Polar Convexity and a Refinement of the Gauss-Lucas theorem

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The Main Definitions

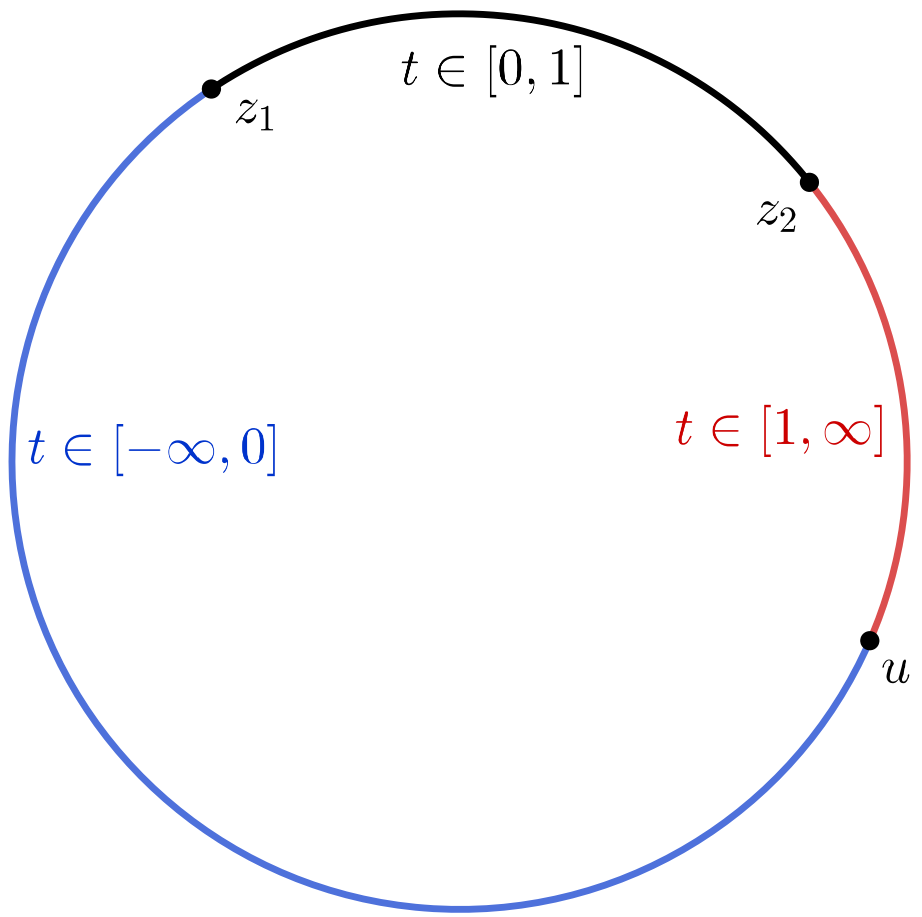
Circles in the plane

Let $\bar{\mathbb{C}}$ be the extended complex plane

Let z_1, z_2 and u be distinct points in $\bar{\mathbb{C}}$

The unique circle through these three points can be parametrized as

$$z = u + \frac{1}{\dfrac{1-t}{z_1 - u} + \dfrac{t}{z_2 - u}}, \text{ where } t \in \bar{\mathbb{R}}$$



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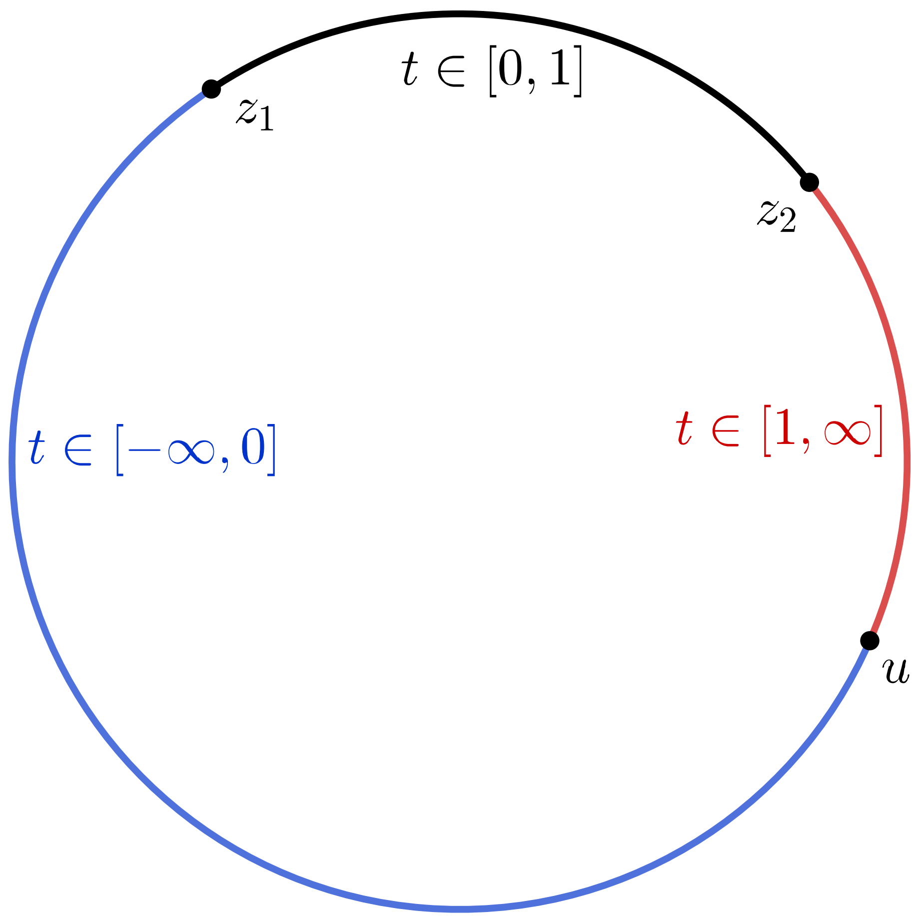

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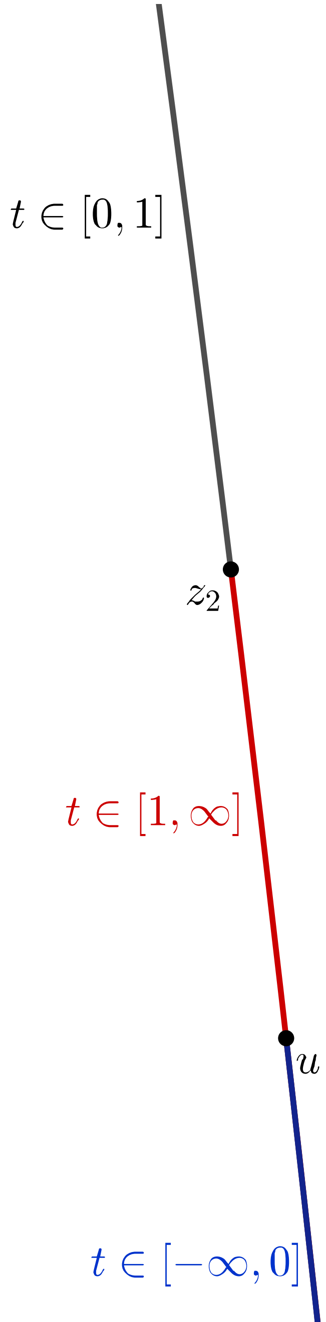

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If $z_1 = \infty$, we have $z = u + \dfrac{z_2 - u}{t}$, where $t \in \bar{\mathbb{R}}$

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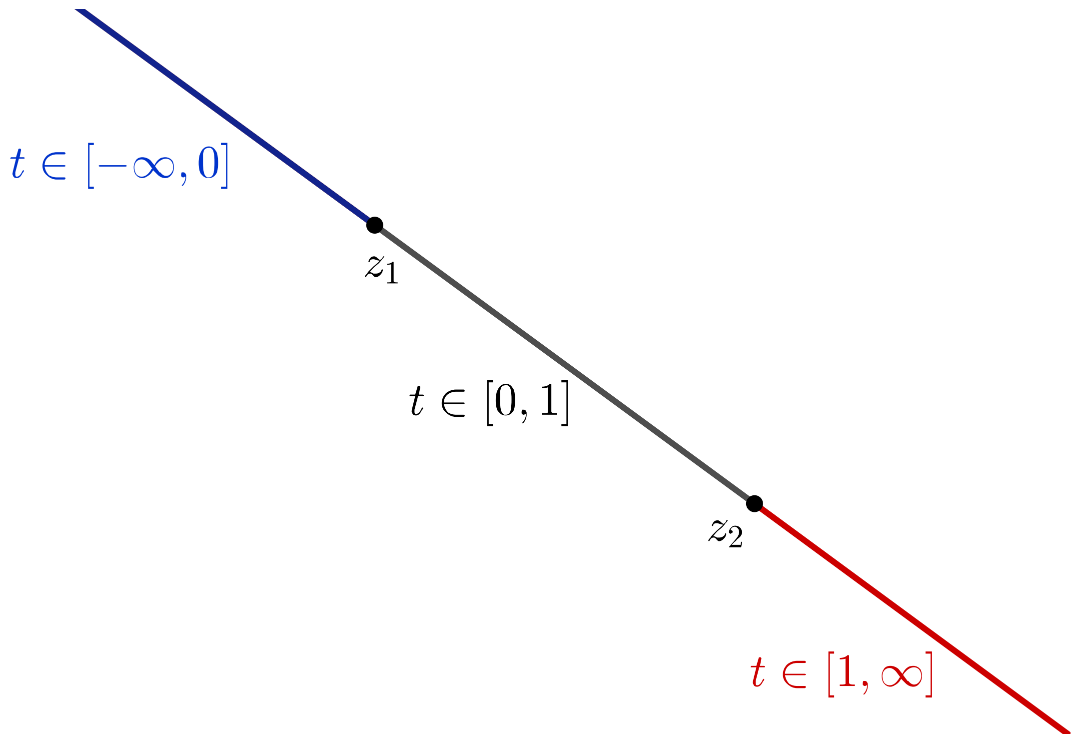

If $u = \infty$, we have $z = (1-t)z_1 + tz_2$, where $t \in \bar{\mathbb{R}}$

Circles in the plane

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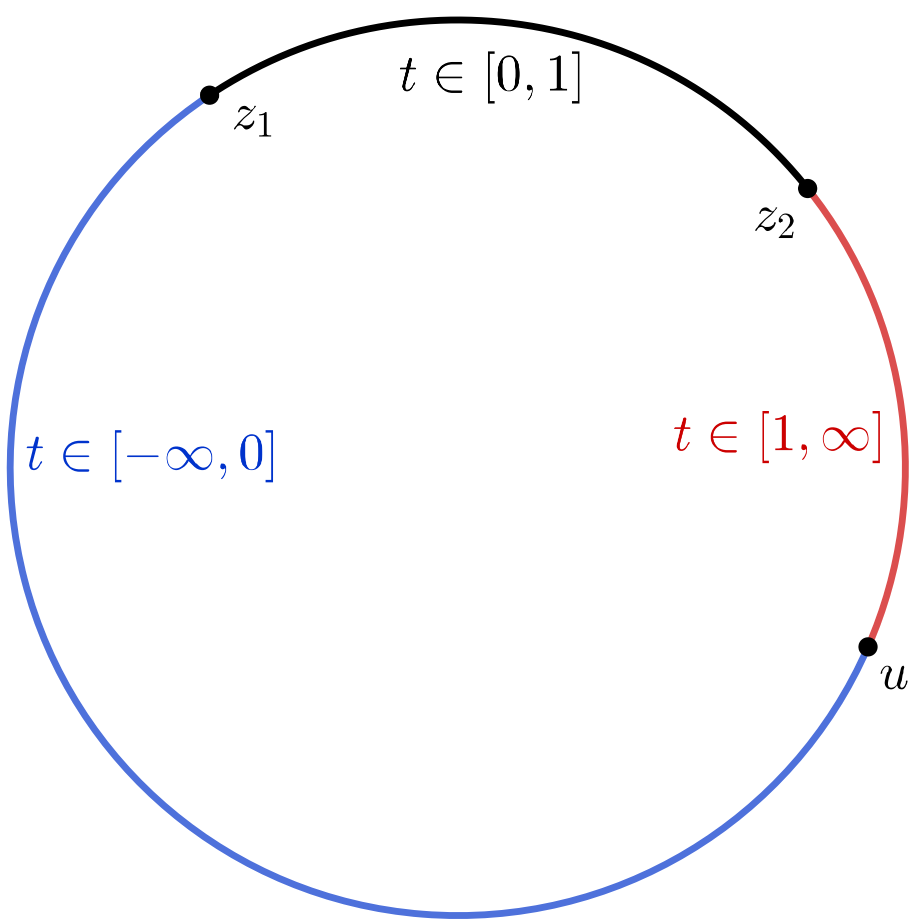
Denote by $\text{arc}_u[z_1, z_2] := \left\{ u + \dfrac{1}{\frac{1-t}{z_1 - u} + \frac{t}{z_2 - u}} : t \in [0,1] \right\}$

If any of the points z_1, z_2 and u coincide with common value v, then define

$\text{arc}_u[z_1, z_2] := \{v\}$

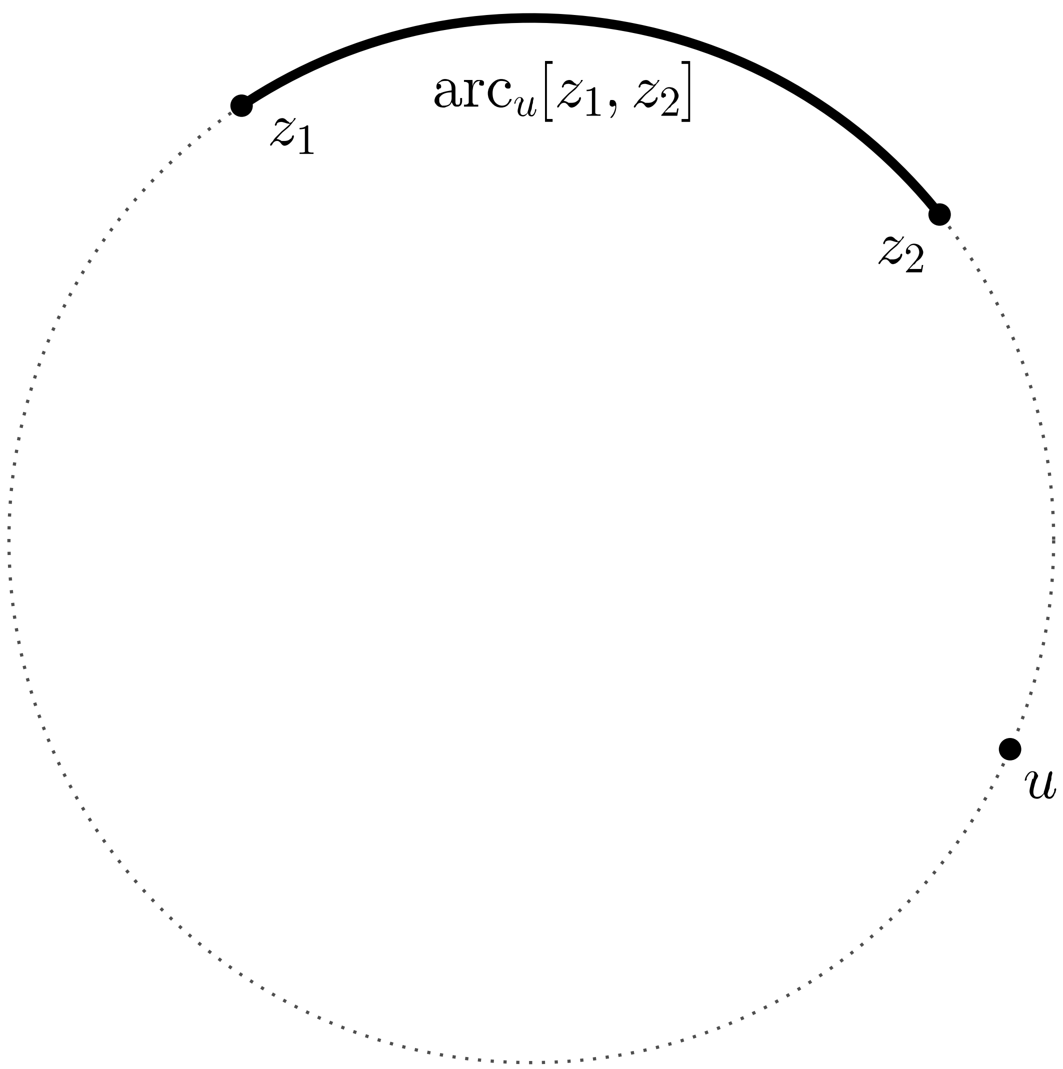


Polar Convexity

Definition: A set $A \subset \bar{\mathbb{C}}$ is *convex w.r.t. the pole* $u \in \bar{\mathbb{C}}$

