Variational Convexity of Functions and Variational Sufficiency in Optimization

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$$\nabla f(\bar{x}) = 0, \quad \text{and} \;\; \nabla^2 f(\bar{x}) \text{ is positive definite},$$

which is equivalent to the local strong convexity of f around $\bar{x}$.

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Fundamental question: Do we have such local convexity reduction in nonsmooth optimization, especially in constrained optimization?

No local convexity reduction in constrained optimization

Problems with equality constraints:

$$\text{minimize} \quad f_0(x) \quad \text{subject to} \;\; f_i(x) = 0, i = 1, 2, \ldots, m$$

Lagrangian functions: $L(x, y) = f_0(x) + y_1 f_1(x) + \ldots + y_m f_m(x)$.
The local optimality condition of a feasible solution $\bar{x}$ is

$$\nabla_x L(\bar{x}, \bar{y}) = 0, \quad \nabla_y L(\bar{x}, \bar{y}) = 0$$

$\nabla^2_{xx} L(\bar{x}, \bar{y})$ is positive definite relative to the subspace

$$S = \{\xi \in \mathbb{R}^n | \; \langle \nabla f_i(\bar{x}), \xi \rangle = 0, \; i = 1, 2, \ldots, m\}.$$

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$\Longrightarrow$ Does this reduce to the local convexity of L around $(\bar{x}, \bar{y})$?

$\Longrightarrow$ The answer is no in general!

Maximal Monotonicity and Convexity

Theorem:

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a l.s.c., proper function. Then f is convex if and only if ∂f is maximal monotone.

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Question: Do we have this equivalence in nonsmooth case?

$\Longrightarrow$ The answer is no!

Variational Convexity

The natural following questions arise:

Questions:

- Which property is equivalent to the second-order sufficient optimality condition in NLP, nonsmooth optimization, etc?
- Which property is equivalent to the local maximal monotonicity of subgradient mappings?

[1] R. T. Rockafellar, *Variational convexity and local monotonicity of subgradient mappings*, Vietnam J. Math., 47 (2019), 547–561.

[2] R. T. Rockafellar, *Augmented Lagrangians and hidden convexity in sufficient conditions for local optimality*, Math. Program., 192 (2022), DOI 10.1007/s10107-022-01768-w.

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$\Longrightarrow$ We need a property more subtle than local convexity.

$\Longrightarrow$ This has been answered by Rockafellar[1,2], and this property is called variational convexity.

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2. Tools of Variational Analysis

Tools of Variational Analysis

See[3,4] to find more detail.

Regular normal cone to $\Omega \subset \mathbb{R}^n$ at $\bar{x} \in \Omega$ is

$$\widehat{N}_\Omega(\bar{x}) := \left\{ v \in \mathbb{R}^n \middle| \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \le 0 \right\}$$

Limiting normal cone to $\Omega \subset \mathbb{R}^n$ at $\bar{x} \in \Omega$ is

$$N_\Omega(\bar{x}) := \left\{ v \in \mathbb{R}^n \middle| \exists\, x_k \xrightarrow{\Omega} \bar{x},\ v_k \to v,\ v_k \in \widehat{N}_\Omega(x_k) \right\}$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ and $x \in \Omega$

[3] B. S. Mordukhovich, Variational Analysis and Applications, Springer (2018)

[4] R. T. Rockafellar and R. J-B. Wets, Variational Analysis, Springer (1998)

Tools of Variational Analysis

Regular coderivative and limiting coderivative of $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ are defined, respectively by

$$\widehat{D}^* F(\bar{x},\bar{y})(v) := \left\{ u \in \mathbb{R}^n \middle| (u,-v) \in \widehat{N}_{\operatorname{gph} F}(\bar{x},\bar{y}) \right\},\ v \in \mathbb{R}^m$$

$$D^* F(\bar{x},\bar{y})(v) := \left\{ u \in \mathbb{R}^n \middle| (u,-v) \in N_{\operatorname{gph} F}(\bar{x},\bar{y}) \right\},\ v \in \mathbb{R}^m$$

