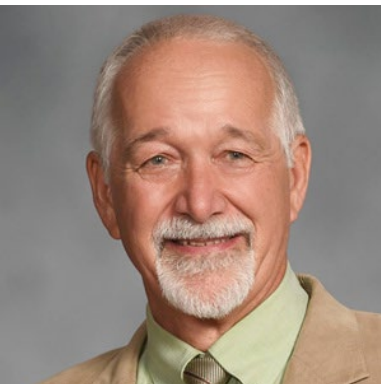


Quantum Interior Point Methods (QIPMs) with Iterative Refinement for Linear and Semidefinite Optimization

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ISE

Industrial and
Systems Engineering



QC & QC

Quantum Computers are here
& we need more optimizers & OR in
Quantum Computing

UN: **IYQST** -- INFORMS: **QCOR**

The United Nation declared 2025
as the International Year of
Quantum Science and Technology

Quantum Computing (QC) is Here!

NISQ

Is QC really computing?

Hybrid classic-QC

Why is it important?

Why QC optimization?

Why to get involved?



NISQ devices v/s
quantum annealers

Pro's and Con's

Opportunities
& challenges

Quantum supremacy
Quantum advantage

QC Capacity Grows Exponentially-IBM

Scaling IBM Quantum technology

Superconducting qubits



IBM Q System One (Released)

(In development)

Next family of IBM Quantum systems

2019

2020

2021

2022

2023

and beyond

27 qubits

65 qubits

127 qubits

433 qubits

1,121 qubits

Path to 1 million qubits

Falcon

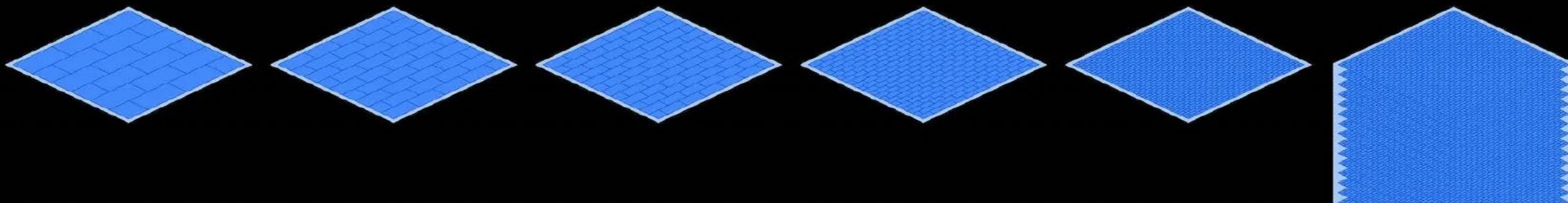
Hummingbird

Eagle

Osprey

Condor

and beyond
Large scale systems



Key advancement

Key advancement

Key advancement

Key advancement

Key advancement

Key advancement

Optimized lattice

Scalable readout

Novel packaging and controls

Miniaturization of components

Integration

Build new infrastructure,
quantum error correction

Quantum Computing Challenges

We are having **NISQ**
Noisy Intermediate-Scale Quantum devices
QC challenges

The promise:
Fault Tolerant Logical Qubits
Are coming!

(Not only on QC simulators)

Conic Linear Optimization

Primal-dual pair of CLO problems is given as

$$\begin{array}{ll} (P) \min & c^T x \\ & \text{s.t. } Ax - b \in \mathcal{C}_1 \\ & \quad \quad x \in \mathcal{C}_2 \\ (D) \max & b^T y \\ & \text{s.t. } c - A^T y \in \mathcal{C}_2^* \\ & \quad \quad y \in \mathcal{C}_1^*, \end{array}$$

where $b, y \in \mathbb{R}^m$, $c, x \in \mathbb{R}^n$, $A : m \times n$ matrix, $\mathcal{C}_1, \mathcal{C}_2$ are convex cones and $\mathcal{C}_i^* = \{s \in \mathbb{R}^n : x^T s \geq 0, \forall x \in \mathcal{C}_i\}$ are the dual cones for $i = 1, 2$.

These are solvable efficiently (in polynomial time) by using Interior Point Methods. LO is based on polyhedral cones.

Be careful! Perfect duality, strict complementarity lost.

Are all convex cones good???