if for any $z_1, z_2 \in A$, we have $\mathrm{arc}_u[z_1, z_2] \subset A$

The set A is called u-*convex*, for short



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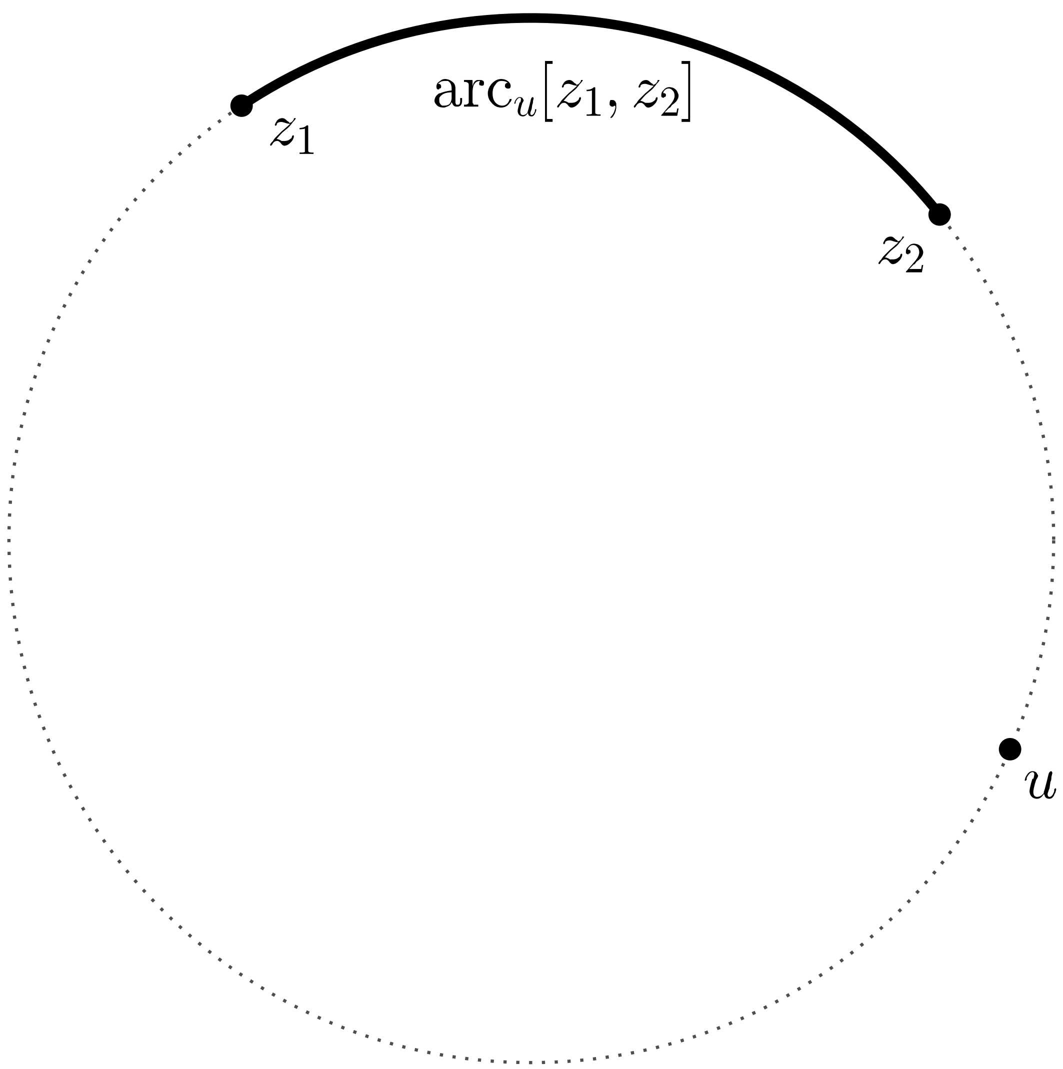

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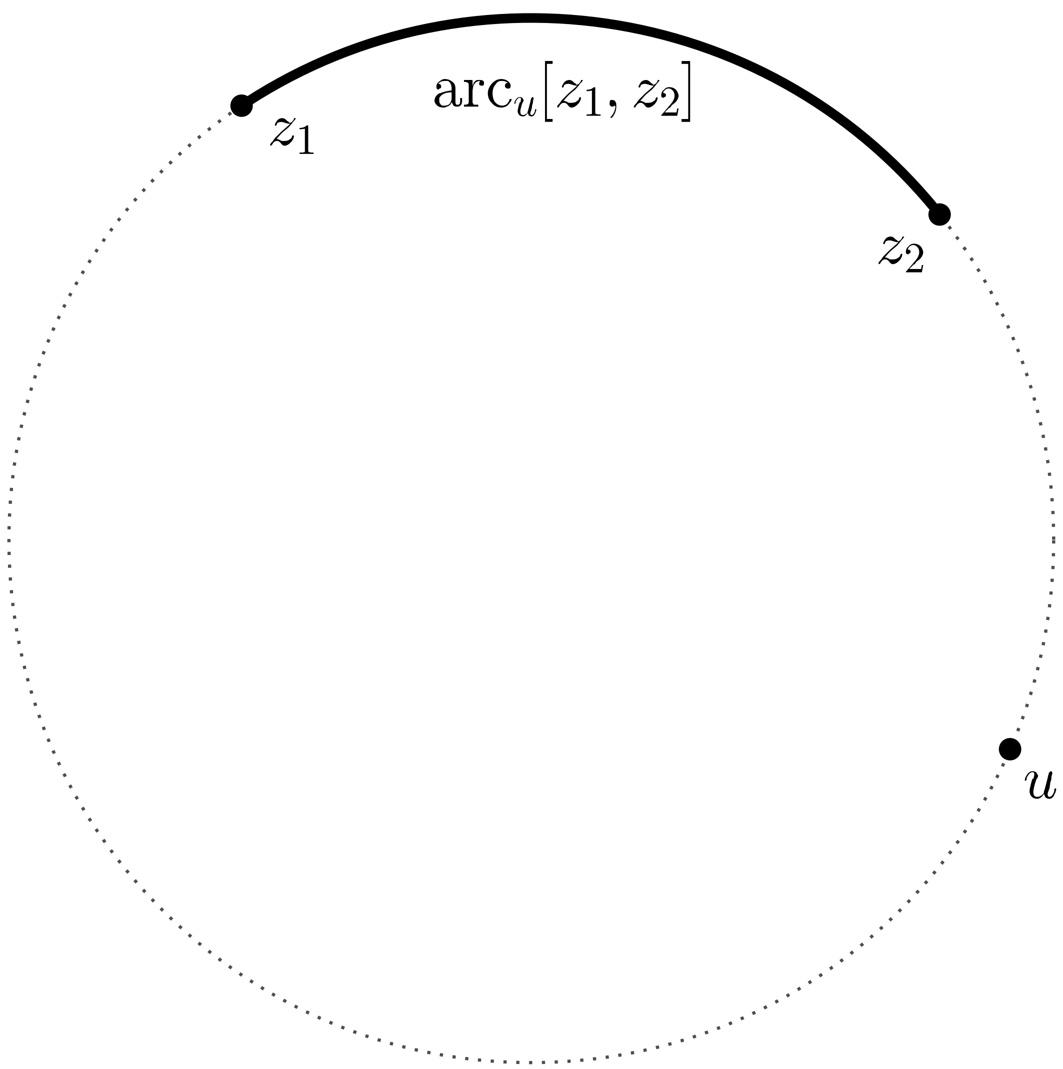

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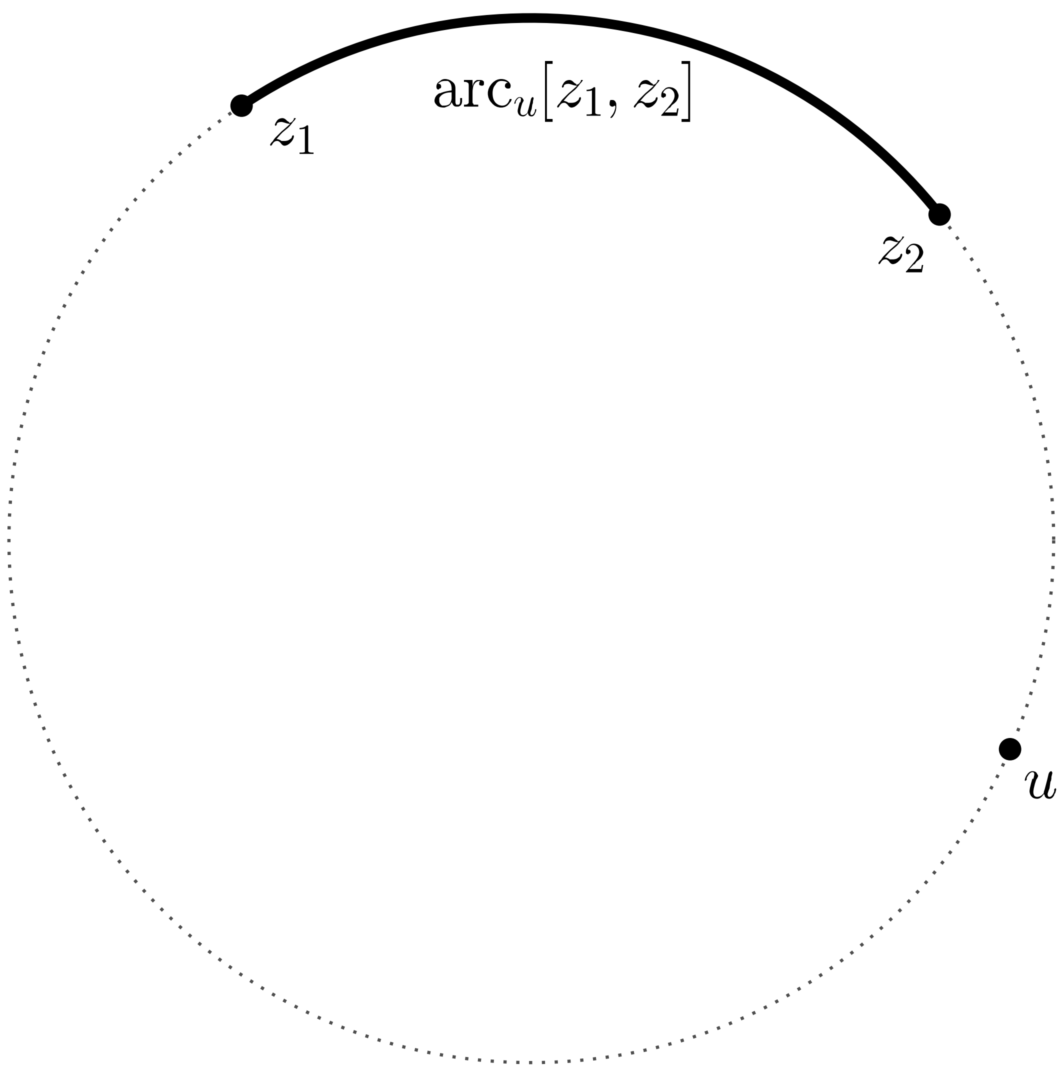

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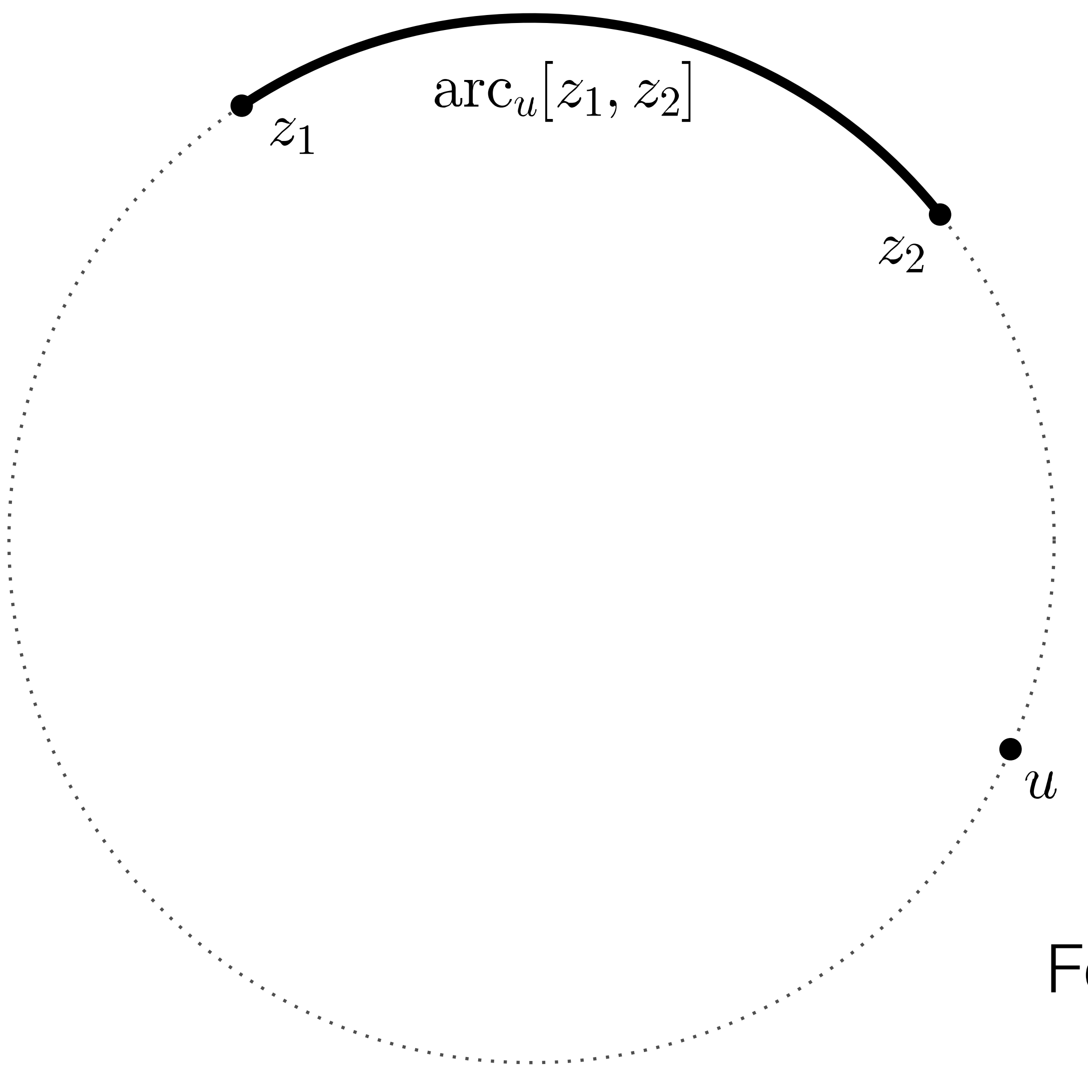