Subdifferential of $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty,\infty]$ at $\bar{x} \in \operatorname{dom}\varphi$ is

$$\partial\varphi(\bar{x}) := \left\{ v \in \mathbb{R}^n \middle| \ (v,-1) \in N_{\operatorname{epi}\varphi}\big(\bar{x},\varphi(\bar{x})\big) \right\}$$

Combined second-order subdifferential and limiting second-order subdifferential of φ at $\bar{x}$ relative to $\bar{v} \in \partial\varphi(\bar{x})$ are

$$\breve{\partial}^2\varphi(\bar{x}, \bar{x})(u) := \big(\widehat{D}^*\partial\varphi\big)(\bar{x}, \bar{v})(u), \quad u \in \mathbb{R}^n$$

$$\partial^2\varphi(\bar{x}, \bar{x})(u) := \big(D^*\partial\varphi\big)(\bar{x}, \bar{v})(u), \quad u \in \mathbb{R}^n$$

Note that, we have the inclusion

$$\breve{\partial}^2\varphi(\bar{x}, \bar{x})(u) \subset \partial^2\varphi(\bar{x}, \bar{x})(u) \quad \text{for all} \quad u \in \mathbb{R}^n.$$

If $\varphi \in \mathcal{C}^2$-smooth around $\bar{x}$, then

$$\breve{\partial}^2\varphi(\bar{x}, \bar{x})(u) = \partial^2\varphi(\bar{x}, \bar{v})(u) = \big\{\nabla^2\varphi(\bar{x})u\big\}, \quad u \in \mathbb{R}^n$$

Prox-regularity

Definition

$\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is prox-regular[a][b] at $\bar{x} \in \operatorname{dom}\varphi$ for $\bar{v} \in \partial\varphi(\bar{x})$ if φ is lower semicontinuous and there are $\varepsilon > 0$ and $\rho \geq 0$ such that for all $x \in \mathbb{B}_\varepsilon(\bar{x})$ with $\varphi(x) \leq \varphi(\bar{x}) + \varepsilon$ we have

$$\varphi(x) \geq \varphi(u) + \langle \bar{v}, x - u\rangle - \frac{\rho}{2}\|x - u\|^2 \;\forall\, (u, v) \in (\operatorname{gph}\partial\varphi) \cap \mathbb{B}_\varepsilon(\bar{x}, \bar{v})$$

[a] R. A. Poliquin and R. T. Rockafellar, Prox-regular functions in variational analysis, Trans. Amer. Math. Soc. 348, 1805–1838 (1996)

[b] R. T. Rockafellar and R. J-B. Wets, Variational Analysis, Springer (1998)

φ is subdifferentially continuous at $\bar{x}$ for $\bar{v}$ if the convergence $(x_k, v_k) \to (\bar{x}, \bar{v})$ with $v_k \in \partial\varphi(x_k)$ yields $\varphi(x_k) \to \varphi(\bar{x})$. If both properties hold, φ is continuously prox-regular. This is the major class in second-order variational analysis

Variationally Convex Functions

Variational Convexity

An l.s.c. function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ is called variationally convex at $\bar{x}$ for $\bar{v} \in \partial\varphi(\bar{x})$ if for some convex neighborhood $U \times V$ of $(\bar{x}, \bar{v})$ there exist an l.s.c. convex function $\psi \leq \varphi$ on U and a number $\varepsilon > 0$ such that

$$(U_\varepsilon \times V) \cap \operatorname{gph} \partial\varphi = (U \times V) \cap \operatorname{gph} \partial\psi \quad \text{and} \quad \varphi(x) = \psi(x), \tag{1}$$

at the common elements (x, v), where $U_\varepsilon := \{x \in U \mid \varphi(x) < \varphi(\bar{x}) + \varepsilon\}$. We say that φ is variationally strongly convex at $\bar{x}$ for $\bar{v}$ with modulus $\sigma > 0$ if (1) holds with ψ being strongly convex on U with this modulus.

Some first-order characterizations of variationally convex functions can be found in [5]. The characterizations via augmented Lagrangian functions and second subderivative can be found in [6].

[5] R. T. Rockafellar, *Variational convexity and local monotonicity of subgradient mappings*, Vietnam J. Math., 47 (2019), 547–561.

