# Multiplicative Gradient Method: When and Why It Works

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- (MG) does not fall under any "well-known" optimization frameworks, e.g., Newton-type method, mirror descent, etc.

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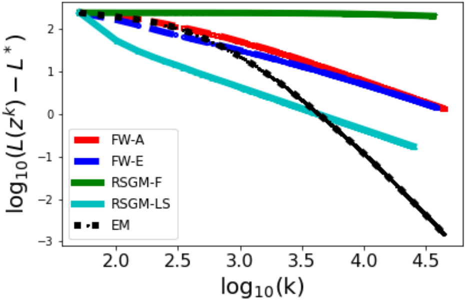
- Impressive numerical performance: $x^0 = (1/n)e$




FW-A & FW-E [Dvu20; ZF22]: Frank-Wolfe (FW) method for logarithmically-homogeneous self-concordant barriers (with adaptive stepsize and exact line search)

RSGM-F & RSGM-LS [BBT17; LFN18]: Relatively smooth gradient method (with fixed stepsize and backtracking line search)

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  - And what is the interaction between the complexity of (MG) and the problem structure?

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- ▷ To facilitate understanding, we first show our results when they are specialized to the following applications:
  - D-optimal design
  - Quantum state tomography
  - Semidefinite relaxation of Boolean QP
- ▷ In all of these applications, the objective functions involve "$\ln(\cdot)$", and hence are neither Lipschitz nor smooth (i.e., have Lipschitz gradients) on the feasible sets.
- ▷ Certain first-order methods for these applications have been developed recently [Nes11; BBT17; LFN18; Dvu20; ZF22] — our generalized MG method contributes to this line of research from a different viewpoint.

# D-Optimal Design (D-OPT)

$$\max_x F(x) := m^{-1} \ln\det\left(\textstyle\sum_{i=1}^n x_i a_i a_i^\top\right) \quad \text{s.t.} \quad x \in \Delta_n \qquad \text{(D-OPT)}$$

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$$\begin{aligned} \max_X \; & F(X) := m^{-1} \textstyle\sum_{j=1}^q n_j \ln(\langle X, a_j a_j^H \rangle) \\ \text{s. t.} \quad & X \in \mathbb{H}_+^n, \;\; \mathrm{tr}(X) = \langle I_n, X \rangle = 1 \end{aligned} \tag{QST}$$

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$$\boxed{\begin{aligned} \hat{X}^{t+1} &= \exp\{\ln(X^t) + \ln(\nabla F(X^t))\} \\ X^{t+1} &= \hat{X}^{t+1} / \mathrm{tr}(\hat{X}^{t+1}) \end{aligned}}$$

(For any $X = \sum_{i=1}^n \lambda_i u_i u_i^H \succ 0$, $\ln(X) := \ln(\lambda_i) u_i u_i^H$.)

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- ▷ Nesterov [Nes11] later showed that (SDP) above can be equivalently written in the dual form:

$$\begin{aligned} \max_X \quad & F(X) := 2\ln\left(\textstyle\sum_{i=1}^n \langle X, r_i r_i^\top\rangle^{1/2}\right) \\ \text{s.t.} \quad & X \in \mathbb{S}_+^n,\ \langle I_n, X\rangle = 1 \end{aligned} \tag{RBQP}$$

where $A = R^\top R$ and $R := [r_1 \ \cdots \ r_n]$, and $\mathbb{S}_+^n$ denotes the cone of $n \times n$ real symmetric PSD matrices.

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# Comparison of Computational Guarantees

RSGM [BBT17; LFN18]: Relatively smooth gradient method
FW [Dvu20; ZF22]: FW method for logarithmically-homogeneous self-concordant barriers
MG: (Generalized) Multiplicative gradient method (this work)
BSG [Nes11]: Barrier subgradient method

Table 1: Comparison of operations complexities (with $x^0 = (1/n)e$ or $X^0 = (1/n)I_n$)

| | RSGM | FW | MG | BSG | Regime |
|---|---|---|---|---|---|
| PET | $O\left(\frac{mn^2}{\varepsilon}\ln\left(\frac{\ln(n)}{\varepsilon}\right)\right)$ | $O\left(\frac{m^2 n}{\varepsilon}\right)$ | $O\left(\frac{mn\ln(n)}{\varepsilon}\right)$ | $O\left(\frac{mn^2}{\varepsilon^2}\ln^2\left(\frac{n}{\varepsilon}\right)\right)$ | $n = O(\exp(m))$ |
| D-OPT | $O\left(\frac{mn^2}{\varepsilon}\ln\left(\frac{\ln(n/m)}{\varepsilon}\right)\right)$ | $O\left(\frac{m^2 n}{\varepsilon}\right)$ | $O\left(\frac{m^2 n\ln(n)}{\varepsilon}\right)^\dagger$ | $O\left(\frac{m^2 n^2}{\varepsilon^2}\ln^2\left(\frac{n}{\varepsilon}\right)\right)$ | |
| QST | x? | $O\left(\frac{m^2 n^2}{\varepsilon}\right)$ | $O\left(\frac{mn^2\ln(n)}{\varepsilon}\right)^\ddagger$ | $O\left(\frac{mn^3}{\varepsilon^2}\ln^2\left(\frac{n}{\varepsilon}\right)\right)$ | $n = O(\exp(m))$ |
| RBQP | x? | x? | $O\left(\frac{n^3\ln(n)}{\varepsilon}\right)$ | $O\left(\frac{n^4}{\varepsilon^2}\ln^2\left(\frac{n}{\varepsilon}\right)\right)$ | |

† [Coh19] ‡ [LCL21]

# A General Problem Class

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- $e \in \mathsf{int}\,\mathcal{K}_1$ is the "center" of $\mathcal{K}_1$, e.g., $e = \mathbf{1}_n := (1, \ldots, 1)$ if $\mathcal{K}_1 = \mathbb{R}^n_+$ and $e = I_n$ if $\mathcal{K}_1 = \mathbb{S}^n_+$.