NOT

IPMs ~ 70-40: Why IPMs “failed” by ~ 1970

Computers ~ 1970

IBM360: top computer that time

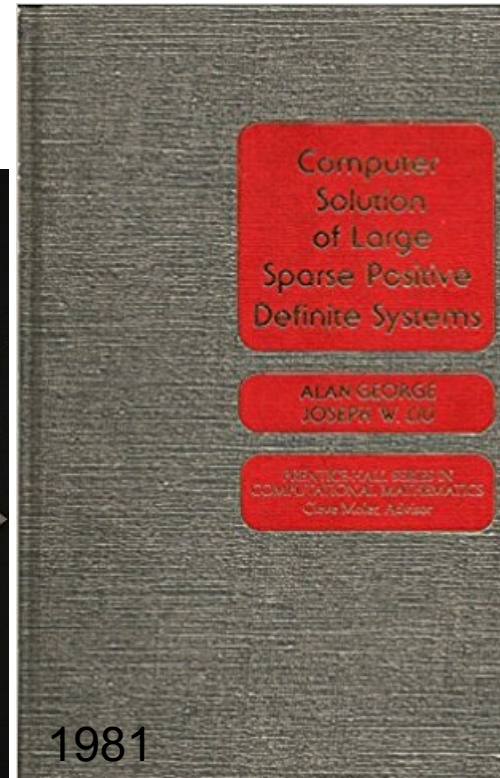
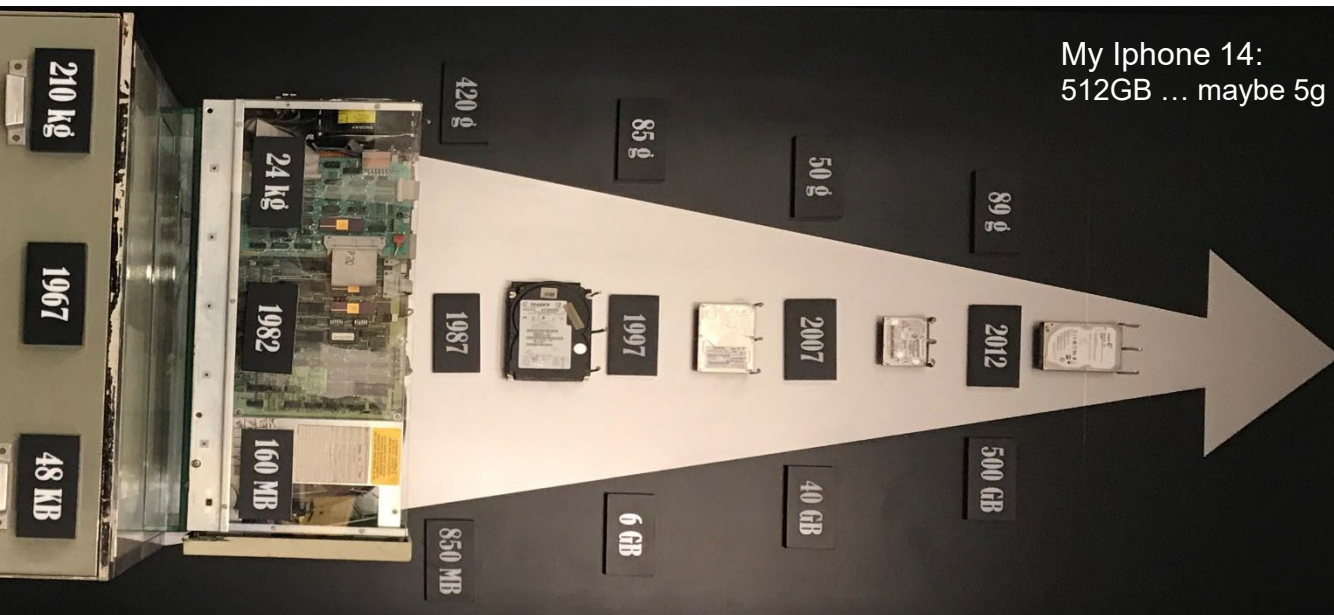
- Gym size room, Operators
- Memory: **128-256 KB**
- Hard disk: **7.2MB-400MB**
- Large Storage: **tape drives**
- Input: punch cards

- NO:**
- Double precision arithmetic
 - Regularization
 - Sparse matrix methods
 - Automatic differentiation



40: What made the IPM revolution possible?

- Plenty of memory
 - Complexity Theory \longrightarrow Control of Algorithms
 - Polynomial time algorithm – Ellipsoid Method
 - Mainframe \longrightarrow Workstations \longrightarrow Desktop PC
 - Sparse matrix theory and packages
 - Automatic Differentiation
- All stars lined up JIT for the “IPM Revolution”**



The New York Times

NEW YORK, WEDNESDAY, NOVEMBER 2, 1981

A Soviet Discovery Rocks World of Mathematics

By MALCOLM W. BROWNE

A surprise discovery by an obscure Soviet mathematician has rocked the world of mathematics and computer analysis, and experts have begun exploring practical applications.

Mathematicians describe the discovery by L.G. Khachiyan as a method by which computers can find guaranteed solutions to a class of very difficult problems that have hitherto been tackled on a kind of trial-and-error basis.

Apart from its profound theoretical interest, the discovery may be applicable to weather prediction, complicated industrial processes, petroleum refining, the scheduling of workers at large factories, secret codes and many other things.

"I have been deluged with calls from virtually every department of government for an interpretation of the significance of this," a leading expert on computer methods, Dr. George B. Shostak of Stanford University, said in an interview.

The solution of mathematical problems by computer must be broken down into a series of steps. One class of problems, so-called linear programming, involves solving a set of linear equations and inequalities to find the best solution to a problem. The Russian discovery offers a way by which the number of steps in a solution can be drastically reduced. It also offers the mathematician a way of learning quickly whether a problem has a solution or not, without having to complete the entire intensive computation that may be required.

According to the American journal Science, the discovery was published in the November issue of the journal.

Continued on Page A26, Column 3

Russian's Surprise Problem-Solving Discovery Reported

Called from Prague, the discovery was first reported last January in the Soviet journal Doklady. It was also in the West, however, partly because the author, Leonid Khachiyan, was a mathematician at the University of California at Berkeley, in the month before then, two mathematicians working at Harvard University, along with other colleagues, had been studying the Russian paper by Dr. Eugene Lashin of the University of California at Berkeley, in the month before then, two mathematicians working at Harvard University, along with other colleagues, had been studying the Russian paper.