[6] R. T. Rockafellar, *Augmented Lagrangians and hidden convexity in sufficient conditions for local optimality*, Math. Program., 192 (2022), DOI 10.1007/s10107-022-01768-w.

3. Variational Convexity via Moreau Envelopes

Moreau Envelopes and Proximal Mappings

[7]Given an extended-real-valued, proper, l.s.c. function $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ and a positive number γ, the *Moreau envelope* $e_\gamma\varphi$ and the *proximal mapping* $\mathrm{Prox}_{\gamma\varphi}$ are defined by, respectively,

$$e_\gamma\varphi(x) := \inf_{y\in\mathbb{R}^n} \left\{ \varphi(y) + \frac{1}{2\gamma}\|y-x\|^2 \right\}, \tag{2}$$

$$\mathrm{Prox}_{\gamma\varphi}(x) := \operatorname*{argmin}_{y\in\mathbb{R}^n} \left\{ \varphi(y) + \frac{1}{2\gamma}\|y-x\|^2 \right\}. \tag{3}$$

[7]Rockafellar, R.T., Wets R.J-B.: Variational Analysis. Springer, Berlin (1998)

Characterization of Variational Convexity via Moreau Envelopes

Theorem 1[8]**:** Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an l.s.c. and prox-bounded function with $\bar{x} \in \operatorname{dom} \varphi$ and $\bar{v} \in \partial\varphi(\bar{x})$. The following assertions are equivalent:

(i) φ is variationally convex at $\bar{x}$ for $\bar{v}$.

(ii) φ is prox-regular at $\bar{x}$ for $\bar{v}$, and the Moreau envelope $e_\lambda \varphi$ is locally convex around $\bar{x} + \lambda\bar{v}$ for small $\lambda > 0$.

[8] P. D. Khanh, B. S. Mordukhovich, V. T. Phat, Variational convexity of functions and variational sufficiency in optimization, arXiv: 2208.14399

Quantitative Characterization of Variational Strong Convexity via Moreau Envelopes

Theorem 2[9]**:** Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an l.s.c. and prox-bounded function with $\bar{x} \in \operatorname{dom} \varphi$ and $\bar{v} \in \partial\varphi(\bar{x})$. The following assertions are equivalent:

(i) φ is variationally strongly convex at $\bar{x}$ for $\bar{v}$ with modulus $\sigma > 0$.

(ii) φ is prox-regular at $\bar{x}$ for $\bar{v}$ and $e_\lambda\varphi$ is locally strongly convex around $\bar{x} + \lambda\bar{v}$ with modulus $\frac{\sigma}{1+\sigma\lambda}$ for all numbers $\lambda > 0$ sufficiently small.

[9] P. D. Khanh, B. S. Mordukhovich, V. T. Phat, Variational convexity of functions and variational sufficiency in optimization, arXiv: 2208.14399

Equivalence Between Variational Strong Convexity and Local Strong Convexity of Moreau envelopes

Theorem 3[10]**:** Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an l.s.c. and prox-bounded function with $\bar{x} \in \operatorname{dom} \varphi$ and $\bar{v} \in \partial\varphi(\bar{x})$. The following assertions are equivalent:

(i) φ is variationally strongly convex at $\bar{x}$ for $\bar{v}$.

(ii) φ is prox-regular at $\bar{x}$ for $\bar{v}$ and $e_\lambda\varphi$ is locally strongly convex around $\bar{x} + \lambda\bar{v}$ for all numbers $\lambda > 0$ sufficiently small.

[10] P. D. Khanh, B. S. Mordukhovich, V. T. Phat, Variational convexity of functions and variational sufficiency in optimization, arXiv: 2208.14399

4. Coderivative-Based Characterizations of Variational Convexity

Second-order Subdifferential Characterizations of Variational Convexity

Theorem 4:[11] Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be subdifferentially continuous at $\bar{x} \in \operatorname{dom} \varphi$ and $\bar{v} \in \partial\varphi(\bar{x})$. Then the following assertions are equivalent:
(i) φ is variationally convex at $\bar{x}$ for $\bar{v}$.
(ii) φ is prox-regular at $\bar{x}$ for $\bar{v}$ and there exist neighborhoods U of $\bar{x}$ and V of $\bar{v}$ such that

$$\langle z, w\rangle \geq 0 \ \text{ whenever } z \in \breve{\partial}^2\varphi(x,y)(w),\ (x,y) \in \operatorname{gph}\partial\varphi \cap (U \times V),\ w \in \mathbb{R}^n. \tag{4}$$