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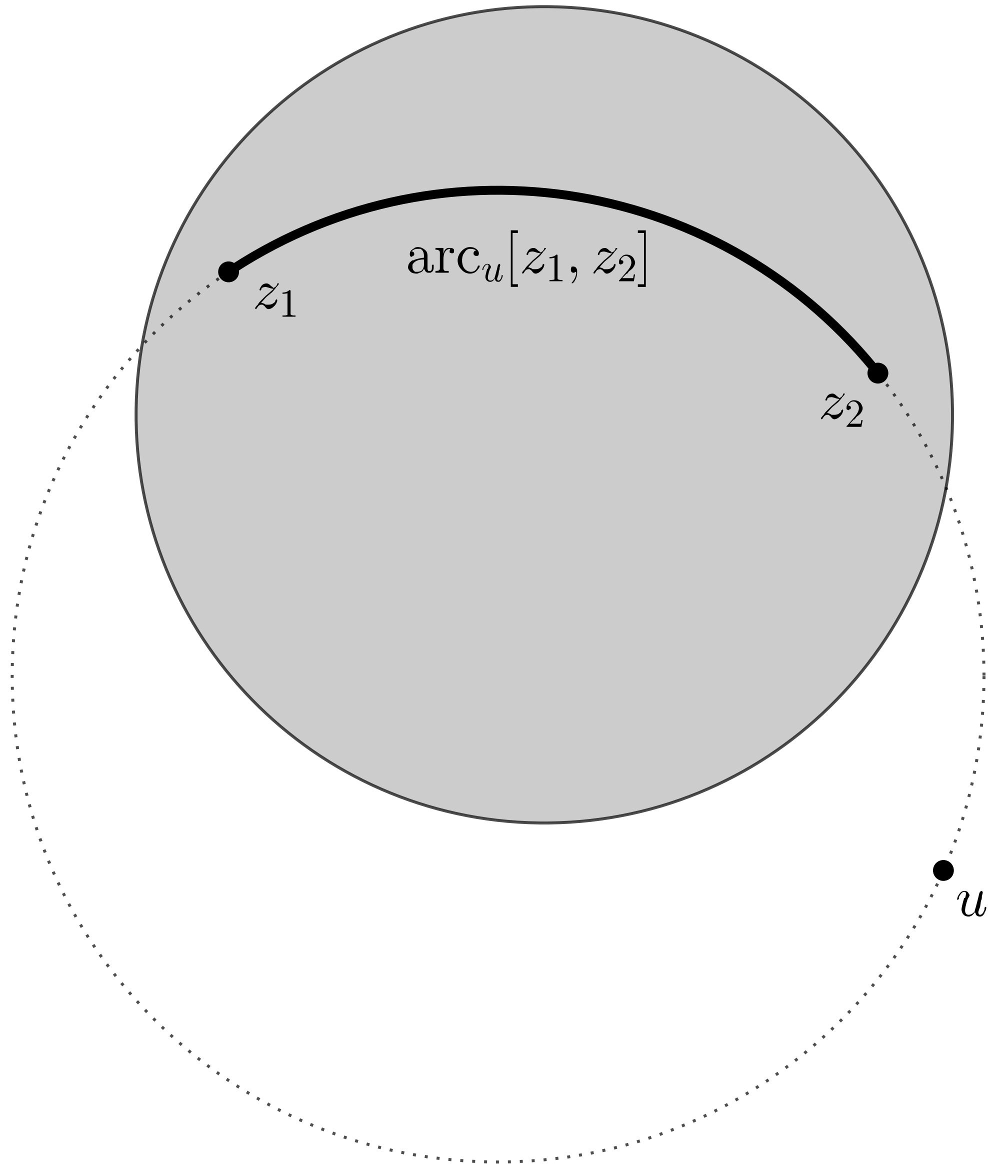
For any $A \subset \bar{\mathbb{C}}$ and any $u \notin$ int A

conv$_u(A) :=$ the smallest u-convex set containing A

Example

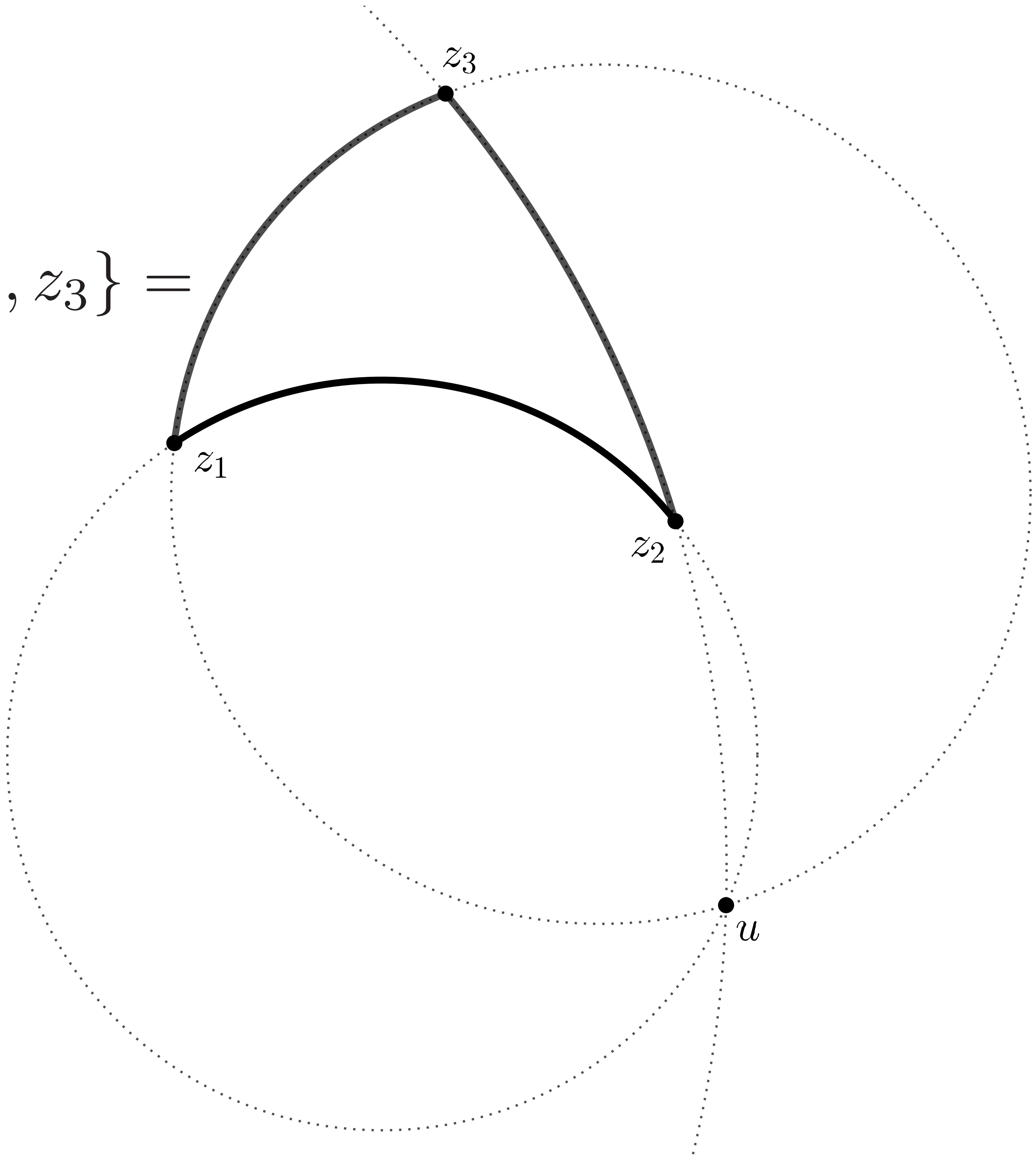
If D is a disk and u is not in its interior, then D is u-convex



Examples

If $u, z_1, z_2, z_3 \in \mathbb{C}$ are distinct, then

$$\mathrm{conv}_u\{z_1, z_2, z_3\} =$$



Provided that $u \notin \mathrm{conv}\{z_1, z_2, z_3\}$

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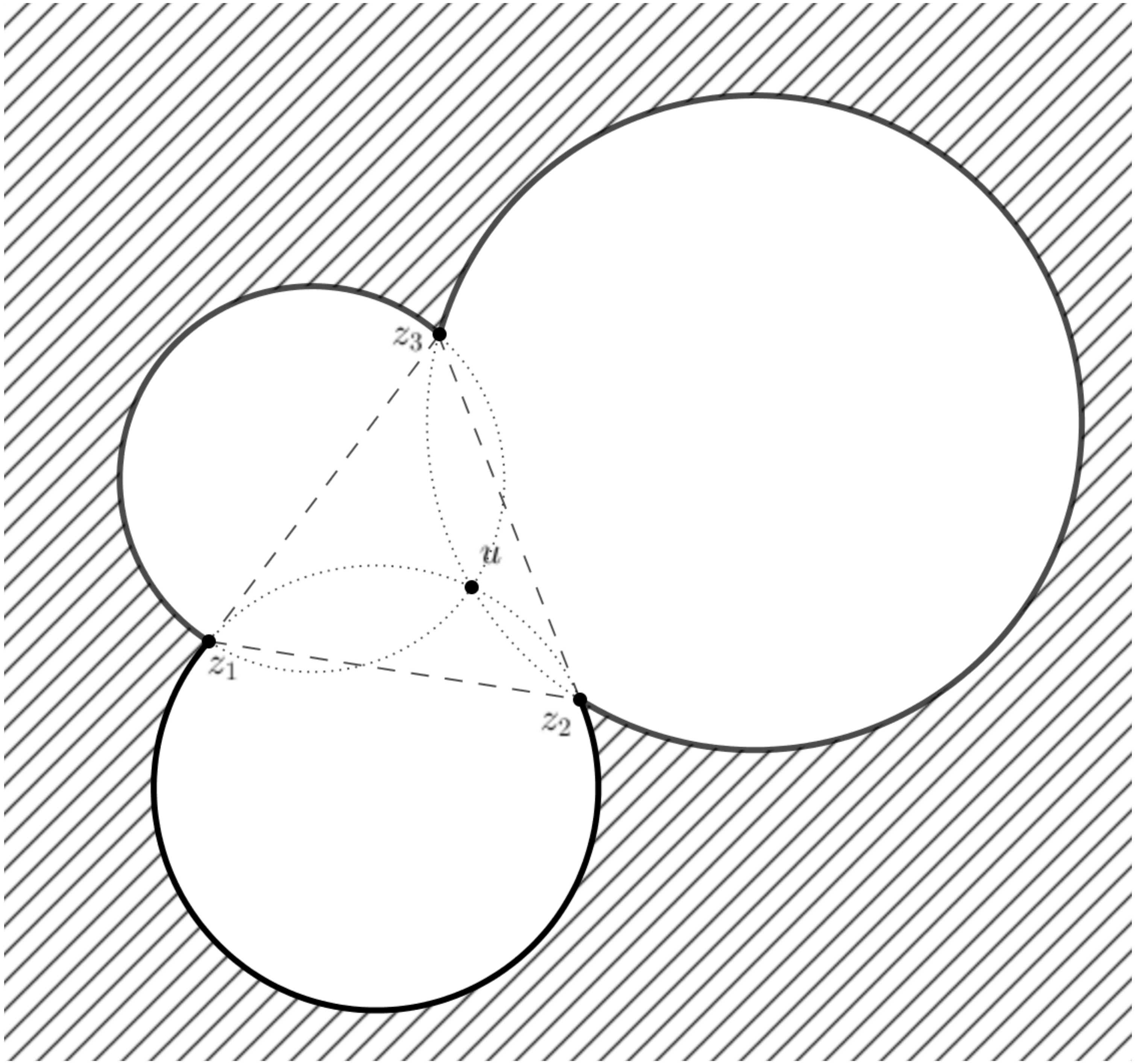

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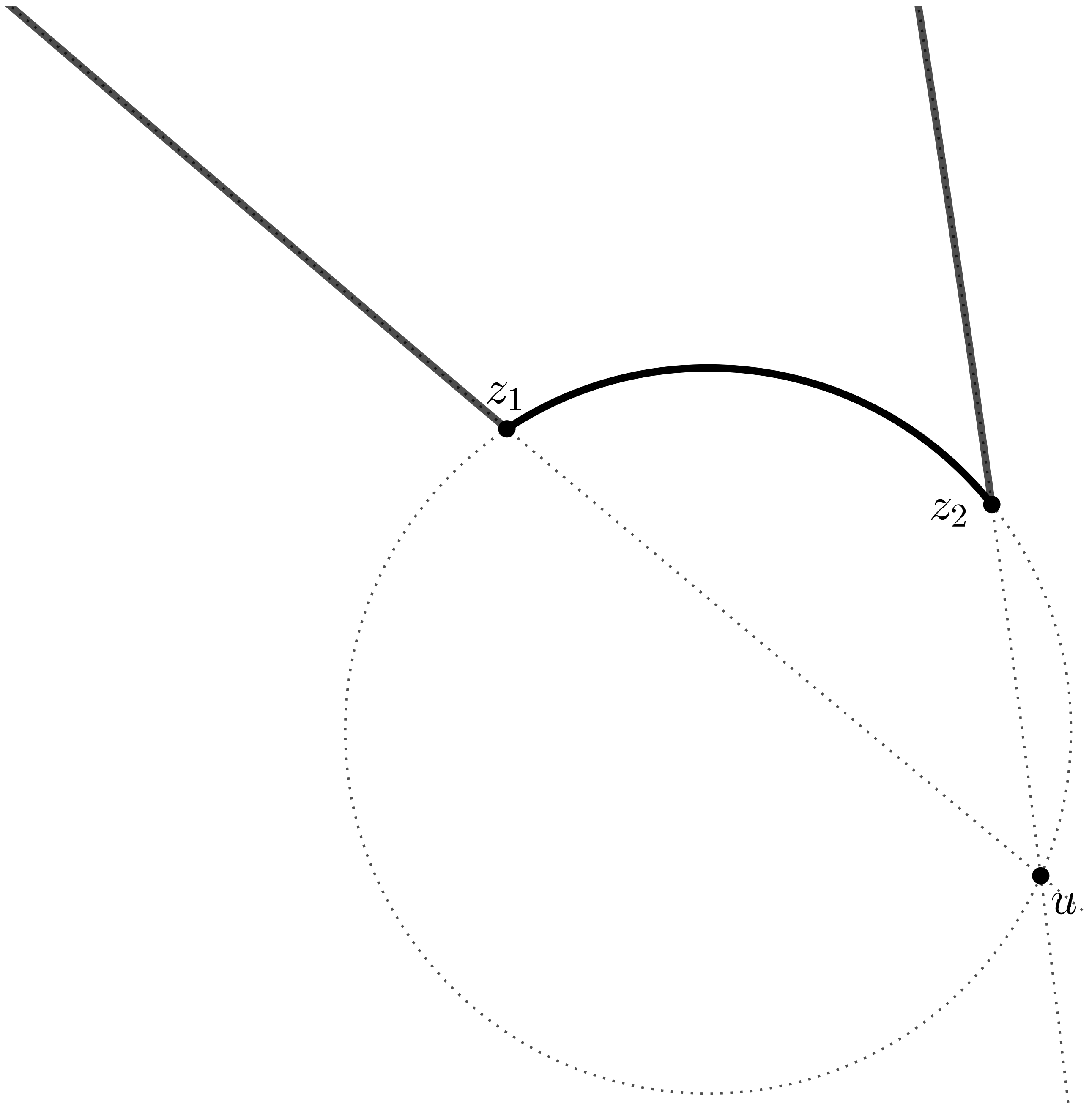

Mobius Transformations

Mobius transformation $T(z) = \dfrac{az+b}{cz+d}$ sends circles onto circles

An easy consequence of that fact is $T(\text{conv}_u\{z_1, \ldots, z_n\}) = \text{conv}_{T(u)}\{T(z_1), \ldots, T(z_n)\}$

In particular, if $T(z) = \dfrac{1}{z-u}$ then $T(\text{conv}_u\{z_1, \ldots, z_n\}) = \text{conv}\{T(z_1), \ldots, T(z_n)\}$

If $u \notin \{z_1, \ldots, z_n\}$, then the set $\text{conv}_u\{z_1, \ldots, z_n\}$ is the intersection of all closed circular domains that contain $z_1, \ldots, z_n$ and have u on the their boundary, with u removed

$$\text{conv}_u\{z_1, \ldots, z_n\} = \left\{ u + \frac{1}{\sum_{i=1}^n \frac{t_i}{z_i - u}} : t_i \geq 0 \text{ with } \sum_{i=1}^n t_i = 1 \right\}$$

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Polya and Szego: Problems and Theorems in Analysis II (1976)

Finishing the definition

If $u \in \{z_1, \ldots, z_n\}$, then $\text{conv}_u\{z_1, \ldots, z_n\} = \{u\} \cup \text{conv}_u\{z_i : z_i \neq u \text{ for } i = 1, \ldots, n\}$

In this way we have $u \in \text{conv}_u\{z_1, \ldots, z_n\}$ if and only if $u \in \{z_1, \ldots, z_n\}$

The set $\text{conv}_u\{z_1, \ldots, z_n\}$ does not behave in a continuous way

when u converges to a point in $\{z_1, \ldots, z_n\}$

How polar convexity appears? - First example

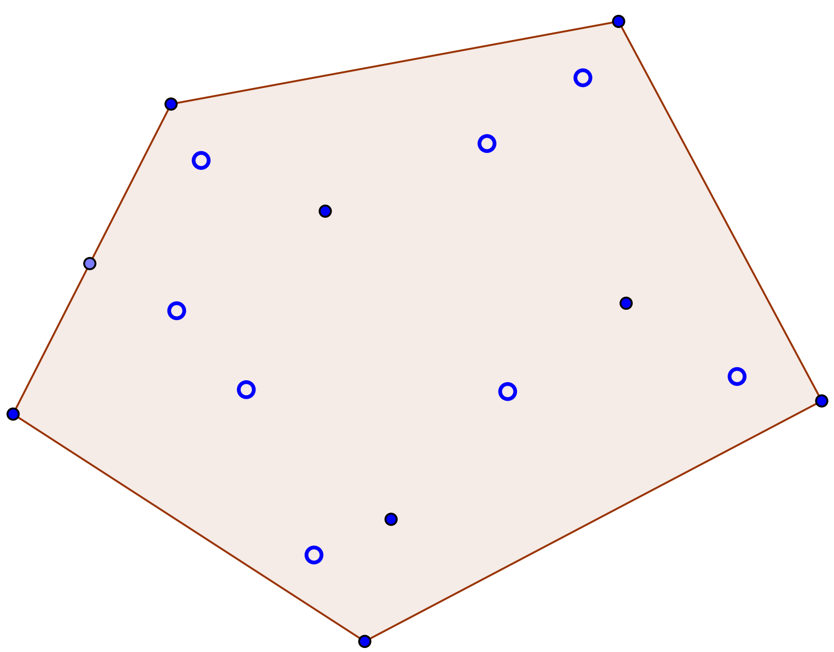
The Gauss-Lucas theorem

Let $p(z)$ be a complex polynomial of degree n

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

Recall the classical Gauss-Lucas theorem:

The convex hull of the zeros of $p(z)$, contains all zeros of $p'(z)$



That is: if a half-plane contains the zeros of $p(z)$, then it contains the zeros of $p'(z)$