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- $\mathsf{A} : \mathcal{K}_1 \to \mathcal{K}_2$ is a linear operator, where $\mathcal{K}_2$ is any regular cone.
  - We require both $\mathsf{A} : \mathsf{int}\,\mathcal{K}_1 \to \mathsf{int}\,\mathcal{K}_2$ and $\mathsf{A}^* : \mathsf{int}\,\mathcal{K}_2^* \to \mathsf{int}\,\mathcal{K}_1$.

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  or the Cartesian product of these primitive cones, e.g.,$\mathbb{R}^m_+$.
- $\mathcal{C}$ is sometimes referred to as the “generalized unit simplex”, including unit simplex, unit $\ell_2$-ball and spectrahedron.

# The Class of Gradient Log-Convex Functions

We call $F$ in (P) *gradient log-convex* if $\nabla F : \mathsf{int}\,\mathcal{K}_1 \to \mathsf{int}\,\mathcal{K}_1$ satisfies

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(includes PET and D-OPT).
- ▷ $\mathcal{K}_1$ is any *representable* symmetric cone (all except the 27-dimensional exceptional one), $\mathcal{K}_2 = \mathbb{R}^m_+$ and
  - $f(y) = \sum_{j=1}^m w_j \ln y_j$, for all $y > 0$ and $w \in \mathsf{ri}\,\Delta_m$ (includes QST).
  - $f(y) = \ln |||y|||_p := p^{-1} \ln(\sum_{j=1}^m y_j^p)$, for all $y > 0$ and $p \in (0,1]$ (includes RBQP).

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- Our GMG method was developed and analyzed under the framework of EJA.
- To understand this method, we will briefly review some basics of EJA.

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- ▷ We call $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$ (or simply $\mathbb{V}$) a EJA with rank $n$ if
  - $\mathbb{V}$ is a vector space
  - $\circ : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ is a *bilinear* operation on $\mathbb{V}$ such that $x \circ y = y \circ x, \quad x^2 \circ (x \circ y) = x \circ (x^2 \circ y), \quad \forall\, x, y \in \mathbb{V}.$
  - $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ is *associative*: $\langle x, y \circ z \rangle = \langle x \circ y, z \rangle$, $\forall\, x, y, z \in \mathbb{V}$.
  - Let $\| \cdot \|$ be the norm induced by $\langle \cdot, \cdot \rangle$.

- ▷ Let $e$ be the identity element in $\mathbb{V}$, so that $x \circ e = e \circ x = x$.

- ▷ Any $x \in \mathbb{V}$ has the spectral decomposition $\sum_{i=1}^n \lambda_i(x) q_i(x)$:
  - the eigenvalues $\{\lambda_i(x)\}_{i=1}^n$ are *real*
  - the "eigenvectors" $\{q_i(x)\}_{i=1}^n \subseteq \mathbb{V}$ form a *Jordan frame*.

- ▷ A Jordan frame $\{q_i\}_{i=1}^n \subseteq \mathbb{V}$ satisfy
  - (Completeness) $\sum_{i=1}^n q_i = e$.
  - (Orthogonality) $q_i \circ q_j = 0$, $\forall\, i \neq j,\;\; i, j \in [n]$,
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> For each symmetric cone $\mathcal{K}$, there exists a unique EJA $\mathbb{V}$ such that $\mathcal{K} \subseteq \mathbb{V}$ and
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> - $x \in \mathcal{K} \Longleftrightarrow \lambda_1(x), \ldots, \lambda_n(x) \geq 0$
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> The rank of $\mathcal{K}$ is defined to be the rank of $\mathbb{V}$.

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- In general, (GMG) updates both eigenvalues and the "eigenvectors", and specializes to all the methods we've seen earlier.

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Theorem (Convergence rate of (GMG))

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- ▷ The convergence rate is data independent — it does not depend on $\mathsf{A}$.
- ▷ The optimal choice for the above bound is $x^0 = (1/n)e$, and we have

  $$F^* - F(\bar{x}^T) \le \frac{\ln(n)}{T}, \qquad \forall\, T \ge 1$$

  Recall that $n$ is the rank of $\mathcal{K}_1$.

# Future Work

- Develop other forms of the generalized MG method.
- Discover more applications of (P), particularly when $\mathcal{K}_1$ or $\mathcal{K}_2$ is a Cartesian product of second-order cones.
- Modify the GMG method to accommodate more complicated feasible sets.
- Efficient numerical implementation of GMG method for problems involving matrix variables.

# Thank you!

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