The Soviet discovery promises to provide a way, computers to solve a class of problems related to the "Traveling Salesman Problem," one of the most intractable and difficult mathematical problems. The class of problems is too difficult for the fastest present computers to solve, but it is always possible to derive such solutions for almost any case.

The theoretical discovery, however, cannot be used to solve such problems in practice, because it is not possible to find a solution in a finite amount of time.

Practical applications of the discovery are being explored by a group of computer scientists at the University of California at Berkeley.

The discovery is a breakthrough in the field of linear programming, a branch of mathematics that deals with the optimization of a linear function subject to a set of linear constraints. The Russian discovery offers a way by which the number of steps in a solution can be drastically reduced. It also offers the mathematician a way of learning quickly whether a problem has a solution or not, without having to complete the entire intensive computation that may be required.

According to the American journal Science, the discovery was published in the November issue of the journal.

Continued on Page A26, Column 3

QIPMs

Quantum Interior Point Methods

We also have **Quantum Central Path Methods** ...

Thanks to joint work with the QCOL team:

Brandon Augustino, *Mohammadhossein Mohamadisiahroudi*, Ramin Fakhimi,

Arielle Carr, Pouya Sampourmahani, Zeguan Wu, Luis Zuluaga

Giacomo Nannicini (USC), Xiaodi Wu, Jiaqi Leng (UMd)

Quantum Interior Point Methods (QIPMs)

- QIPMs are Hybrid Classic-Quantum algorithms
- For solving real world optimization problems

Needed:

Efficient, reliable

Quantum Computers & Quantum Linear Algebra

Data flow in hybrid classic-quantum algorithms:

- Classic → QRAM → QSolve → QTomography → Classic solution
- 

Quantum Linear Algebra

Need to solve Newton svstems – “accurately, fast”

$$Mz = \sigma, M|z\rangle = |\sigma\rangle$$

Algorithm	Complexity
Factorization (e.g. Cholesky)	$\mathcal{O}(p^3)$
Conjugate Gradient	$\mathcal{O}(pd\sqrt{\kappa}\log(\frac{1}{\epsilon}))$
HHL + QTA	$\mathcal{O}(\text{polylog}(p) \frac{d^2 \kappa^2 \ \sigma\ }{\ M\ _\epsilon}) \times \mathcal{O}(\frac{p\ \sigma\ }{\ M\ _\epsilon})$
VTAA-HHL + QTA	$\mathcal{O}(\text{polylog}(p) \frac{d^2 \kappa \ \sigma\ }{\ M\ _\epsilon}) \times \mathcal{O}(\frac{p\ \sigma\ }{\ M\ _\epsilon})$
QLSA (Wossnig, et al. 2018) + QTA	$\mathcal{O}(\text{polylog}(p) \frac{\kappa \ \sigma\ }{\ M\ _\epsilon}) \times \mathcal{O}(\frac{p\ \sigma\ }{\ M\ _\epsilon})$
QLSA (Childs, et al. 2017) + QTA	$\mathcal{O}(\text{polylog}(\frac{p\kappa \ \sigma\ }{\ M\ _\epsilon}) d\kappa) \times \mathcal{O}(\frac{p\ \sigma\ }{\ M\ _\epsilon})$
QLSA (Carrera, et al. 2020) + QTA	$\mathcal{O}(\text{polylog}(\frac{p\kappa \ \sigma\ }{\ M\ _\epsilon}) d\kappa) \times \mathcal{O}(\frac{p\ \sigma\ }{\ M\ _\epsilon})$
QLSA (Chakraborty, et al. 2018) + QTA	$\mathcal{O}(\text{polylog}(\frac{p\ \sigma\ }{\epsilon}) \kappa \ M\ _F) \times \mathcal{O}(\frac{p\ \sigma\ }{\epsilon})$

* κ is the condition number of M ,

d is maximum number of non-zero elements in each row and column in M ,

p is the number of rows/columns of M , and

ϵ is the error of Linear Equation Solver.

with tomography

Novel IR-IF-QIPMs

QLSAs have **quantum advantage** w.r.t. the dimension, and **quantum disadvantage** w.r.t. **condition number & accuracy** of solution

Tomography (extracting the solution) is costly

... **direct use Leads to Inexact Infeasible QIPMs (II-QIPMs)**



Best complexity QIPMs feature:

- **Novel Inexact Feasible IPMs (IF-QIPMs)**
- Error (inexactness) & condition number dependence need to be addressed
- **Solution: Iterative Refinement (IR) “inside and outside”**
 - IR inside: for the Newton System
 - IR outside: at the problem level – limits condition number
- **IR-IF-QIPMs: High precision solution through low precision computation.**

Semidefinite Optimization

Let

- $b \in \mathbb{R}^m$
- matrices $A_1, \dots, A_m, C \in \mathcal{S}^n$

Then, the primal-dual Semidefinite Optimization (SDO) pair is given by:

$$z_P = \inf_X \{ \text{tr}(CX) : \text{tr}(A_i X) = b_i, \forall i \in [m], X \succeq 0 \}$$

$$z_D = \sup_{y, S} \left\{ b^\top y : \sum_{i=1}^m y_i A_i + S = C, S \succeq 0, y \in \mathbb{R}^m \right\}$$

where

- $[m] = \{1, \dots, m\}$
- $S = C - \sum_{i \in [m]} y_i A_i \succeq 0$ is the slack matrix of the dual problem
- \mathcal{S}^n is the cone of $n \times n$ symmetric matrices
- We assume that the matrices A_1, \dots, A_m are linearly independent