Polar Derivatives and Laguerre's Theorem

Let $p(z)$ be a polynomial of degree n

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

The *polar derivative of* $p(z)$ *with pole* $u \in \mathbb{C}$ is defined by

$$D_u p(z) = np(z) - (z - u)p'(z)$$

It can be shown deg $D_u p(z) \leq n - 1$ and $\lim_{u \to \infty} \frac{D_u p(z)}{u} = p'(z)$

So, define $D_\infty p(z) = p'(z)$

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Gauss-Lukas is a corollary of Laguerre: fix a half plane containing the zeros of $p(z)$ and let $u \to \infty$

An Extention of Laguerre's Theorem

Theorem: Let $p(z)$ be a polynomial of degree at most n and zeros $z_1, \ldots, z_n \in \bar{\mathbb{C}}$

Let $u \in \bar{\mathbb{C}}$. Then, the zeros of $D_u p(z)$ are in $\text{conv}_u\{z_1, \ldots, z_n\}$

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Laguerre is a corollary of this theorem: fix a circular domain D containing the zeros $z_1, \dots, z_n$ of $p(z)$. Let $u \notin D$. Then, the zeros of $D_u p(z)$ are in conv$_u\{z_1, \dots, z_n\} \subset D$

How polar convexity appears? - Second example

Example 10

Take any polynomial of degree 3 with complex coefficients

$$p(z) = z^3 + a_2 z^2 + a_1 z + a_0$$

Symmetrize it with 3 complex variables

$$P(z_1, z_2, z_3) = z_1 z_2 z_3 + \frac{a_2}{3}(z_1 z_2 + z_1 z_3 + z_2 z_3) + \frac{a_1}{3}(z_1 + z_2 + z_3) + a_0$$

Note that $P(z, z, z) = p(z)$

Fix z_3, then solve $P(z_1, z_2, z_3) = 0$ for z_2, we get the Mobius transformation in z_1:

$$T_{z_3}(z_1) := -\frac{(a_2 z_3 + a_1)z_1 + (a_1 z_3 + 3a_0)}{(a_2 + 3z_3)z_1 + (a_2 z_3 + a_1)}$$

Recall that Mobius transformation map circles into circles

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Let C be the circle with centre 0 and radius 1

Consider the family of circles $\{T_{z_3}(C) : z_3 \in C\}$

Here are two striking facts

1. The circles $\{T_{z_3}(C) : z_3 \in C\}$ pass through a common point, call it u

2. Each connected component of

$$\big(\cup \{T_{z_3}(C) : z_3 \in C\}\big)^c \text{ is } u\text{-convex}$$

How polar convexity appears? - Third example

A refinement of Gauss-Lucas

The Gauss-Lucas theorem

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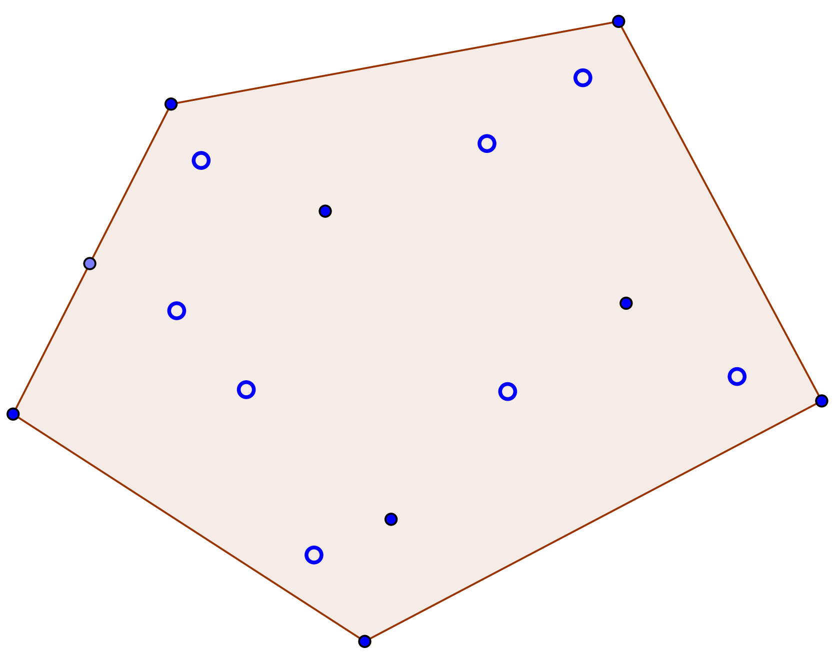

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Krawtchouk's lemma

Consider the polynomial $p(z) = (z - z_1)^{k_1} \cdots (z - z_m)^{k_m}$ of degree n

The distinct zeros $z_1, \ldots, z_m$ have respective multiplicities $k_1, \ldots, k_m$

For all $1 \leq j, k \leq m$, define the points $\gamma_{j,k} := \begin{cases} ((n - k_j) z_j + k_j z_k)/n & \text{if } j \neq k \\ \infty & \text{if } j = k \end{cases}$

Krawtchouk (1929): Let M be an open disk such that $z_j \in$ cl M, but $\gamma_{j,1}, \ldots, \gamma_{j,m} \notin M$
Then M does not contain non-trivial critical points of $p(z)$

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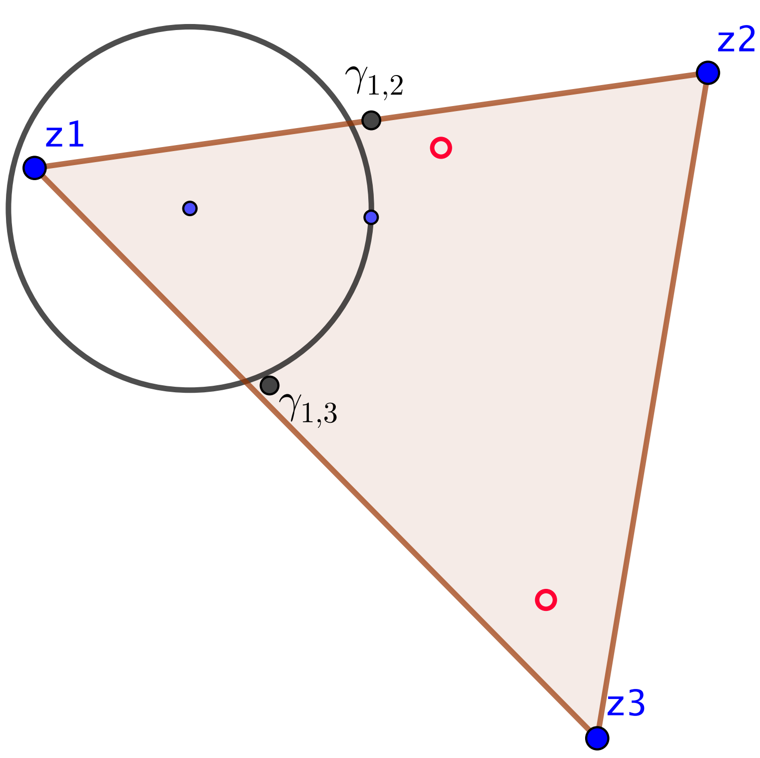
Example: $p(z) = (z - z1)^3(z - z2)^2(z - z3)$ has two non-trivial critical points

$$\gamma_{1,1} = \infty, \qquad \gamma_{1,2} = \frac{(6-3)z1 + 3z2}{6}, \qquad \gamma_{1,3} = \frac{(6-3)z1 + 3z3}{6}$$



There are many disks, containing z_j and none of $\gamma_{j,1}, \ldots, \gamma_{j,m}$

A natural problem is to find the union of all such disks

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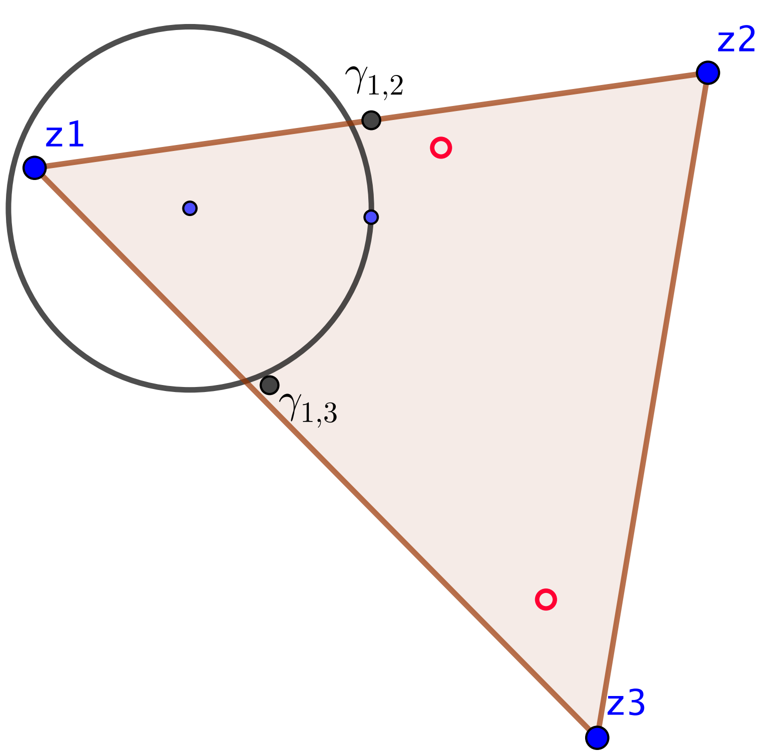

Answer: The union of all such disks is $\left(\text{conv}_{z_j}\{\gamma_{j,1}, \ldots, \gamma_{j,m}\}\right)^c$

The main result

Consider the degree n polynomial $p(z) = (z - z_1)^{k_1} \cdots (z - z_m)^{k_m}$

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Theorem (2021): All non-trivial critical points of $p(z)$ are in

$$\text{conv}\{z_1, \ldots, z_m\} \bigcap \bigcap_{j=1}^{m} \text{conv}_{z_j}\{\gamma_{j,1}, \ldots, \gamma_{j,m}\}$$

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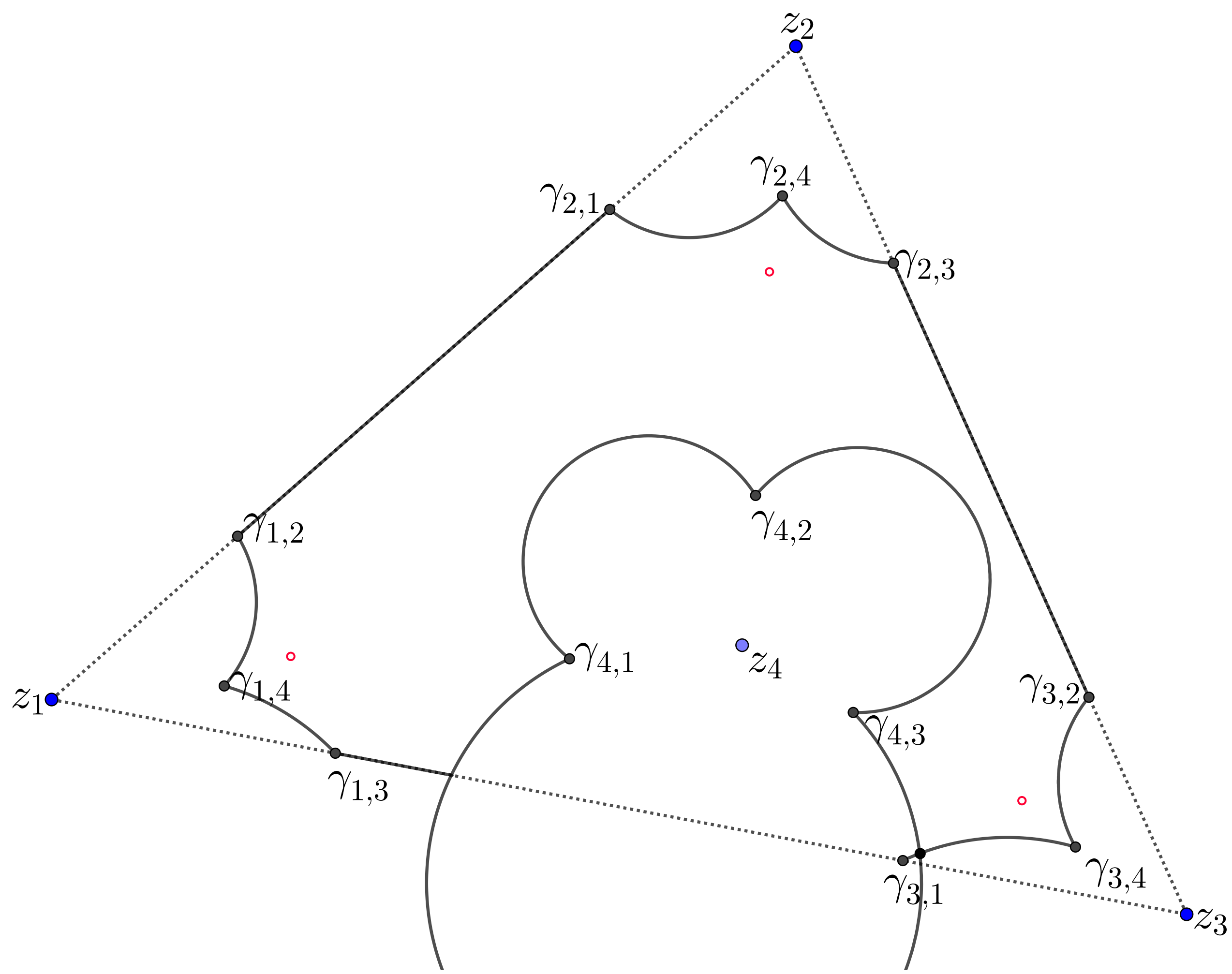
Example: $p(z) = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$ has two non-trivial critical points

$$\gamma_{1,1} = \infty, \quad \gamma_{1,2} = \frac{3z1 + z2}{4},$$

$$\gamma_{1,3} = \frac{3z1 + z3}{4}, \; \gamma_{1,4} = \frac{3z1 + z4}{4}, \ldots$$



Corollary

Consider the degree n polynomial $p(z) = (z - z_1)^{k_1} \cdots (z - z_m)^{k_m}$

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Corollary (2021): Let ζ be a non-trivial critical point of $p(z)$. For all $j, k \in \{1, \ldots, m\}$

there are numbers $t_{j,k} \geq 0$, satisfying $\sum_{k=1}^{m} t_{j,k} = 1$ such that

$$\frac{k_j}{n} \frac{1}{\zeta - z_j} = \sum_{\substack{k=1 \\ k \neq j}}^{m} \frac{t_{j,k}}{z_k - z_j}$$

If z_j is not an extreme point of conv$\{z_1, \ldots, z_m\}$, then $t_{j,j}$ can be taken to be 0 above

The set of all poles of a set

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Denote by $\mathscr{P}(A)$ the set of all poles of a set $A \subset \bar{\mathbb{C}}$

Realization: If the zeros of $p(z)$ are in A and $u \in \mathscr{P}(A)$, then the zeros of $D_u p(z)$ are in A

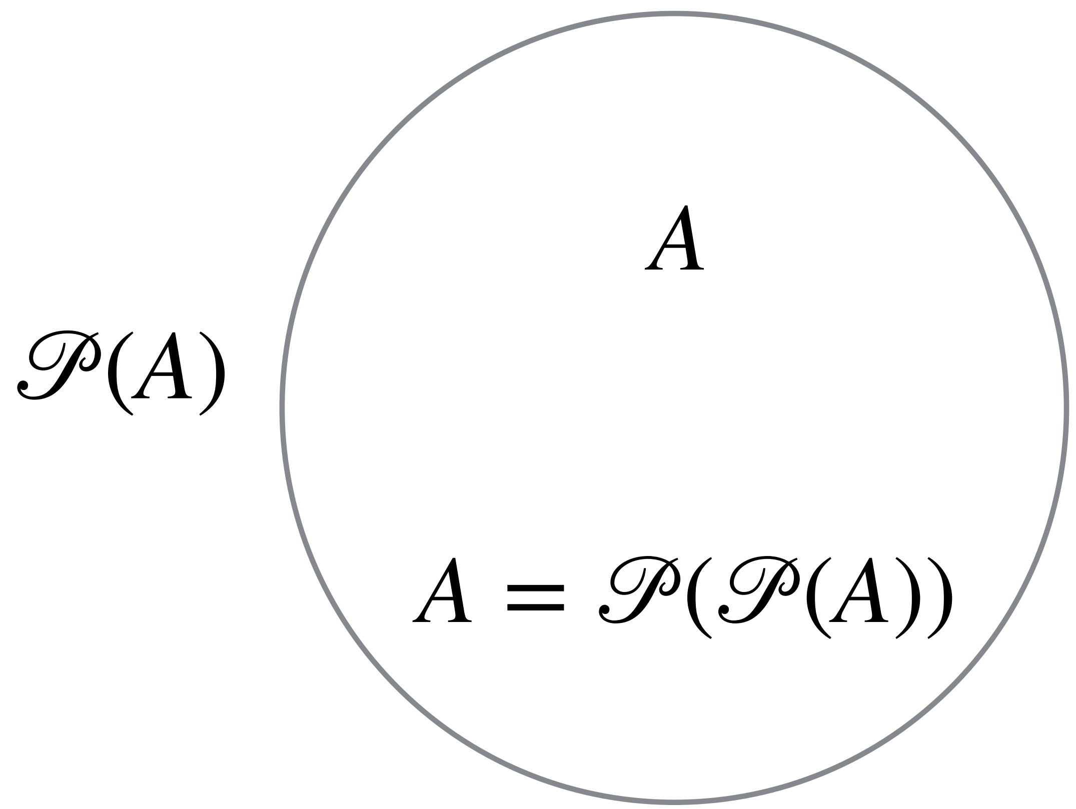
Thus, of interest is to calculate the set of poles of a given set