(iii) φ is prox-regular at $\bar{x}$ for $\bar{v}$ and there exist neighborhoods U of $\bar{x}$, and V of $\bar{v}$ such that

$$\langle z, w\rangle \geq 0 \ \text{ whenever } \ z \in \partial^2\varphi(x,y)(w),\ (x,y) \in \operatorname{gph}\partial\varphi \cap (U \times V),\ w \in \mathbb{R}^n. \tag{5}$$

[11] P. D. Khanh, B. S. Mordukhovich, V. T. Phat, Variational convexity of functions and variational sufficiency in optimization, arXiv: 2208.14399

Second-order Characterizations of Variational Strong Convexity

Theorem 5: [12] Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be subdifferentially continuous at $\bar{x} \in \operatorname{dom}\varphi$ and $\bar{v} \in \partial\varphi(\bar{x})$. Then the following assertions are equivalent:
(i) φ is variationally strongly convex at $\bar{x}$ for $\bar{v}$ with modulus $\sigma > 0$.
(ii) φ is prox-regular at $\bar{x}$ for $\bar{v}$ and there exist neighborhoods U of $\bar{x}$ and V of $\bar{v}$ such that

$$\langle z, w\rangle \geq \sigma\|w\|^2 \text{ whenever } z \in \breve{\partial}^2\varphi(x,y)(w),\ (x,y) \in \operatorname{gph}\partial\varphi \cap (U \times V),\ w \in \mathbb{R}^n. \tag{6}$$

(iii) φ is prox-regular at $\bar{x}$ for $\bar{v}$ and there are neighborhoods U of $\bar{x}$ and V of $\bar{v}$ such that

$$\langle z, w\rangle \geq \sigma\|w\|^2 \text{ whenever } z \in \partial^2\varphi(x,y)(w),\ (x,y) \in \operatorname{gph}\partial\varphi \cap (U \times V),\ w \in \mathbb{R}^n. \tag{7}$$

Furthermore, the strong variational convexity in (i) with some modulus $\sigma > 0$ is equivalent to the prox regularity of φ at $\bar{x}$ for $\bar{v}$ together with the fulfillment of the pointbased condition

$$\langle z, w\rangle > 0 \text{ whenever } z \in \partial^2\varphi(\bar{x},\bar{v})(w),\ w \neq 0. \tag{8}$$

[12] P. D. Khanh, B. S. Mordukhovich, V. T. Phat, Variational convexity of functions and variational sufficiency in optimization, arXiv: 2208.14399

5. Variational Sufficiency in Composite Optimization

Variational Sufficiency in Optimization

Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a lower semicontinuous function. Suppose that $\bar{x} \in \operatorname{dom} \varphi$ is a stationary point, i.e., $0 \in \partial\varphi(\bar{x})$. We have the following implications:

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- variational convexity of φ at $\bar{x} \Longrightarrow \bar{x}$ is a local minimizer.

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- variational convexity of φ at $\bar{x} \Longrightarrow \bar{x}$ is a local minimizer.
- variational strong convexity of φ at $\bar{x} \Longrightarrow \bar{x}$ is a tilt-stable local minimizer.

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Definition

Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$, and consider the unconstrained optimization problem:

$$\text{minimize} \quad \varphi(x) \quad \text{subject to} \quad x \in \mathbb{R}^n. \tag{9}$$

It is said that the variational sufficient condition for local optimality in (9) holds at $\bar{x}$ if φ is variationally convex at $\bar{x}$ for $0 \in \partial\varphi(\bar{x})$. If φ is variationally strongly convex at $\bar{x}$ for 0 with modulus $\sigma > 0$, then we say that the strong variational sufficient condition for local optimality at $\bar{x}$ holds with modulus σ.