SDO

First SDO formed by
Bellman-Fang 1963
Craven-Mond 1981

**General Theory of
IPMs by Nesterov
and Nemirovskii**

Central Path, Newton System

For $\nu > 0$, assuming *interior point condition*, and linear independence of the matrices $A^{(i)}$, the central path is the solution set of equation system

$$\begin{aligned} \operatorname{tr}(A_i X) &= b_i \quad \forall i \in [m], \quad X \succ 0 \\ \sum_{i \in [m]} y_i A_i - S &= C, \quad S \succ 0 \\ XS &= \nu I, \end{aligned}$$

- Linearizing the central path eqns gives the Newton linear system:

$$\begin{aligned} X \Delta S + \Delta X S &= \sigma \nu I - XS \\ \Delta S \in L \quad \Delta X &\in L^\perp. \end{aligned}$$

**No symmetric solution
for this linear system.**

Symmetrization needed!

First proposal for a QIPM

- Kerenidis and Prakash (2018) made the first effort at a quantum interior point method
- Use quantum random access memory (QRAM) and block encodings to solve the Newton linear system
- Small neighborhood IPM:

$$\mathcal{N}_F(\gamma) = \left\{ (X, y, S) \in \mathcal{P}^0 \times \mathcal{D}^0 : \left\| X^{1/2} S X^{1/2} - \nu I \right\|_F \leq \gamma \nu \right\}$$

- They posit a worst case running time of

$$\tilde{\mathcal{O}}_{n, \kappa, \frac{1}{\xi}, \frac{1}{\epsilon}} \left(\frac{n^{2.5}}{\xi^2} \mu \kappa^3 \log \frac{1}{\epsilon} \right)$$

for SDPs

- The term $\tilde{\mathcal{O}}_{n, \kappa, \frac{1}{\xi}} \left(\frac{n^2 \kappa^2}{\xi^2} \right)$ comes from a tomography subroutine
- $\mu \leq n$ and κ are factors corresponding to the QLSA
- For solving LPs, the running time is

$$\tilde{\mathcal{O}}_{n, \kappa, \frac{1}{\xi}, \frac{1}{\epsilon}} \left(\frac{n^{1.5}}{\xi^2} \mu \kappa^3 \log \frac{1}{\epsilon} \right)$$

No symmetrization.
Not taking account QC error
→ inexact, infeasible QIPM.
Condition number dependence.

Symmetrizing the Newton System

Symmetrization is a linear transformation:

$$H_P(M) = \frac{1}{2} [PMP^{-1} + P^{-T}M^T P^T].$$

for a given invertible matrix P

The Alizadeh-Haeberly-Overton (AHO) direction is given by

$$P = I$$

The Nesterov-Todd (NT) direction is given by

$$P = W^{-1/2}$$

where

$$\begin{aligned} W &= S^{-1/2}(S^{1/2}XS^{1/2})^{1/2}S^{-1/2} \\ &= X^{-1/2}(X^{1/2}SX^{1/2})^{1/2}X^{-1/2} \end{aligned}$$

Additionally there is the so called HKM direction for which

$$P = S^{1/2}$$

Quantum Tomography introduces error.

Gives **Inexact Infeasible QIPMs (II-QIPM)**

$$\begin{pmatrix} 0 & \mathcal{A} & 0 \\ \mathcal{A}^\top & 0 & \mathcal{I} \\ 0 & \mathcal{E} & \mathcal{F} \end{pmatrix} \begin{pmatrix} \Delta y \\ \Delta X \\ \Delta S \end{pmatrix} = \begin{pmatrix} \xi_p \\ \xi_d \\ \sigma\nu I - H_p(XS) + \xi_c \end{pmatrix}$$

II-IPMs have worse iteration complexity than Feasible IPMs!

We cannot avoid tomography thus:

Need to develop

Inexact Feasible IPMs and QIPMs

Inexact-Feasible QIPMs for SDO

Feasibility implies
where

$$\Delta X \in \text{Null}(\mathcal{A}) \text{ and } \Delta S \in \mathcal{R}(\mathcal{A})$$

$$\text{Null}(\mathcal{A}) \equiv \text{nullspace of } \mathcal{A}_s$$

$$\mathcal{R}(\mathcal{A}) \equiv \text{rowspace of } \mathcal{A}_s$$

Let Q_2 be a basis of the null space of \mathcal{A}_s , then we set:

$$\text{svec}(\Delta X) = \text{svec}(Q_2 \Delta z)$$

$$\text{svec}(\Delta S) = \text{svec}(-\mathcal{A}_s^\top \Delta y)$$

So we get the new Newton system, called OSS (Orthogonal Subspace System)

$$\begin{bmatrix} \mathcal{E}Q_2 & \mathcal{F}(-\mathcal{A}_s^\top) \end{bmatrix} \begin{bmatrix} \Delta z \\ \Delta y \end{bmatrix} = \text{svec}(\sigma\nu I - H_p(XS)) \quad (\text{OSS})$$

Regardless of QLSA+QTA error, primal-dual feasibility is preserved!

Analysis of Feasible IPMs can be recovered!