Note that if T is a Mobius transformation such that $T(u) = \infty$

then $T(A)$ is a convex set if and only if $u \in \mathscr{P}(A)$

Thus, if we know the poles $\mathscr{P}(A)$ of a set A we have a description of all Mobius transformations that map A onto a convex set

A few simple examples



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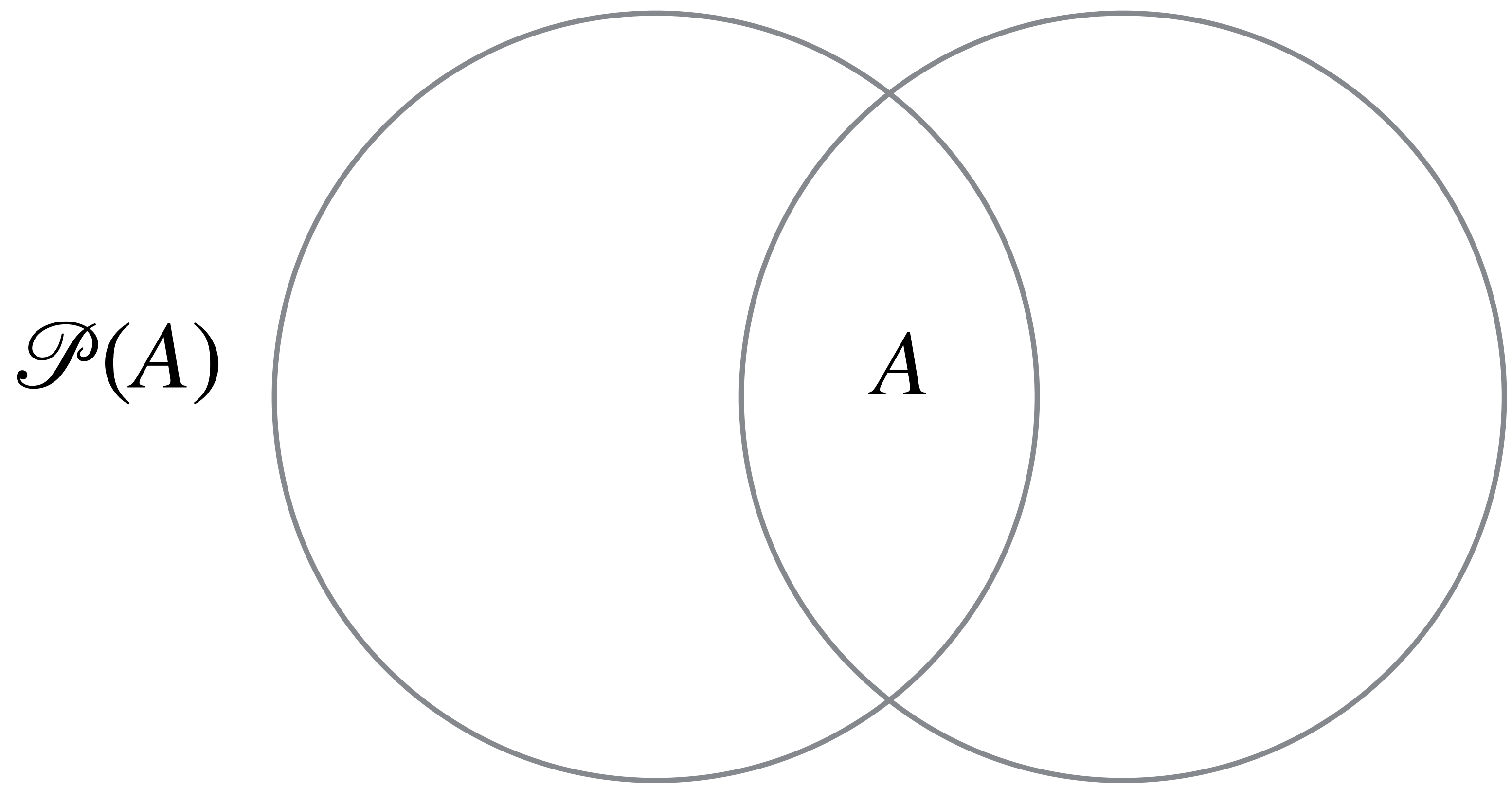

$$A = \mathscr{P}(\mathscr{P}(A))$$

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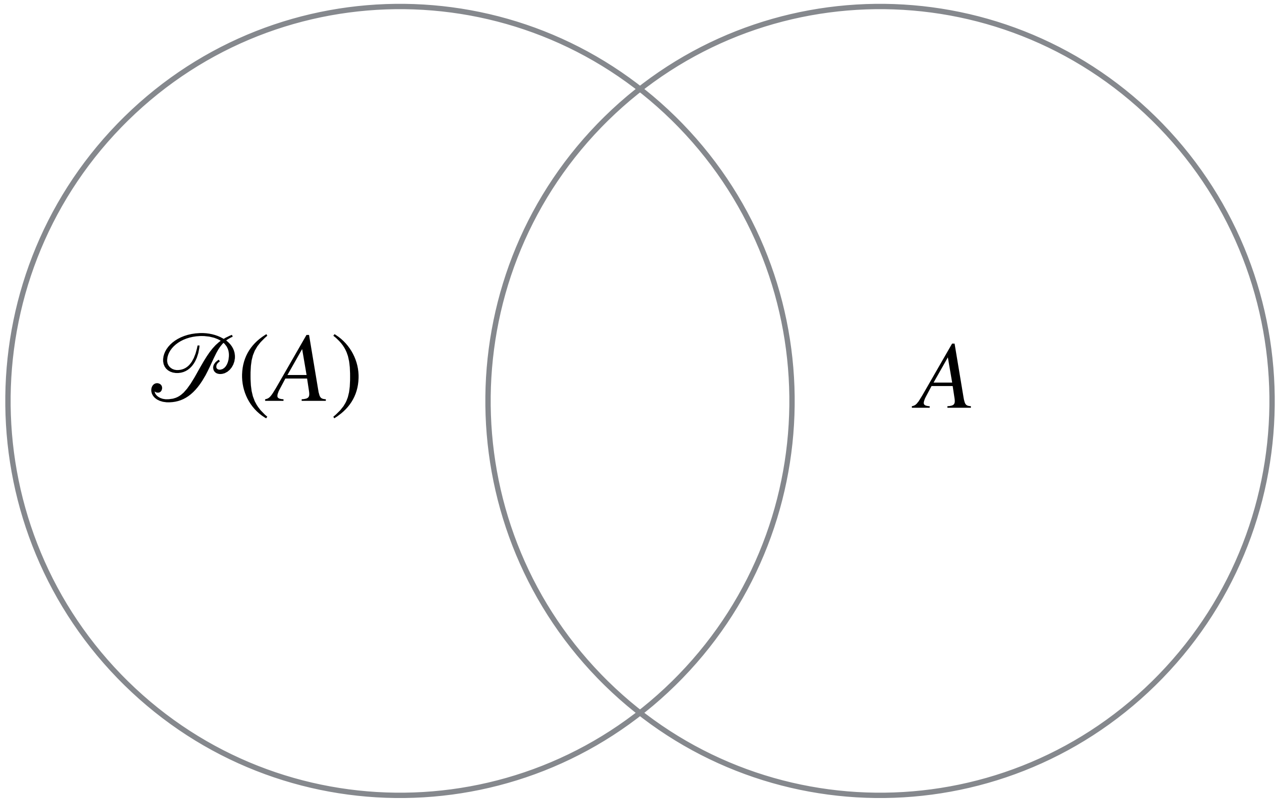

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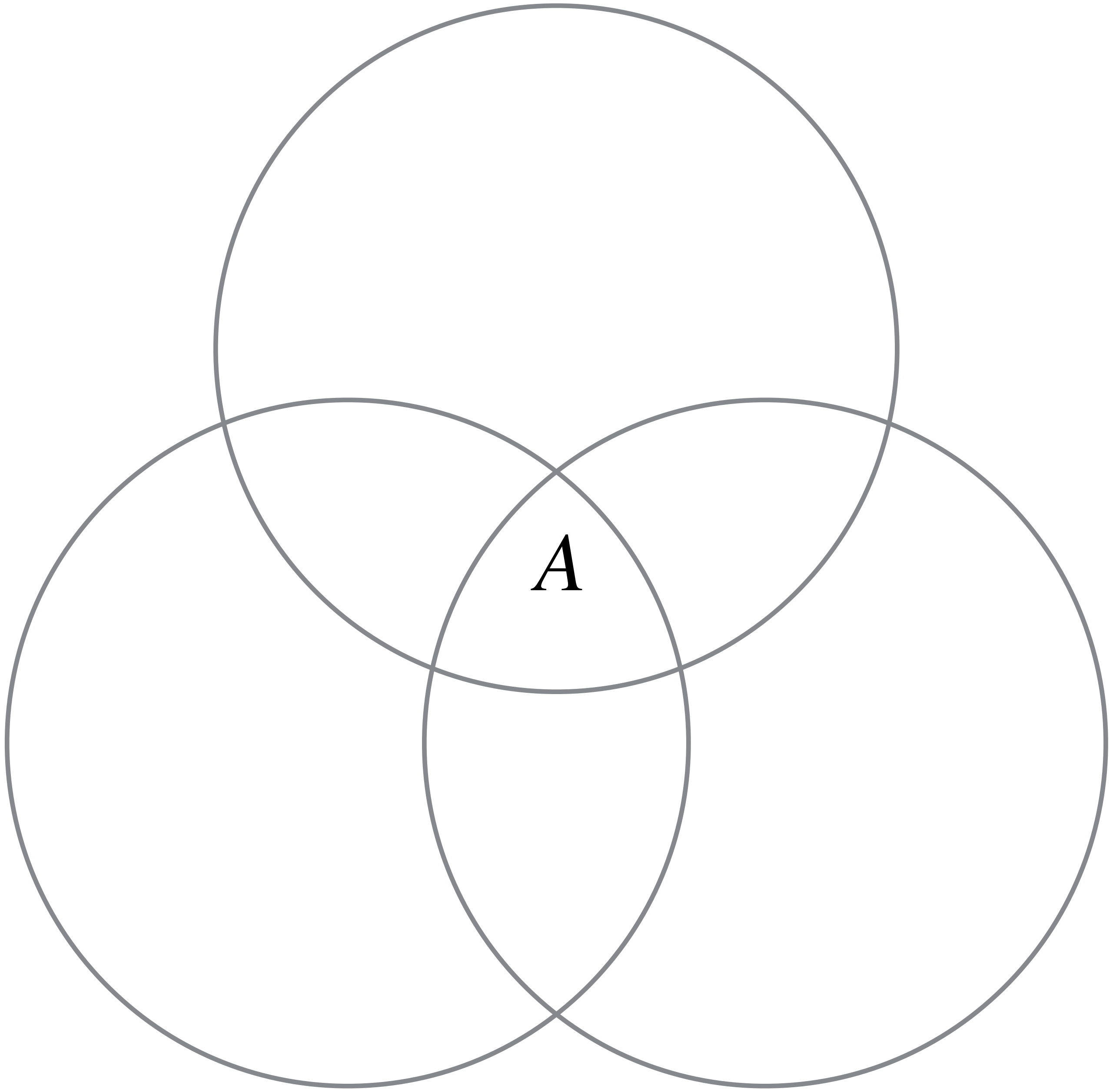
A few simple examples

$\mathscr{P}(A)$



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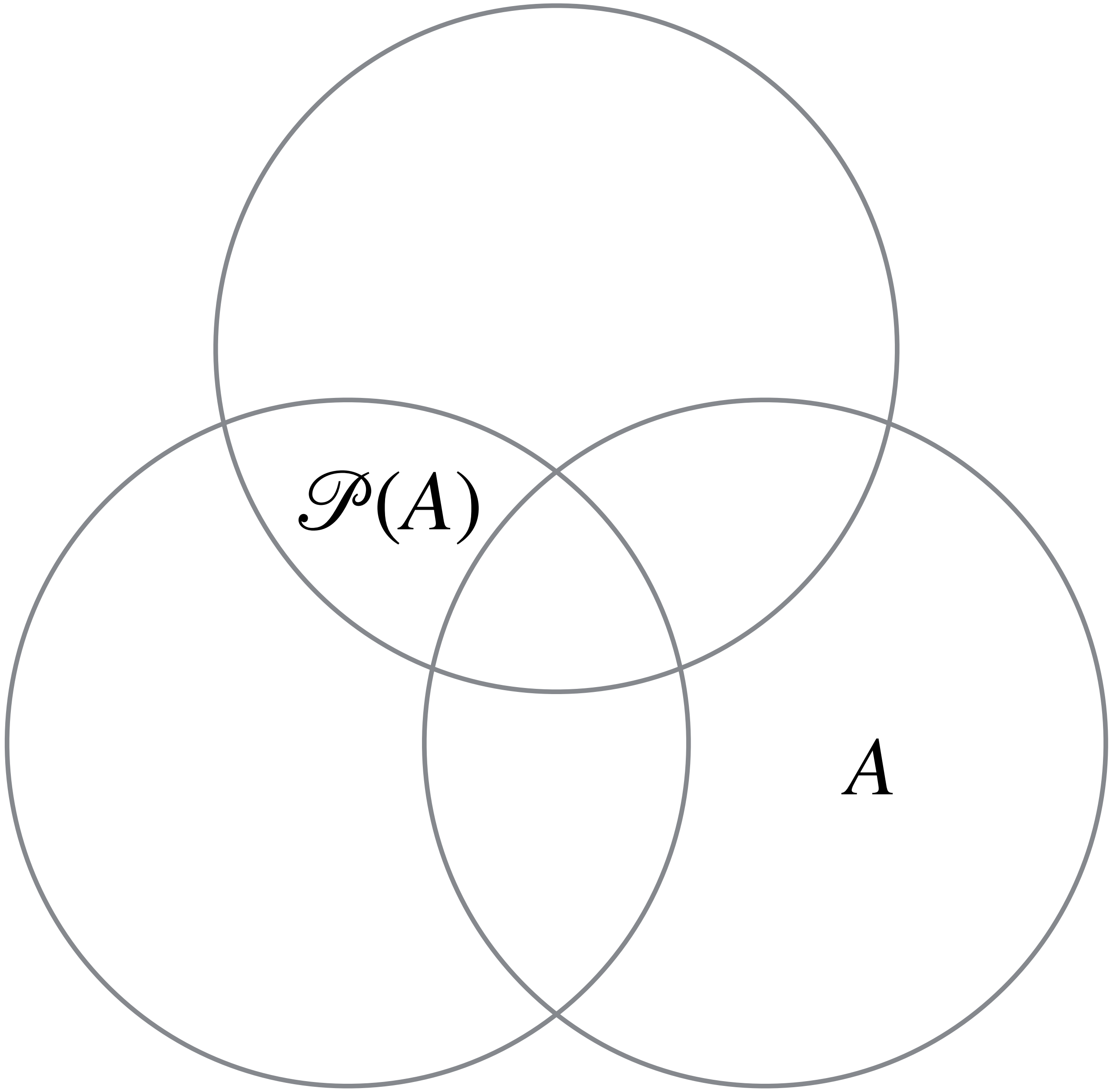