Here we consider the class of composite optimization problems given by:

$$\text{minimize } \varphi(x) := \varphi_0(x) + \psi\big(g(x)\big) \quad \text{subject to } \; x \in \mathbb{R}^n, \tag{10}$$

where $\psi : \mathbb{R}^m \to \overline{\mathbb{R}}$ is an extended-real-valued l.s.c. function, $\varphi_0 : \mathbb{R}^n \to \mathbb{R}$ is a $\mathcal{C}^2$-smooth function, and g is a $\mathcal{C}^2$-smooth mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$.

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$\Longrightarrow$ The characterizations of variational sufficiency in (10)?

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$\Longrightarrow$ The characterizations of variational sufficiency in (10)?
For each $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$ define the set of multipliers

$$\Lambda(x, v) := \left\{ y \in \mathbb{R}^m \;\middle|\; v = \nabla\varphi_0(x) + \nabla g(x)^* y, \; y \in \partial\psi\big(g(x)\big) \right\}. \quad (11)$$

Theorem 6: Let $\bar{x} \in \mathbb{R}^n$ be a stationary point of the composite optimization problem at which $\operatorname{rank} \nabla g(\bar{x}) = m$ and hence there exists a unique vector $\bar{y} \in \mathbb{R}^m$ with

$$\nabla \varphi_0(\bar{x}) + \nabla g(\bar{x})^* \bar{y} = 0 \text{ and } \bar{y} \in \partial \psi\big(g(\bar{x})\big). \tag{12}$$

Suppose in addition that ψ is subdifferentially continuous at $g(\bar{x})$ for $\bar{y}$. Then we have the following assertions:

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Suppose in addition that ψ is subdifferentially continuous at $g(\bar{x})$ for $\bar{y}$. Then we have the following assertions:
(i) The variational sufficiency holds at $\bar{x}$ if and only if ψ is prox-regular at $g(\bar{x})$ for $\bar{y}$ and there exist neighborhoods U of $\bar{x}$ and V of 0 such that

$$\langle \nabla^2 \varphi_0(x) w, w \rangle + \langle \nabla^2 \langle y, g \rangle (x) w, w \rangle + \langle u, \nabla g(x) w \rangle \geq 0 \tag{13}$$

for all $x \in U$, $v \in V$, $y \in \Lambda(x, v)$, $u \in \partial^2 \psi(g(x), y)(\nabla g(x) w)$, $w \in \mathbb{R}^n$, where $\Lambda(x, v)$ is a singleton in this case.

Theorem 6: Let $\bar{x} \in \mathbb{R}^n$ be a stationary point of the composite optimization problem at which $\operatorname{rank} \nabla g(\bar{x}) = m$ and hence there exists a unique vector $\bar{y} \in \mathbb{R}^m$ with

$$\nabla \varphi_0(\bar{x}) + \nabla g(\bar{x})^* \bar{y} = 0 \ \text{ and } \ \bar{y} \in \partial \psi(g(\bar{x})). \tag{12}$$

Suppose in addition that ψ is subdifferentially continuous at $g(\bar{x})$ for $\bar{y}$. Then we have the following assertions:

(i) The variational sufficiency holds at $\bar{x}$ if and only if ψ is prox-regular at $g(\bar{x})$ for $\bar{y}$ and there exist neighborhoods U of $\bar{x}$ and V of 0 such that

$$\langle \nabla^2 \varphi_0(x)w, w\rangle + \langle \nabla^2 \langle y, g\rangle(x)w, w\rangle + \langle u, \nabla g(x)w\rangle \geq 0 \tag{13}$$

for all $x \in U$, $v \in V$, $y \in \Lambda(x, v)$, $u \in \partial^2 \psi(g(x), y)(\nabla g(x)w)$, $w \in \mathbb{R}^n$, where $\Lambda(x, v)$ is a singleton in this case.