Inexact-Feasible QIPM

Algorithm Inexact-Feasible Quantum Interior Point Method

Input: $\epsilon, \delta > 0$; $\sigma = 1 - \delta/\sqrt{n}$; $\beta, \gamma \in (0, 1)$

Output: An ϵ -optimal primal dual pair (X, y, S)

Choose $(X^{(0)}, y^{(0)}, S^{(0)}) \in \mathcal{N}_F(\gamma)$

Set $\nu^{(0)} = \frac{X^{(0)} \bullet S^{(0)}}{n}$

Compute bases for $\mathcal{R}(\mathcal{A}_s) = \mathcal{A}_s^\top$ and $\text{Null}(\mathcal{A}_s) = Q_2$

while $\nu > \epsilon$ **do**

1 $\nu^{(k)} \leftarrow \frac{\text{tr}(X^{(k)} S^{(k)})}{n}$

2 Compute matrices $(X^{(k)})^{-1}$ and $(S^{(k)})^{-1}$, $P^{(k)}$ classically.

3 Using block-encodings, solve Newton system to construct inexact search direction $|\Delta z^{(k)} \circ \Delta y^{(k)}\rangle$.

4 Obtain classical estimate $\overline{\Delta z}^{(k)}, \overline{\Delta y}^{(k)}$ of $\Delta z^{(k)}, \Delta y^{(k)}$ using vector state tomography.

5 Use classical estimate $|\overline{\Delta z}^{(k)}, \overline{\Delta y}^{(k)}\rangle$ to obtain classical estimate $\overline{\Delta X}^{(k)}, \overline{\Delta S}^{(k)}$ of $\Delta X^{(k)}, \Delta S^{(k)}$

6 Update current solution

$$X^{(k+1)} \leftarrow X^{(k)} + \overline{\Delta X}^{(k)}, \quad S^{(k+1)} \leftarrow S^{(k)} + \overline{\Delta S}^{(k)} \quad \text{and} \quad y^{(k+1)} \leftarrow y^{(k)} + \overline{\Delta y}^{(k)}$$

$$k \leftarrow k + 1$$

end

Theorem:

Data stored in QRAM.
The IF-QIPM requires at most

$$\tilde{\mathcal{O}}_{n, \kappa, \frac{1}{\epsilon}} \left(n^{3.5} \frac{\kappa^2}{\epsilon} \right)$$

QRAM access and

$$\tilde{\mathcal{O}}_{n, \kappa, \frac{1}{\epsilon}} \left(n^{4.5} \right)$$

arithmetic operations.

Advantage w.r.t. the Dimension.

Disadvantage w.r.t. condition number and precision

Quantum

Quantum

Iterative refinement for LSP

Solve the linear system $Mv=w$

Algorithm Iterative Refinement for LSPs

Input: Error tolerances $0 < \zeta \ll \xi < 1$, bound on norm of solution θ

Output: A ζ -precise solution v to $\frac{M}{\theta}v = \frac{w}{\theta}$

Normalize the data $(\widetilde{M}, \widetilde{w}) \leftarrow \theta^{-1}(M, w)$

Initialize: $x^{(0)} \leftarrow 0, r^{(0)} \leftarrow \widetilde{w}, \eta^{(0)} \leftarrow 1, k \leftarrow 0$

while $\|r\| > \zeta$ **do**

1 $\bar{u}^{(k)} \leftarrow$ **solve** $(\widetilde{M}, \eta^{(k)}r^{(k)})$ using $O_{LS}(\xi)$ **Quantum-solve**

2 $\tilde{u}^{(k+1)} \leftarrow \frac{\|\eta^{(k)}r^{(k)}\|}{\|\widetilde{M}\bar{u}^{(k)}\|} \bar{u}^{(k)}$

3 Update solution: $v^{(k+1)} \leftarrow v^{(k)} + \frac{1}{\eta^{(k)}} \tilde{u}^{(k)}$

4 Update residual $r^{(k+1)} \leftarrow \widetilde{w} - \widetilde{M}x^{(k+1)}$

5 Update scaling factor $\eta^{(k+1)} \leftarrow \|r^{(k+1)}\|^{-1}$

6 $k \leftarrow k + 1$

end

Here O_{LS} is defined as QLSA+QTA.

Theorem: For the iterates of the IR Algorithm we have

$$(a) \quad \eta^{(k)} \geq \frac{1}{\xi^k},$$

$$(b) \quad \left\| \widetilde{w} - \widetilde{M}v^{(k)} \right\| \leq \xi^k.$$

The IR Algorithm terminates in at most

$$\mathcal{O} \left(\log \left(\frac{1}{\zeta} \right) \right)$$

iterations.

IR for LSP: Complexity

Theorem: Let the problem data stored in QRAM, and use fixed precision oracle O_{LS} at each iteration. Then each iteration of the IR Algorithm requires at most

$$\mathcal{O}(d\kappa_M \cdot \text{polylog}(d, \kappa_M))$$

QRAM access and $\mathcal{O}(ds)$ classic arithmetic operations.

Corollary: Let the problem data stored in QRAM, and use fixed precision oracle O_{LS} at each iteration. Then setting $\zeta = \frac{\epsilon}{\theta}$, the IR Algorithm obtains an ϵ -precise solution of the linear system $Mv=w$ with at most

$$\mathcal{O}(d\kappa_M \cdot \text{polylog}(d, \kappa_M, \theta, \epsilon^{-1}))$$

QRAM access and $\mathcal{O}(ds \cdot \text{polylog}(d, \kappa_M, \theta, \epsilon^{-1}))$ arithmetic operations.