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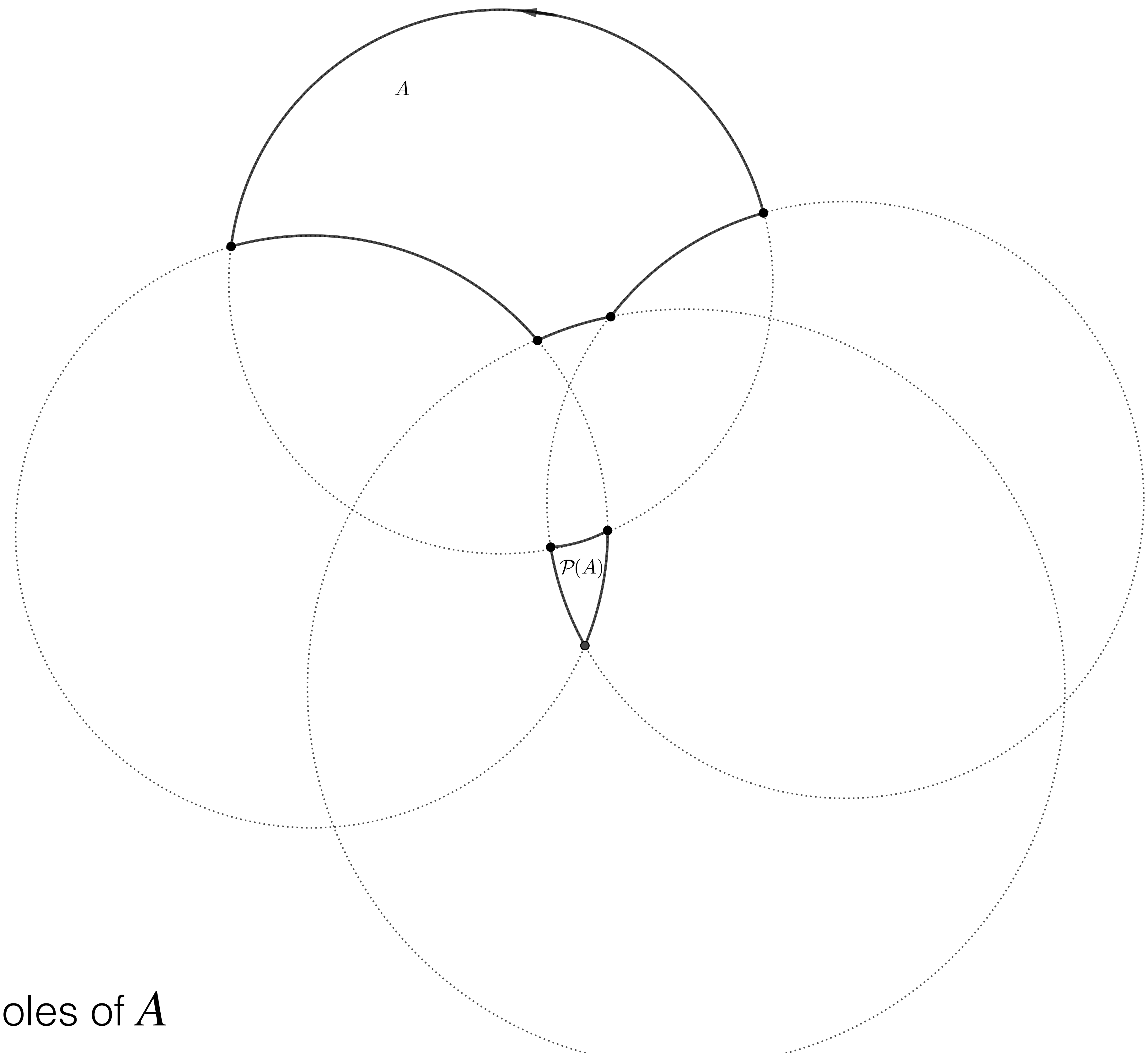
Example 9

Boundary of A is, piece-wise smooth. In fact the boundary is made of circular arcs



$\mathscr{P}(A)$ is the set of all poles of A

Some properties of polar convexity

Denote by $\mathscr{P}(A)$ the set of all poles of $A \subset \bar{\mathbb{C}}$

It is easy to see that $\mathscr{P}(A) \cap (\operatorname{int} A) = \varnothing$, whenever $A \subsetneq \bar{\mathbb{C}}$

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Take any sets $A, U \subset \bar{\mathbb{C}}$

Define $\text{conv}_U(A)$ to be the smallest set containing and convex w.r.t. every $u \in U$

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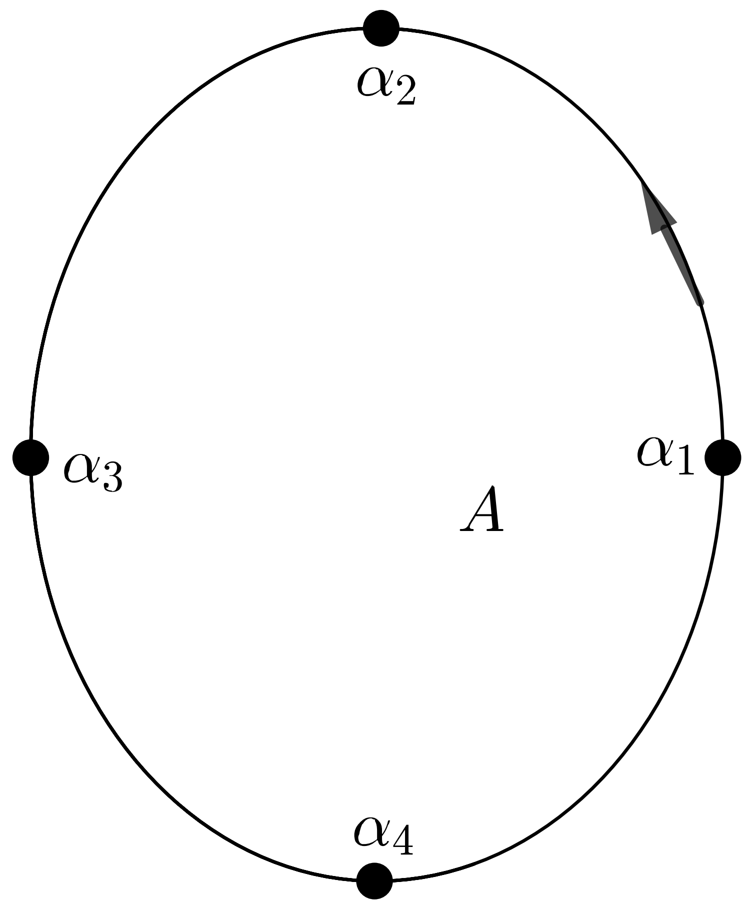
Theorem: For every set $A \subset \bar{\mathbb{C}}$, we have $A \subseteq \mathscr{P}(\mathscr{P}(A))$

Examples: all Poles of a Set

Example 1

What are the poles of the inside of an ellipse?



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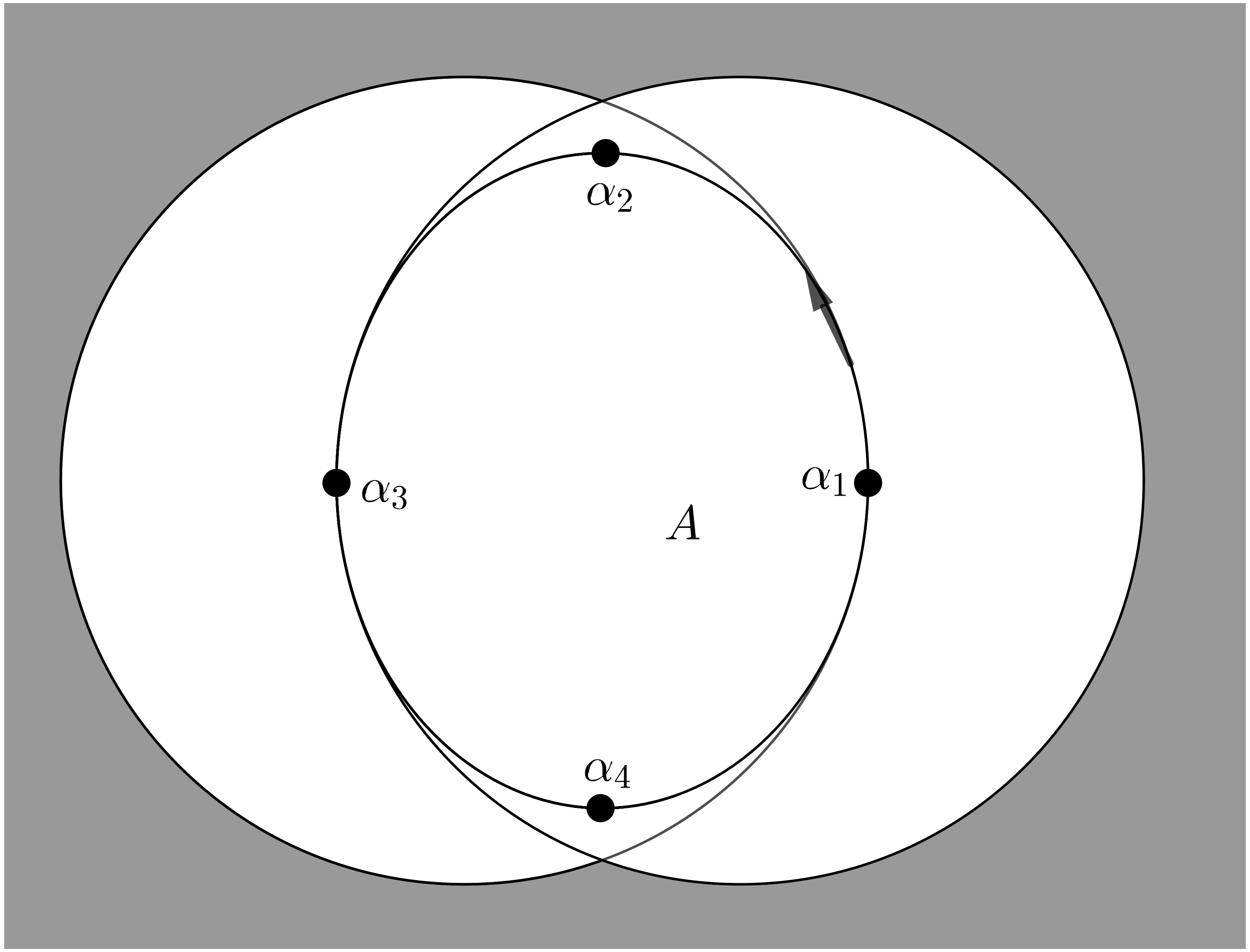
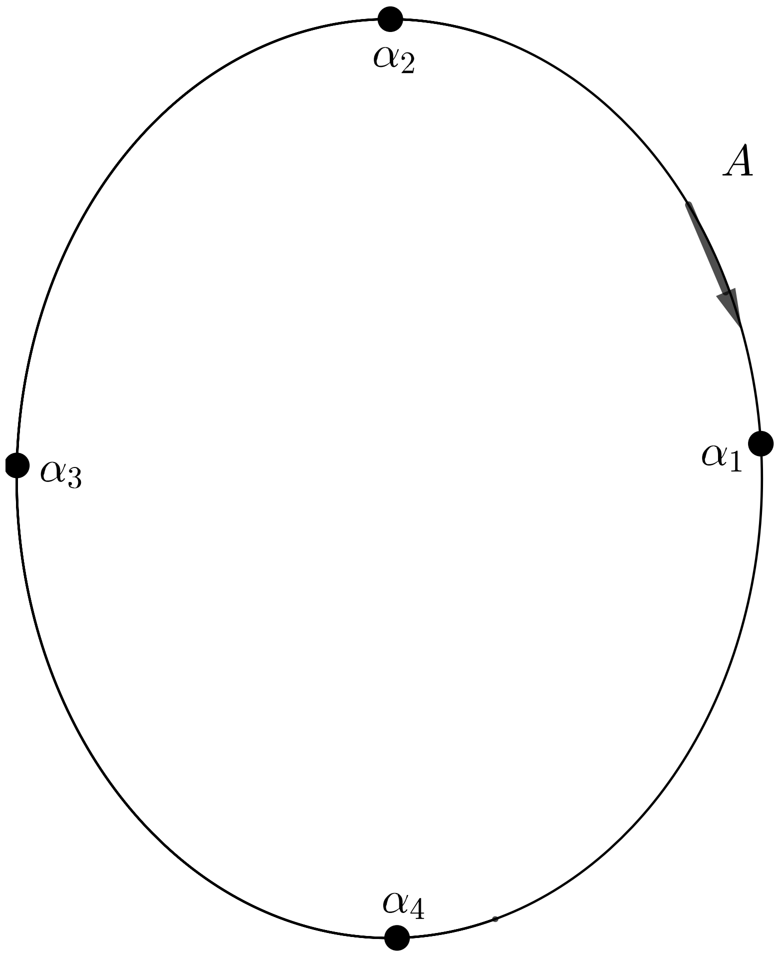


Example 2

What are the poles of the outside of an ellipse?



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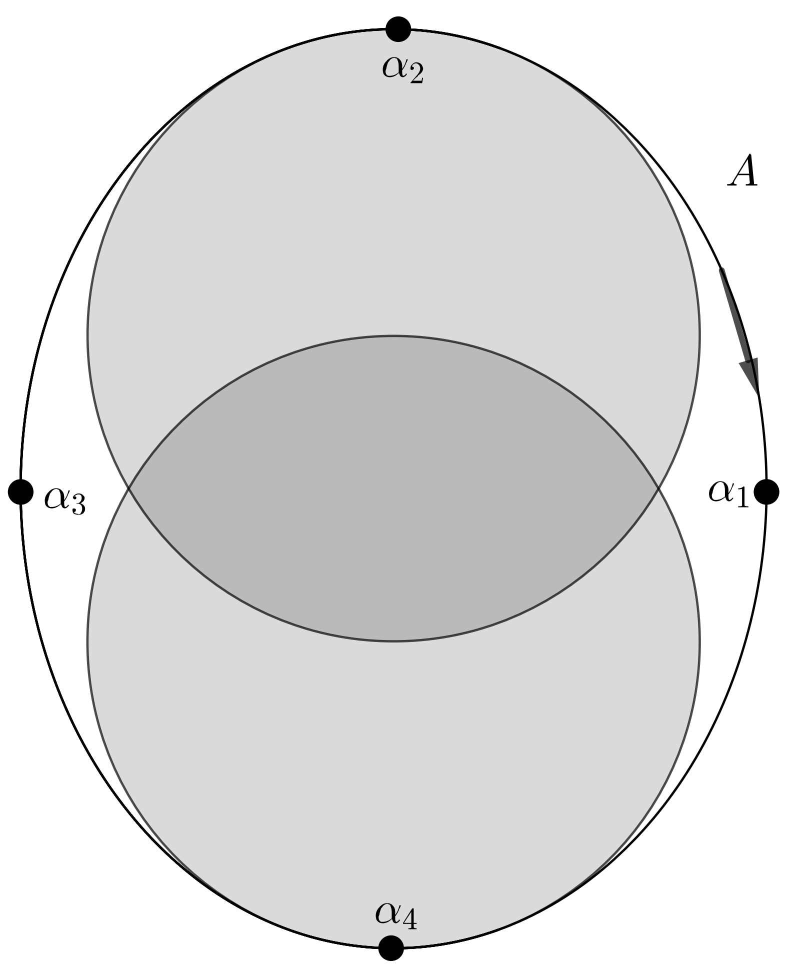
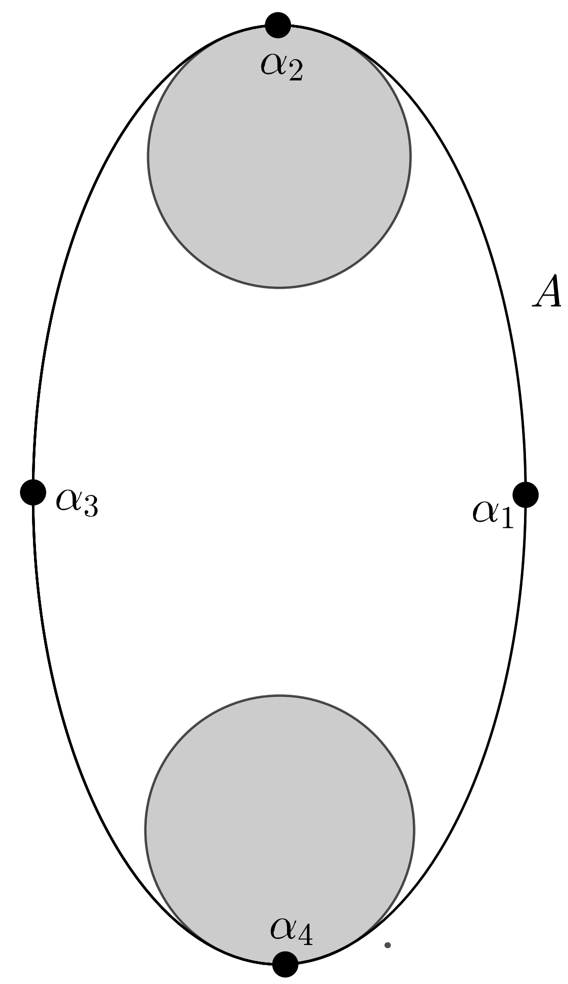