(ii) The strong variational sufficiency holds at $\bar{x}$ with modulus $\sigma > 0$ if and only if ψ is prox-regular at $g(\bar{x})$ for $\bar{y}$ and there exist neighborhoods U of $\bar{x}$ and V of 0 such that

$$\langle \nabla^2 \varphi_0(x)w, w\rangle + \langle \nabla^2 \langle y, g\rangle(x)w, w\rangle + \langle u, \nabla g(x)w\rangle \geq \sigma \|w\|^2 \tag{14}$$

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Furthermore, the strong variational sufficiency in (ii) with some modulus $\sigma > 0$ is equivalent to the prox-regularity of ψ at $g(\bar{x})$ for $\bar{y}$ together with the fulfillment of the pointbased condition

$$\langle \nabla^2 \varphi_0(\bar{x})w, w\rangle + \langle \nabla^2 \langle \bar{y}, g\rangle(\bar{x})w, w\rangle + \langle u, \nabla g(\bar{x})w\rangle > 0 \tag{15}$$

whenever $u \in \partial^2\psi\big(g(\bar{x}), \bar{y}\big)\big(\nabla g(\bar{x})w\big)$ and $w \neq 0$.

Variational Sufficiency Without Full Rank Assumption

An l.s.c. function $\theta : \mathbb{R}^n \to \overline{\mathbb{R}}$ is strongly amenable at $\bar{x}$ if there exists neighborhood U of $\bar{x}$ on which θ can be represented in the composition form $\theta = \psi \circ g$ with a $\mathcal{C}^2$-smooth mapping $g\colon U \to \mathbb{R}^m$ and a proper l.s.c. convex function $\psi : \mathbb{R}^m \to \overline{\mathbb{R}}$ such that the following first-order qualification condition holds:

$$\partial^\infty \psi(\bar{z}) \cap \ker \nabla g(\bar{x})^* = \{0\} \quad \text{with} \quad \bar{z} := g(\bar{x}) \tag{16}$$

The second-order qualification condition (SOQC) for problem (10) at $\bar{x}$, which is formulated as follows:

$$\partial^2 \psi(\bar{z}, \bar{y})(0) \cap \ker \nabla g(\bar{x})^* = \{0\} \quad \text{with} \quad \bar{y} \in \partial \psi(\bar{z}) \quad \text{and} \quad \bar{z} := g(\bar{x}). \tag{17}$$

Theorem 7: Let $\bar{x} \in \mathbb{R}^n$ be a stationary point of the composite optimization problem. Suppose in addition that ψ and g be mappings from the composite representation of a strongly amenable function at $\bar{x}$ and that the second-order qualification condition is satisfied at $\bar{x}$. Then we have the following assertions:

Theorem 7: Let $\bar{x} \in \mathbb{R}^n$ be a stationary point of the composite optimization problem. Suppose in addition that ψ and g be mappings from the composite representation of a strongly amenable function at $\bar{x}$ and that the second-order qualification condition is satisfied at $\bar{x}$. Then we have the following assertions:

(i) The variational sufficiency holds at $\bar{x}$ if there exist neighborhoods U of $\bar{x}$ and V of 0 such that (13) is satisfied for all $x \in U$, $v \in V$, $y \in \Lambda(x, v)$, $u \in \partial^2\psi(g(x), y)(\nabla g(x)w)$, and $w \in \mathbb{R}^n$.

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(ii) The strong variational sufficiency holds at $\bar{x}$ with modulus $\sigma > 0$ if there exist neighborhoods U of $\bar{x}$ and V of 0 such that the neighborhood condition (14) is satisfied for all $x \in U$, $v \in V$, $y \in \Lambda(x, v)$, $u \in \partial^2\psi(g(x), y)(\nabla g(x)w)$, and $w \in \mathbb{R}^n$.

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(iii) The strong variational sufficiency holds at $\bar{x}$ if the pointbased condition (15) is satisfied for any $\bar{y} \in \Lambda(\bar{x}, 0)$.

Theorem 8: In addition to the assumptions of Theorem 7, suppose that
(a) either ψ is piecewise linear,
(b) or ψ is of class

$$\psi(z) := \sup_{v \in P} \left\{ \langle v, z \rangle - \frac{1}{2} \langle Qv, v \rangle \right\}, \tag{18}$$

where $P \subset \mathbb{R}^m$ is a nonempty polyhedral set, Q is positive-definite, and the inner mapping g is open around $\bar{x}$.

Then all the three characterizations (i)–(iii) of Theorem 7 hold.