IF-QIPM with Inner IR

Input: $\epsilon, \delta > 0$; $\sigma = 1 - \delta/\sqrt{n}$; $\beta, \gamma \in (0, 1)$

Output: An ϵ -optimal primal dual pair (X, y, S)

Choose $(X^{(0)}, y^{(0)}, S^{(0)}) \in \mathcal{N}_F(\gamma)$

Set $\nu^{(0)} = \frac{X^{(0)} \bullet S^{(0)}}{n}$

Compute bases for $\mathcal{R}(\mathcal{A}_s) = \mathcal{A}_s^\top$ and $\text{Null}(\mathcal{A}_s) = Q_2$

while $\nu > \epsilon$ **do**

1 $\nu^{(k)} \leftarrow \frac{X^{(k)} \bullet S^{(k)}}{n}$

2 Compute matrices $(X^{(k)})^{-1}$ and $(S^{(k)})^{-1}$, $P^{(k)}$ classically.

3 Obtain classical estimate $\overline{\Delta z}^{(k)}, \overline{\Delta y}^{(k)}$ of $\Delta z^{(k)}, \Delta y^{(k)}$ using IR for the LSP **Quantum-solve**

4 Use classical estimate $\overline{\Delta z}^{(k)}, \overline{\Delta y}^{(k)}$ to obtain classical estimate $\overline{\Delta X}^{(k)}, \overline{\Delta S}^{(k)}$ of $\Delta X^{(k)}, \Delta S^{(k)}$

5 Update current solution

$X^{(k+1)} \leftarrow X^{(k)} + \overline{\Delta X}^{(k)}$, $S^{(k+1)} \leftarrow S^{(k)} + \overline{\Delta S}^{(k)}$ and $y^{(k+1)} \leftarrow y^{(k)} + \overline{\Delta y}^{(k)}$

$k \leftarrow k + 1$

end

Results: Complexity to solve SDO improves to:

$$\tilde{\mathcal{O}}_{n, \kappa, \frac{1}{\epsilon}}(n^{2.5} \kappa)$$

QRAM access and

$$\tilde{\mathcal{O}}_{n, \kappa, \frac{1}{\epsilon}}(n^{4.5})$$

arithmetic operations.

Exponential speedup!

$$\mathcal{O}\left(\frac{n\kappa}{\epsilon}\right)$$

fewer QRAM access than without IR.

Condition number increases

For ease of exposition we consider the case of LO.

- The normal equation system matrix is $A(XS^{-1})A^\top$,
where $X = \text{diag}(x)$ and $S = \text{diag}(s)$ and $XS^{-1} = \text{diag}\left(\frac{x_1}{s_1}, \dots, \frac{x_n}{s_n}\right)$
- By complementary slackness either x_i or s_i goes to 0, while the other goes to its positive (possibly large) optimal value.
- The condition number is bounded by $\kappa = \mathcal{O}\left(\kappa_A \frac{1}{\nu^2}\right)$
- If we stop IPMs with low precision, e.g., $\nu > \epsilon = 10^{-2}$, then we have

$$\kappa = \mathcal{O}\left(\kappa_A \frac{1}{\nu^2}\right) = \mathcal{O}\left(\kappa_A \frac{1}{\epsilon^2}\right) \longrightarrow \kappa = \mathcal{O}(\kappa_A)$$

IR-QIPM with IR for SDO

Input: Problem data $A_1, \dots, A_m, C \in \mathcal{S}^n$, $b \in \mathbb{R}^m$, Error tolerances $0 < \zeta \ll \epsilon < 1$

Output: A ζ -optimal primal-dual solution (X, y, S) to the SDO problem (A_1, \dots, A_m, b, C)

Initialize: $X^{(0)} \leftarrow 0$, $y^{(0)} \leftarrow 0$, $\eta^{(0)} \leftarrow 1$, $\varepsilon^{(0)} \leftarrow n$, $k \leftarrow 0$, $\bar{b} \leftarrow b$, $\bar{C} \leftarrow C$

while $\varepsilon > \zeta$ **do**

1 $(\bar{X}, \bar{y}) \leftarrow \text{solve } (A_1, \dots, A_m, \eta^{(k)} \bar{b}, \eta^{(k)} \bar{C})$ using $O_{\text{SDO}}(\epsilon)$

2 Update solution

$$X^{(k+1)} \leftarrow X^{(k)} + \frac{1}{\eta^{(k)}} \bar{X}, \quad y^{(k+1)} \leftarrow y^{(k)} + \frac{1}{\eta^{(k)}} \bar{y}$$

3 Update refining problem data

$$\bar{b}_i^{(k+1)} \leftarrow b_i - \text{tr}(A_i X^{(k+1)}), \quad \bar{C} \leftarrow C - \sum_{i=1}^m y_i^{(k+1)} A_i$$

4 Compute residual $\varepsilon^{(k+1)} = \frac{\text{tr}(X^{(k+1)} \bar{C})}{n}$

5 Update scaling factor $\eta^{(k+1)} = \frac{1}{\varepsilon^{(k+1)}}$

6 $k \leftarrow k + 1$

end

Condition number bound:

The condition number bound for the OSS is: $\kappa = \mathcal{O}\left(\kappa_T \frac{1}{\nu}\right) = \mathcal{O}\left(\kappa_T \frac{1}{\epsilon}\right)$

Where $T = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{B}_{\text{Null}} \end{pmatrix}$

IR-QIPM: Complexity for SDO :

At most $\tilde{\mathcal{O}}_{n, \kappa_T, \frac{1}{\epsilon}}(n^{2.5} \kappa_T)$

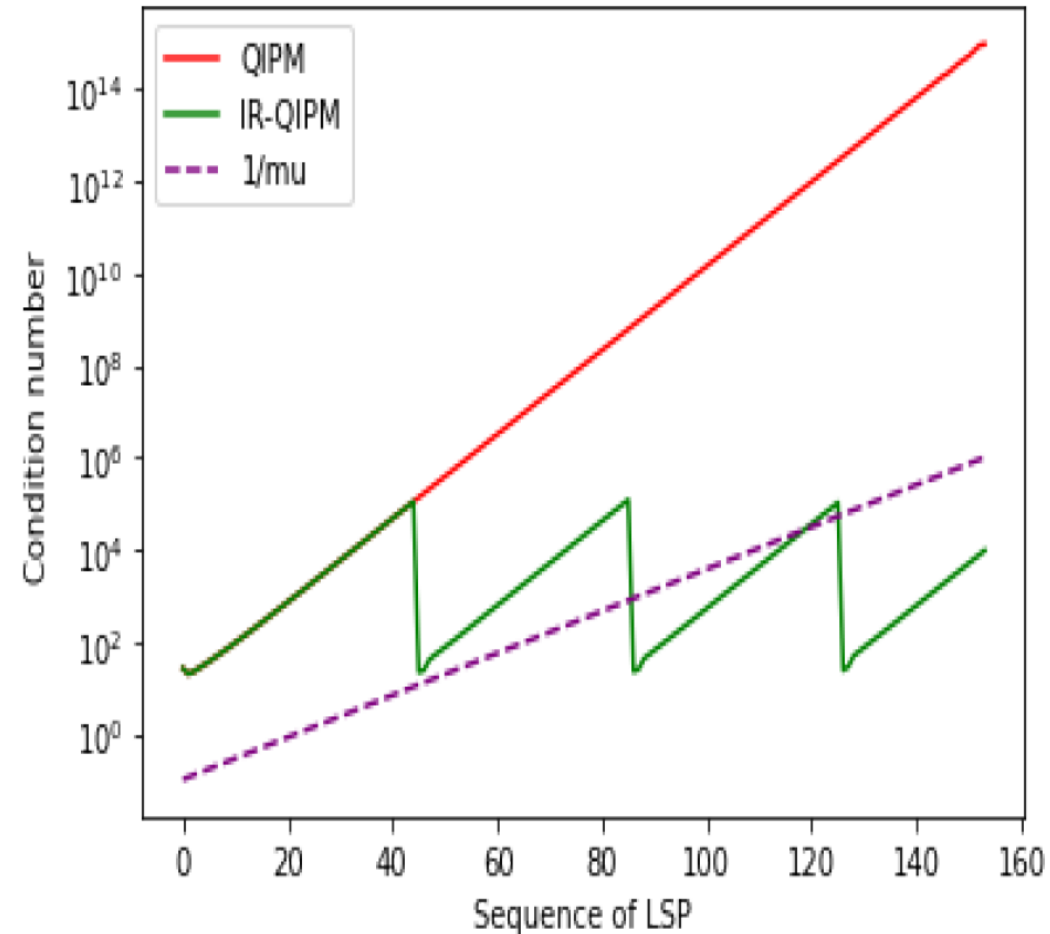
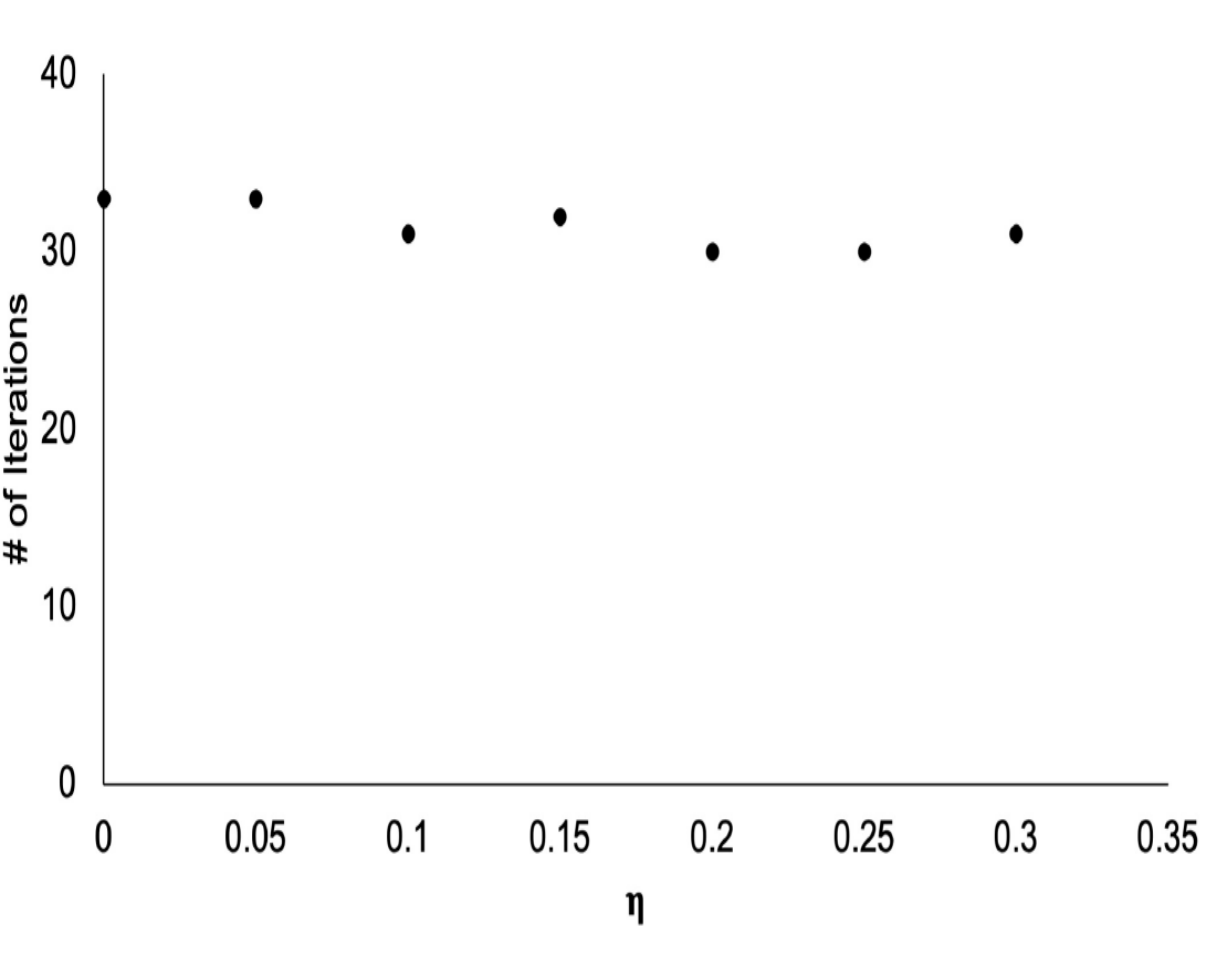
QRAM access and at most $\tilde{\mathcal{O}}_{n, \kappa_T, \frac{1}{\epsilon}}(n^{4.5})$ arithmetic operations.