Example 2

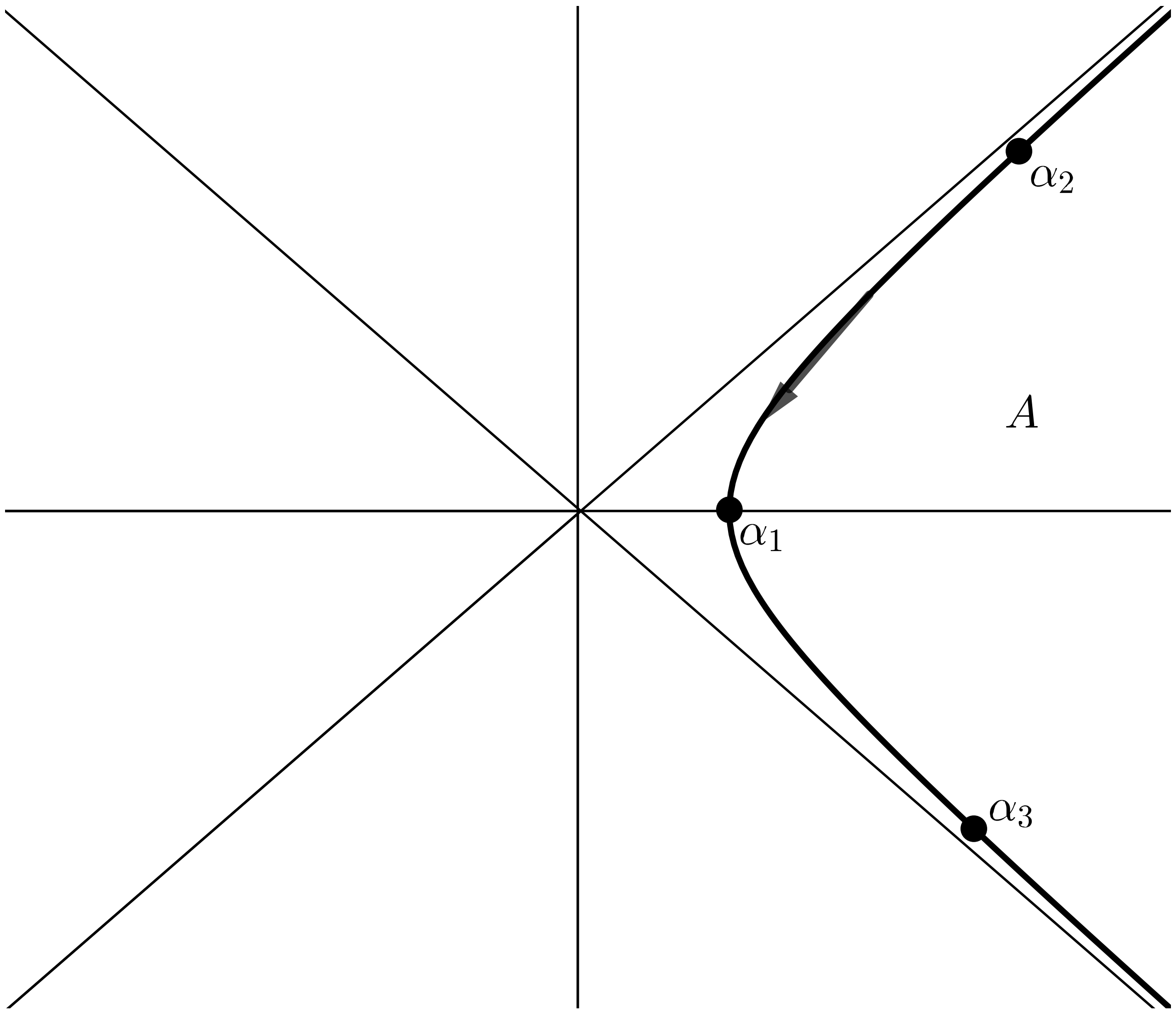
What are the poles of the outside of an ellipse?

It may happen that the intersection is empty



Example 3

What are the poles of the inside of a hyperbola?



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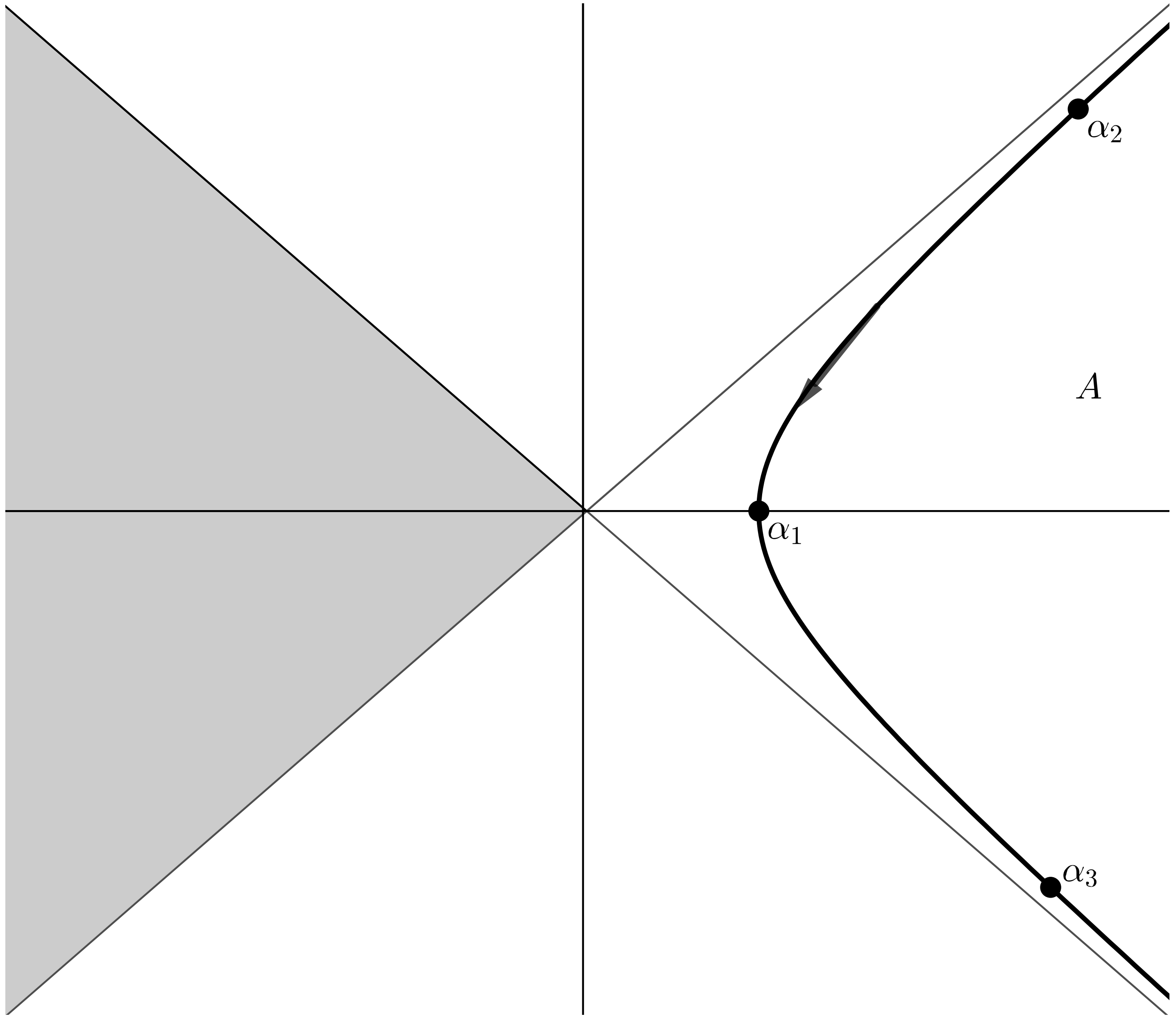

Example 4

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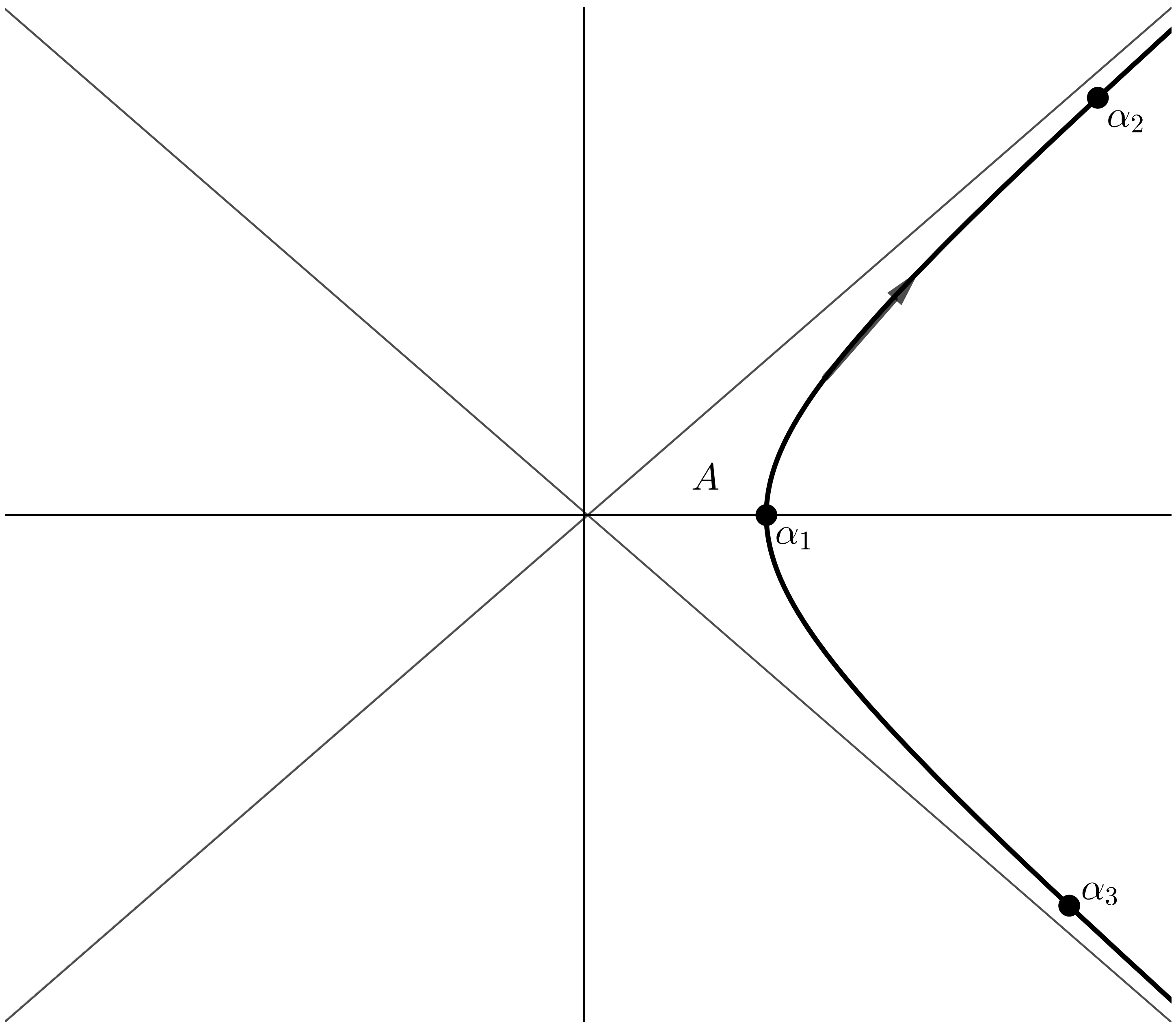
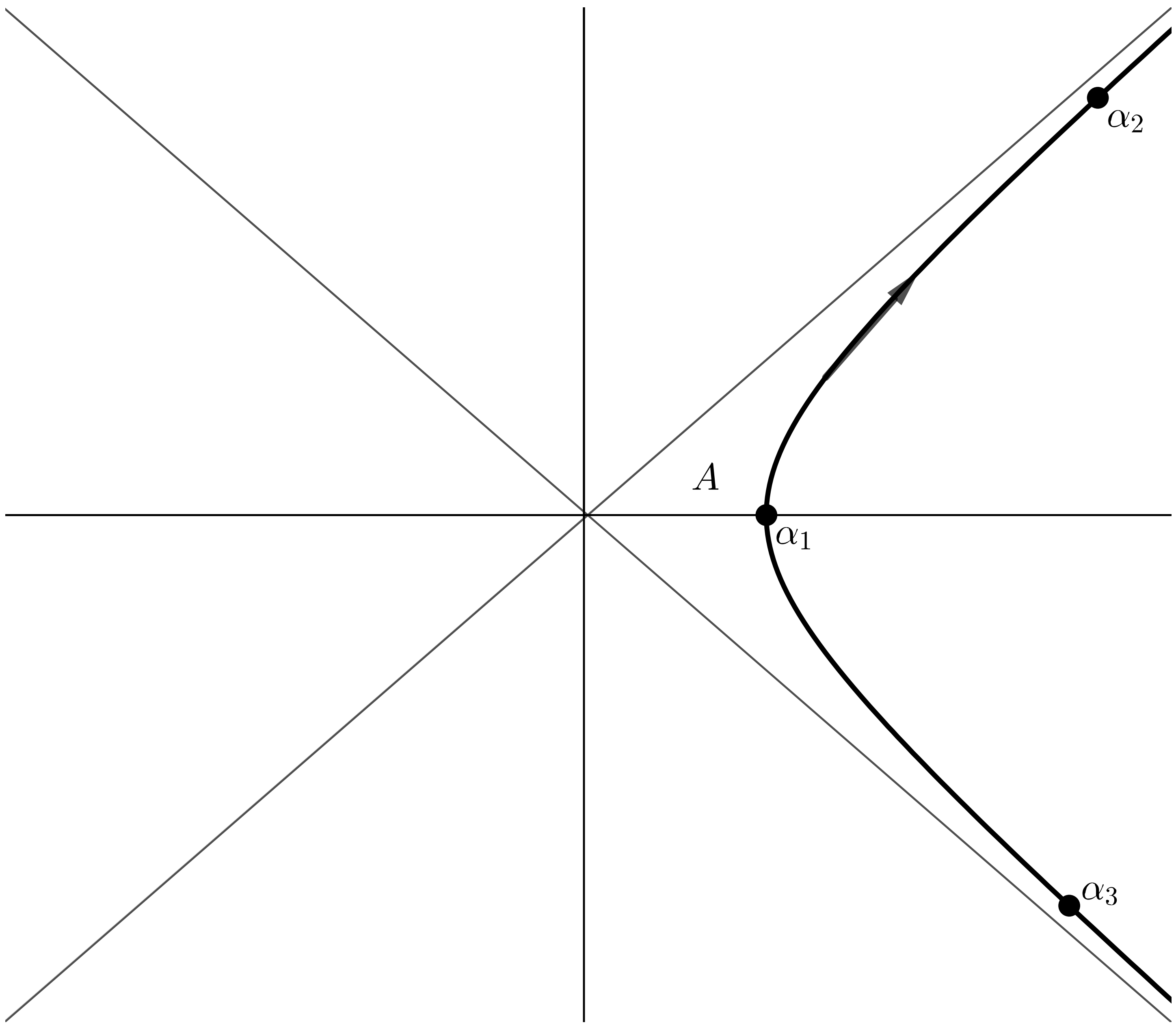


Example 4

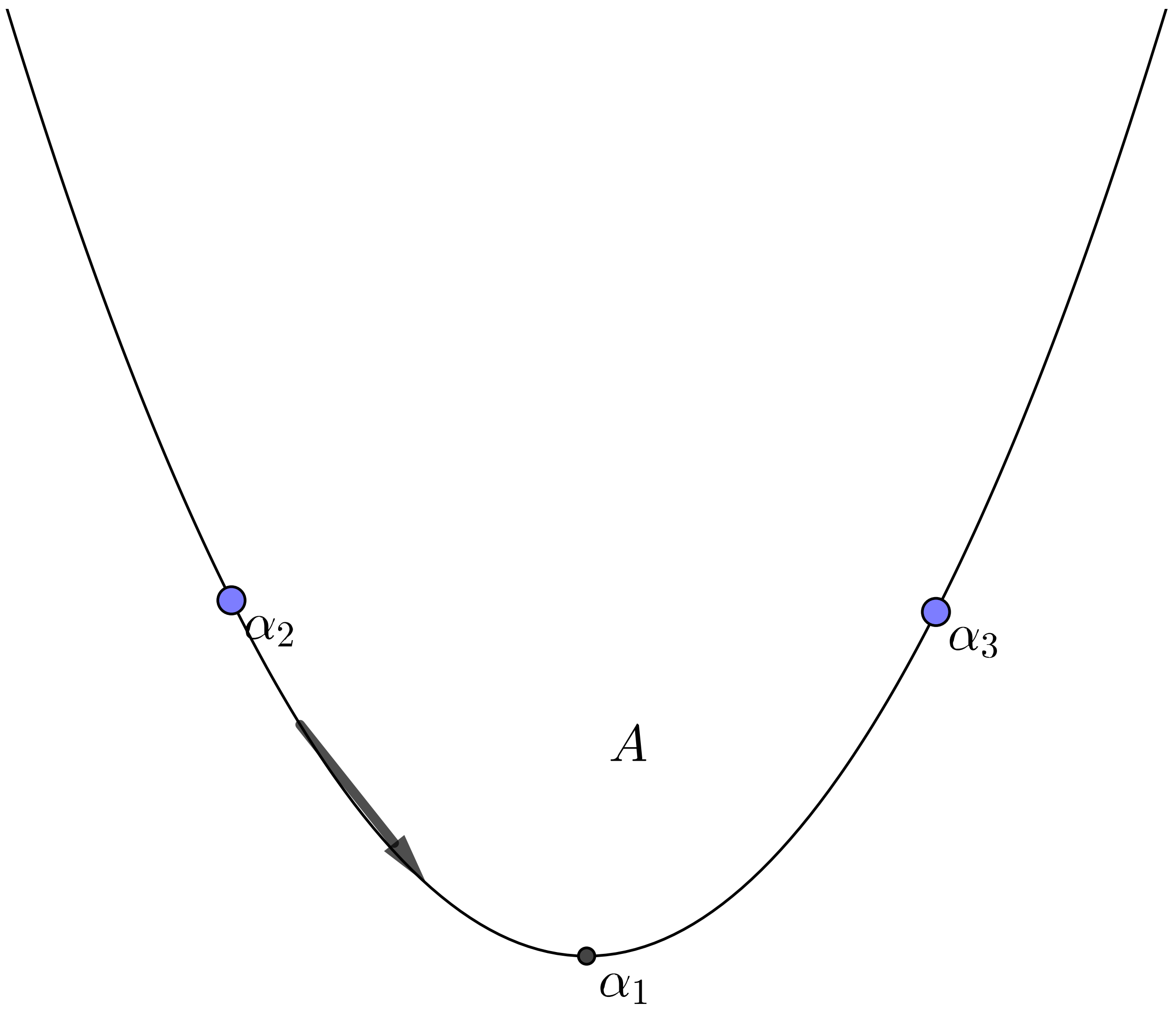
What are the poles of the outside of a hyperbola?