6 Applications to Nonlinear Programming

The conventional model of nonlinear programming (NLP) is formulated as follows:

$$\text{minimize} \quad \varphi_0(x) \;\; \text{subject to} \;\; \begin{cases} \varphi_i(x) \le 0 & \text{for} \;\; i = 1, \ldots, s, \\ \varphi_i(x) = 0 & \text{for} \;\; i = s+1, \ldots, m, \end{cases} \tag{19}$$

where φ_i, $i = 0, \ldots, m$, are $\mathcal{C}^2$-smooth functions around the references points. Problem (19) can be obviously written in the form of composite optimization (10) with $\psi = \delta_\Omega$, where Ω is given by

$$\Omega := \left\{ u \in \mathbb{R}^m \;\middle|\; u_i \le 0 \;\; \text{for} \;\; i = 1, \ldots, s \;\; \text{and} \;\; u_i = 0 \;\; \text{for} \;\; i = s+1, \ldots, m \right\}, \tag{20}$$

and where $g(x) := (\varphi_1(x), \ldots, \varphi_m(x))$.

Lagrangian functions: $L(x,y) = \varphi_0(x) + y_1\varphi_1(x) + \ldots + y_m\varphi_m(x)$.

For each $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$ consider the subspace

$$S(x,y) := \left\{ w \in \mathbb{R}^n \;\middle|\; \langle \nabla\varphi_i(x), w\rangle = 0 \;\text{ for }\; i \in I_+(x,y) \cup \{s+1,\ldots,m\} \right\} \tag{21}$$

together with the index collections

$$I_+(x,y) := \left\{ i \in I(x) \;\middle|\; y_i > 0 \right\} \;\text{ and }\; I(x) := \left\{ i \in \{1,\ldots,s\} \;\middle|\; \varphi_i(x) = 0 \right\}. \tag{22}$$

Variational Sufficiency in Nonlinear Programming

Corollary 9: Let $\bar{x}$ be a feasible solution to the NLP problem satisfying the first-order optimality condition under the fulfillment of LICQ at $\bar{x}$. Then we have the following assertions:

(i) The variational sufficiency holds at $\bar{x}$ if and only if there exist neighborhoods U of $\bar{x}$, V of 0 such that

$$\langle \nabla^2_{xx} L(x,y)w, w\rangle \geq 0 \quad \text{whenever} \quad x \in U,\ v \in V, \quad \text{and} \quad w \in S(x,y), \tag{23}$$

where $y \in \mathbb{R}^s_+ \times \mathbb{R}^{m-s}$ is a unique solution to the system

$$\nabla_x L(x,y) = v, \quad y_1\varphi_1(x) + \ldots + y_m\varphi_m(x) = 0.$$

(ii) The strong variational sufficiency holds at $\bar{x}$ with modulus $\sigma > 0$ if and only if there exist neighborhoods U of $\bar{x}$, V of 0 such that

$$\langle \nabla^2_{xx} L(x,y)w, w\rangle \geq \sigma\|w\|^2 \quad \text{whenever} \quad x \in U,\ v \in V, \quad \text{and} \quad w \in S(x,y), \tag{24}$$

where $y \in \mathbb{R}^s_+ \times \mathbb{R}^{m-s}$ is a unique solution to the system

$$\nabla_x L(x,y) = v, \quad y_1\varphi_1(x) + \ldots + y_m\varphi_m(x) = 0.$$

(iii) The strong variational sufficiency holds at $\bar{x}$ if and only if

$$\langle \nabla_{xx}^2 L(\bar{x}, \bar{y})w, w\rangle > 0 \quad \text{whenever} \quad \bar{y} \in \Lambda(\bar{x}, 0) \quad \text{and} \quad w \in S(\bar{x}, \bar{y}) \setminus \{0\}, \tag{25}$$

where $\bar{y} \in \mathbb{R}_+^s \times \mathbb{R}^{m-s}$ is a unique solution to the system

$$\nabla_x L(\bar{x}, y) = v, \quad y_1\varphi_1(\bar{x}) + \ldots + y_m\varphi_m(\bar{x}) = 0.$$

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THANK YOU FOR YOUR ATTENTION