For LO:

At most $\tilde{\mathcal{O}}_{n, \kappa_T, \frac{1}{\epsilon}}(n^{1.5} \kappa_T)$ QRAM access, and at most $\tilde{\mathcal{O}}_{n, \kappa_T, \frac{1}{\epsilon}}(n^{2.5})$ arithmetic operations.

Impact of IR on condition number

Developed Python package for IR-IF-QIPMs: <https://github.com/QCOL-LU/QIPM>



Quadratic Convergence

Quadratic convergence of the optimality gap without any nondegeneracy or strict complementarity assumption!

Theorem:

IR has quadratic convergence toward the optimal solution set of the SDO problem.

$$X^{(k+1)} \bullet S^{(k+1)} \leq \epsilon (X^{(k)} \bullet S^{(k)})^2$$

and

IR obtains an $\tilde{\epsilon}$ -optimal solution to the primal-dual SDO in at most

$$\mathcal{O} \left(\log \log \left(\frac{1}{\tilde{\epsilon}} \right) \right)$$

Note: IR needs initial solution at IR each step. A good choice: $(\eta^{(k)} X^{(k)}, 0, \eta^{(k)} S^{(k)})$

IR-IF-QIPMs for LO

Complexity of IF-QIPMs with Iterative Refinement

Algorithm	System	LS Solver	Complexity	
Best Theoretical bound	NES	Partial Update	$\mathcal{O}(n^3 L)$	
Feasible IPM	NES	Cholesky	$\mathcal{O}(n^{3.5} L)$	
II-IPM	NES	PCG	$\mathcal{O}(n^4 L \sqrt{\kappa_M})$	
IF-IPM	MNES	PCG	$\mathcal{O}(n^{2.5} L \sqrt{\kappa_M})$	
IR-II-IPM	NES	PCG	$\mathcal{O}(n^4 L \kappa_A)$	
IR-IF-IPM	MNES	PCG	$\mathcal{O}(n^{2.5} L \kappa_A)$	
IR-IF-QIPM	OSS	QTA+QLSA	$\mathcal{O}\left(n^{2.5} L \kappa_A\right)$	<i>Polylog factors suppressed</i>
IR-IF-QIPM	MNES	QTA+QLSA	$\mathcal{O}\left(n^{2.5} L \kappa_A\right)$	

New: *IR-IF-IPMs using PCG to solve Newton Systems*
Analogous properties, complexity as for IR-IF-QIPMs
Quadratic Convergence to the Optimal Set

Further notes on QIPMs

- “Natural” Extensions of II and IF QIPMs:
 - To LO, SDO and SOCO
 - With and without iterative refinements
- “Natural” extension of IF-QIPMs:
 - using the Self-Dual Embedding models
 - also: Quantum Dual Log-barrier Method
- Implementation in the Qiskit environment
 - Solving LO problems up to 16 constraints

The future of Optimization

... is bright!

New computing paradigms

New challenges

New opportunities

Bigger impact

Thanks ...

Open for
Questions and
Discussions

Join and amplify QCOR!



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