$$\mathscr{P}(A) = \varnothing$$



Example 5

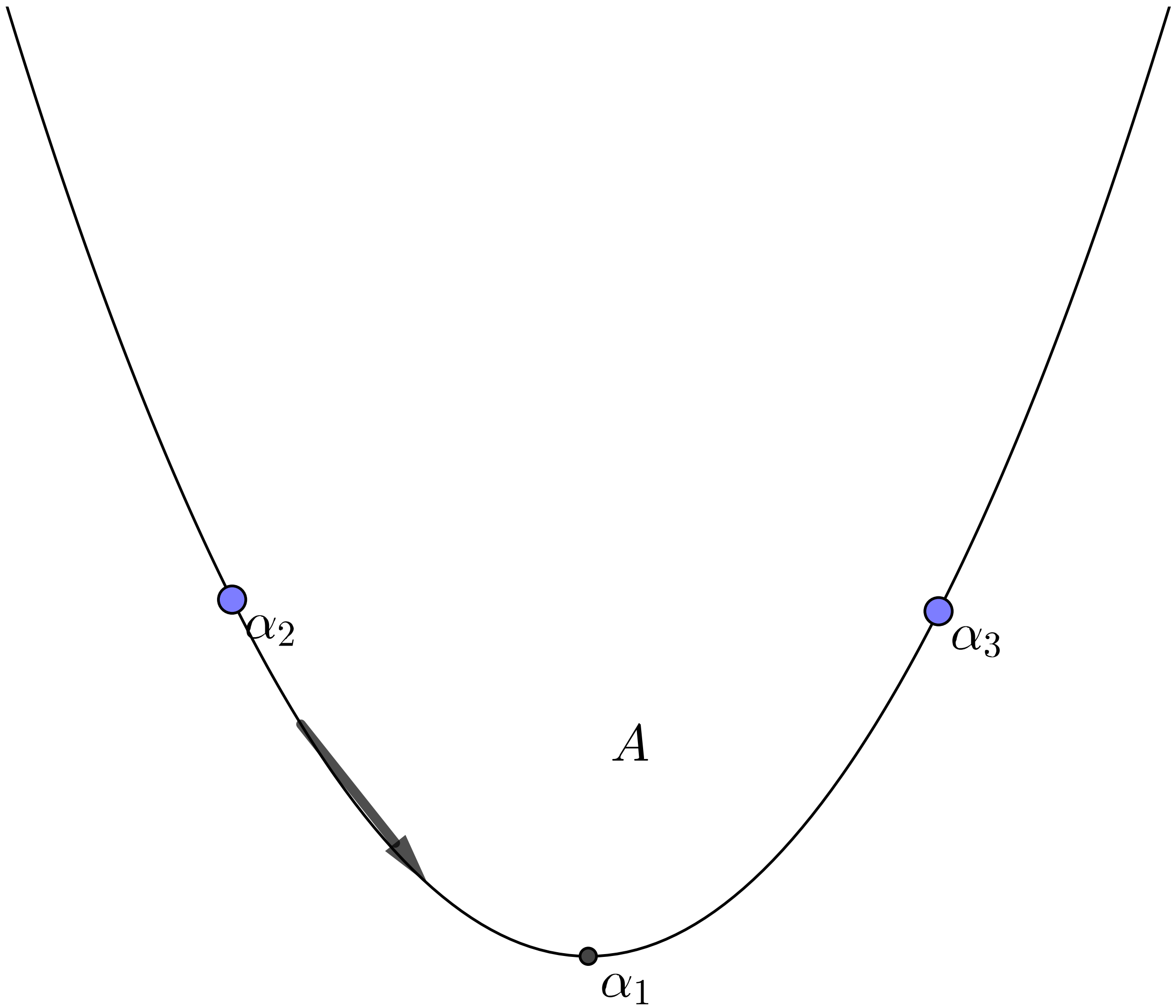
What are the poles of the inside of a parabola?



Example 5

What are the poles of the inside of a parabola?

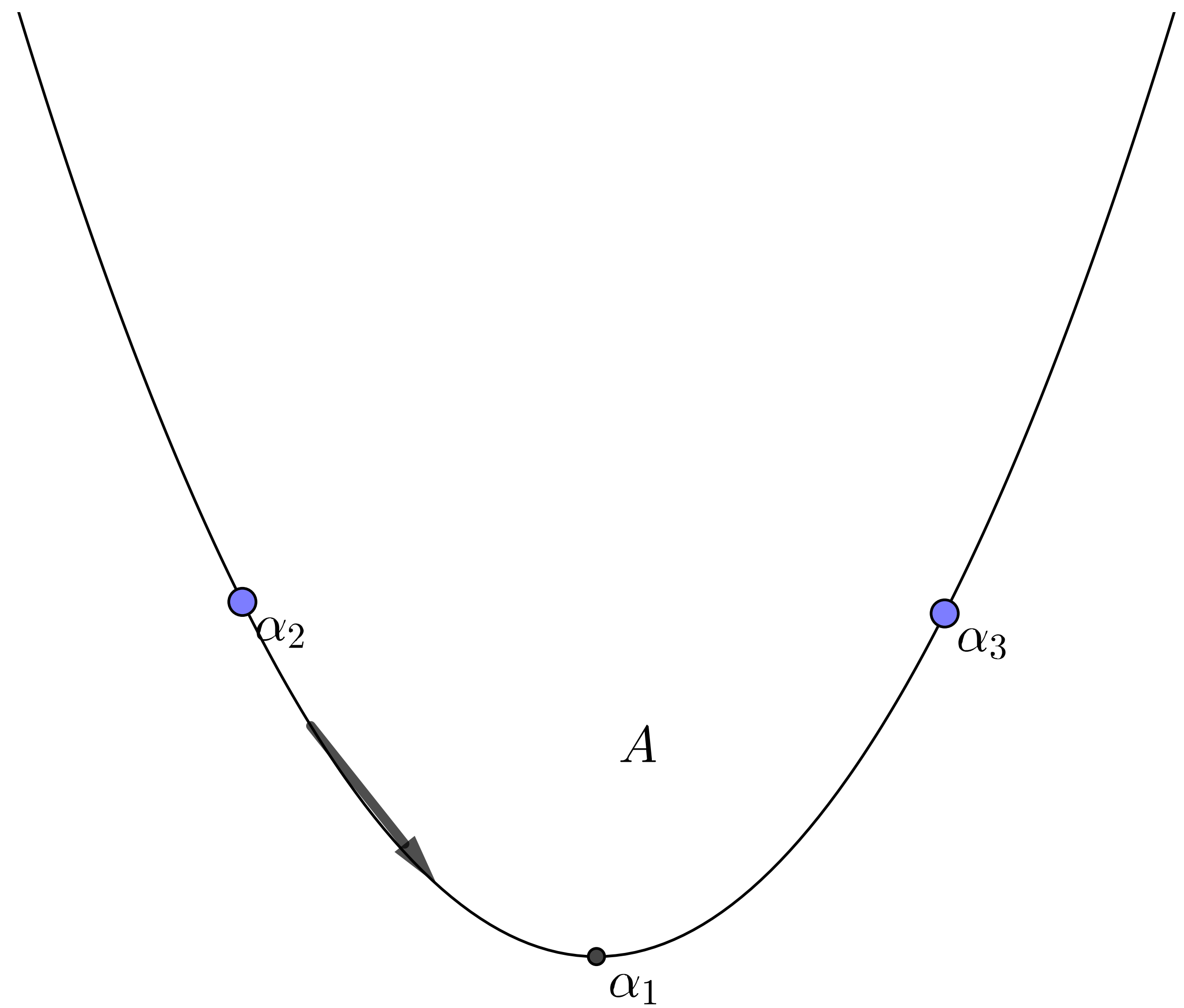
$$\mathscr{P}(A) = \{\infty\}$$



Example 6

What are the poles of the inside of a parabola?

$$\mathscr{P}(A) = \{\infty\}$$

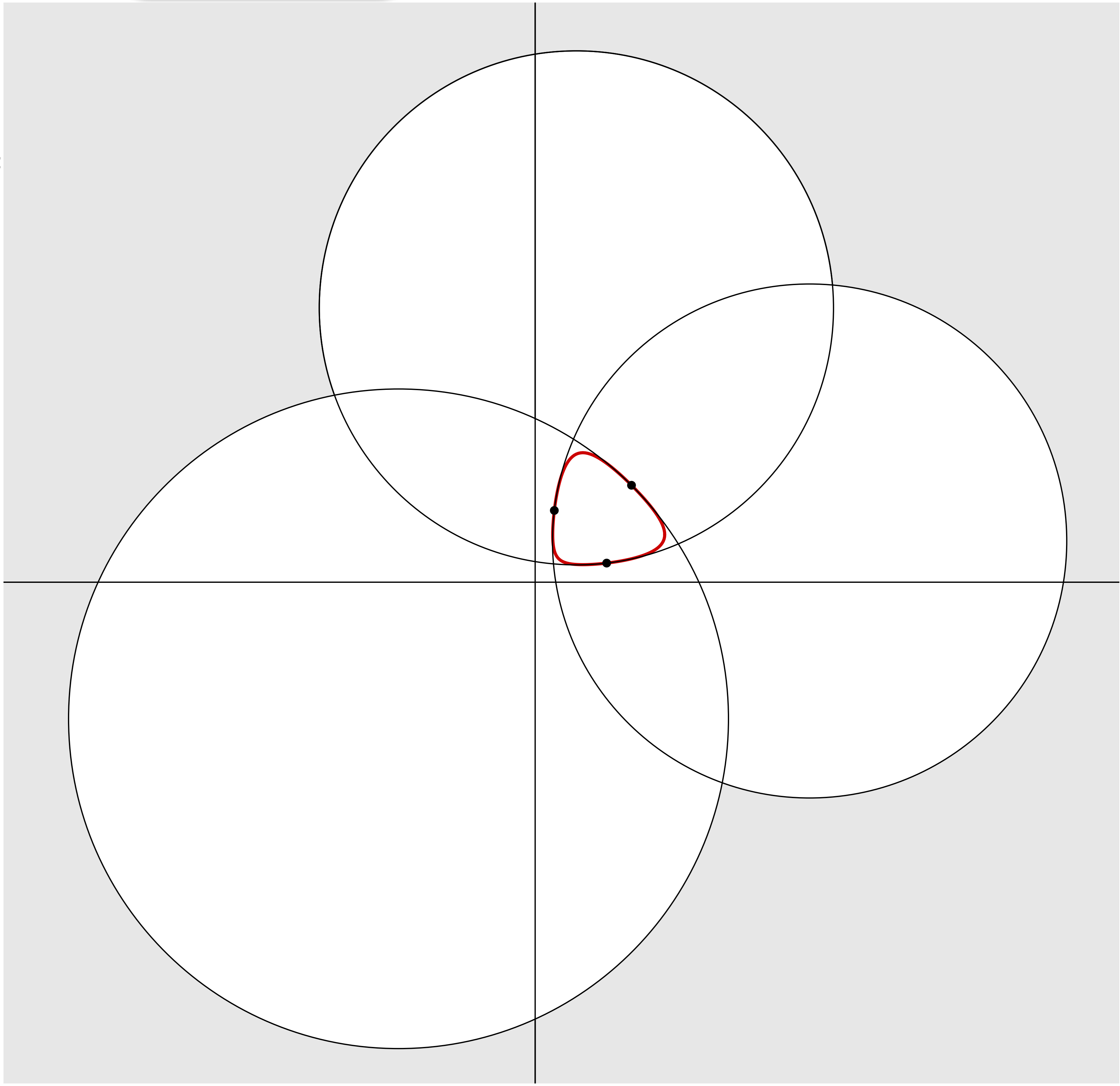


The outside of a parabola has no poles

Example 7

The red curve in is the graph of

$$(e^{\cos(t)}, e^{\sin(t)}) \quad t \in \mathbb{R}$$

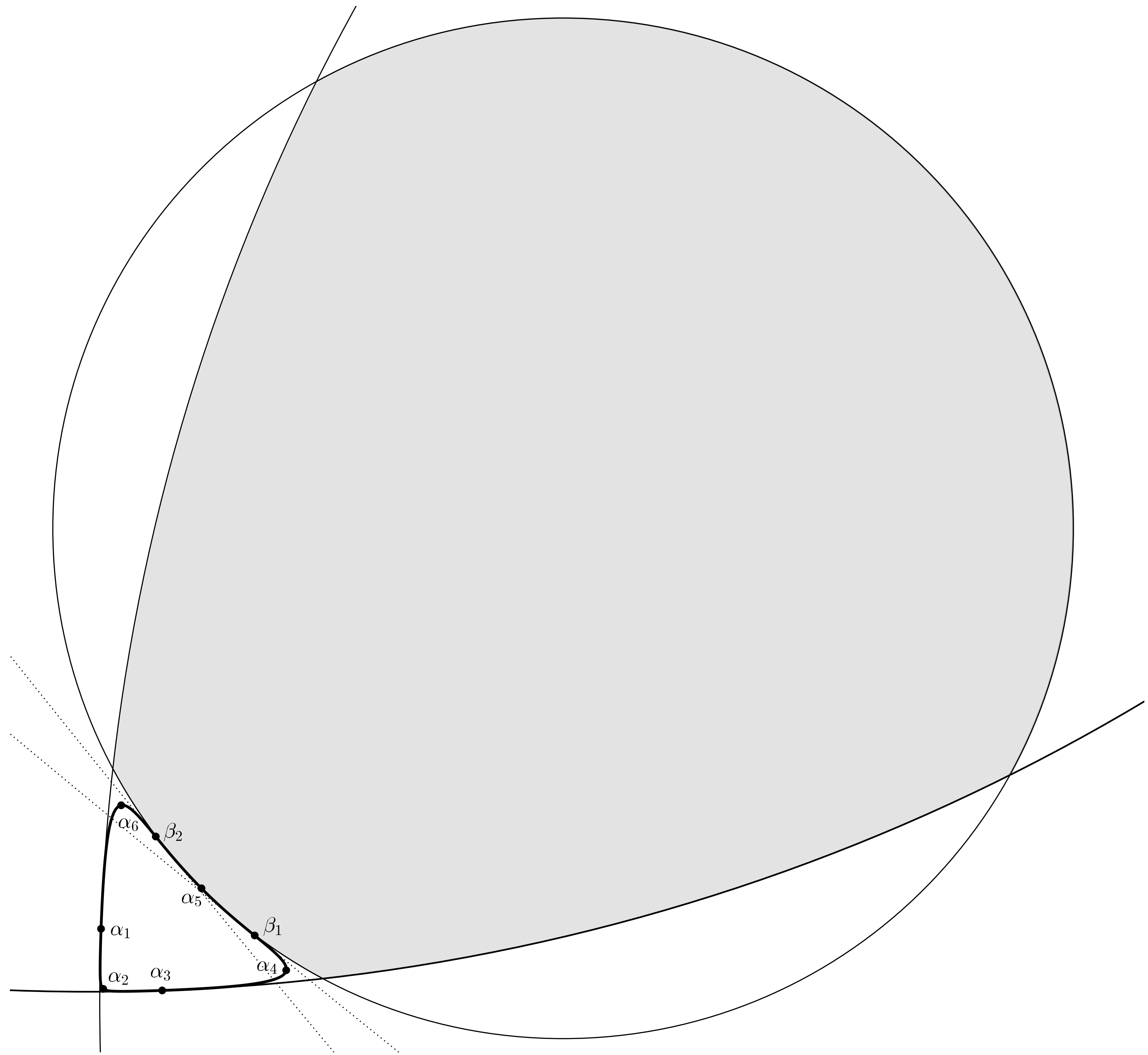


The grey area is the set of poles of its interior

Example 8

The black curve is the graph of

$$(e^{2\cos(t)}, e^{2\sin(t)}) \quad t \in \mathbb{R}$$



The grey area is the set of all poles of its interior


That's all Folks!