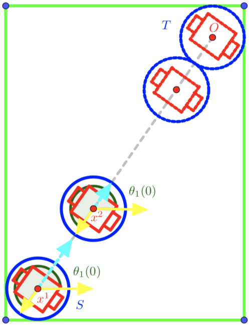
- $x^{01} = (-30, -30)\,,\ x^{02} = (-20, -20)$.
- $T = 6,\ R = 6,\ s_1 = 3,\ s_2 = 1,\ g(x, u) = |u|$.
- $U := \{u = (u^1, u^2) \in \mathbb{R}^2 \big| u^1 = 2u^2,\ -3 \leq u^1 \leq 3\}$.



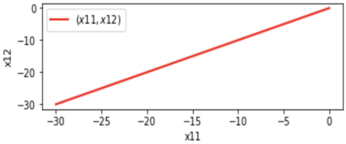
[1] Giovanni Colombo, B. S. Mordukhovich and Dao Nguyen, Optimal Control of Sweeping Processes in Robotics and Traffic Flow Models, J. Optim. Theory Appl.; DOI: 10.1007/s10957-019-01521-y.

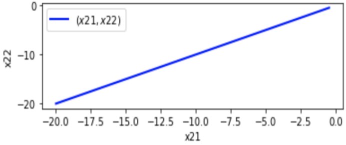
Solving the mobile robot problem with two robots

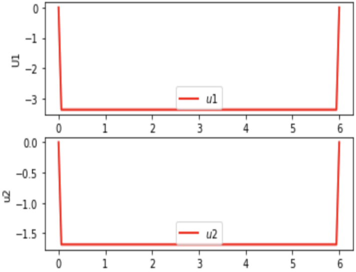
- $(\bar{u}^1, \bar{u}^2) = (3, 1.5)$
- $\bar{x}^1(t) \approx (-30 - 6.36t, -30 - 6.36t)$,
  $t \in [0, 0.38)$
- $\bar{x}^1(t) \approx (-31.01 - 3.71t, -31.01 - 3.71t)$,
  $t \in [0.38, 6]$
- $\bar{x}^2(t) \approx (-20 - 1.06t, -20 - 1.06t)$,
  $t \in [0, 0.38)$
- $\bar{x}^2(t) \approx (-18.99 - 3.71t, -18.99 - 3.71t)$,
  $t \in [0.38, 6]$



Solving the mobile robot problem with two robots





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The final value of the objective function is    552394.481987064

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Solver         :  IPOPT (v3.12)
Solution time  :    11.7053000000014      sec
Objective      :    552394.546801417
Successful solution
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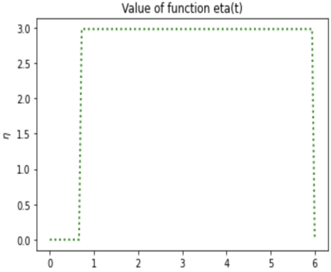
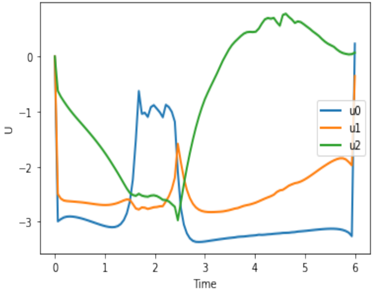
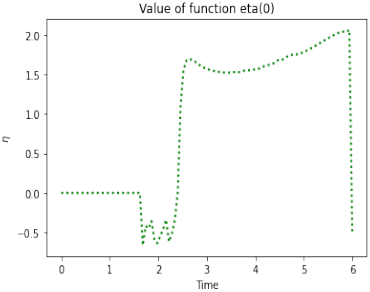
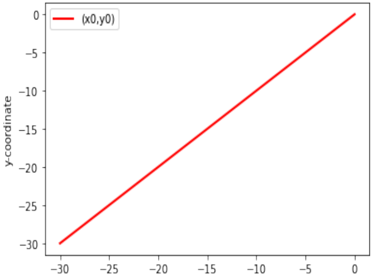
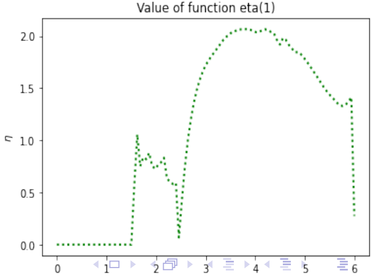


Solving the mobile robot problem with three robots

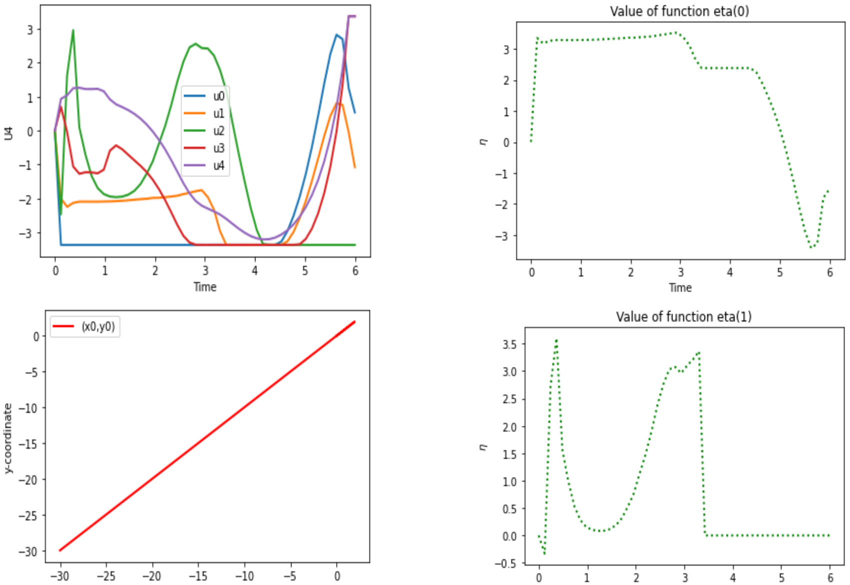








Solving the mobile robot problem with five robots



Current projects

- Consider the general set C and second order necessary optimality conditions.
- Numerical method for random control sweeping processes.
- Stochastic sweeping processes.
- Optimal control for PDEs.
- Lie algebra and Lie brackets.

THANK YOU FOR YOUR KIND ATTENTION!

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