On the Projection-Based Convexification of Some Spectral Sets

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26th Midwest Optimization Meeting

University of Waterloo, Ontario

Nov, 2024

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  (P1) For all $x, y \in \mathbb{E}$, we have $\langle x, y \rangle \le \langle \lambda(x), \lambda(y) \rangle := \sum_{i=1}^n \lambda_i(x)\lambda_i(y)$.

  (P2) For all $\mu \in \mathcal{K}$ and $y \in \mathbb{E}$, there exists $x \in \mathbb{E}$ such that

$$\lambda(x) = \mu \ \text{ and } \ \langle x, y \rangle = \langle \lambda(x), \lambda(y) \rangle .$$

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  We call $\lambda : \mathbb{E} \to \mathcal{K}$ a *spectral map* (with $\operatorname{ran} \lambda = \mathcal{K}$).
- Given λ and $\mathcal{C} \subseteq \mathbb{R}^n$, define the *spectral set*
  $$\mathcal{S} := \lambda^{-1}(\mathcal{C}) := \{x \in \mathbb{E} : \lambda(x) \in \mathcal{C}\}.$$
  We always assume that $\mathcal{C} \cap \mathcal{K} \neq \emptyset$.

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As a result,

(Iso) For all $x \in \mathbb{E}$, $\|x\| = \|\lambda(x)\|_2$, where $\|\cdot\|$ is induced by $\langle \cdot, \cdot \rangle$ on $\mathbb{E}$.

(Res) For any $\omega \in \mathsf{RS}(\mathcal{K}) := \mathcal{K} \cap (-\mathcal{K})$, there exists $d \in \mathbb{E}$ such that

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- Initially proposed by Gowda [Gow19], and subsequently studied by Gowda and Jeong [GJ23; JG23].

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▷ The system induced by complete isometric hyperbolic polynomials [Bau01].

- Let $p : \mathbb{E} \to \mathbb{R}$ be a degree-n homogeneous polynomial that is *hyperbolic* w.r.t. some $d \in \mathbb{E}$, namely, $p(d) \neq 0$ and $t \mapsto p(td - x)$ has only real roots:

$$\lambda_1(x) \geq \cdots \geq \lambda_n(x) \quad \Rightarrow \quad \lambda(x) := (\lambda_1(x), \ldots, \lambda_n(x)).$$

- If p is complete and isometric, then $(\mathbb{E}, \mathbb{R}^n, \lambda)$ is a FTvN system, and $\mathsf{ran}\,\lambda$ is a closed convex cone.
- Special case: λ is the eigenvalue map on a Euclidean Jordan algebra of rank n, and in particular, on $\mathbb{S}^n$ (or $\mathbb{H}^n$), where $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$.

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- ▷ Our approach initially targeted the eigenvalue map on $\mathbb{S}^n$, but then was straightforwardly generalized to the FTvN system.

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- Due to its nonconvex nature, a natural step to obtain its global optimal solutions is to convexify $\mathcal{S}$.

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- However, little is known when $\mathcal{C}$ is "non-invariant".
- The notion of "invariance" is indeed defined w.r.t. $\mathsf{ran}\,\lambda = \mathcal{K}$:

λ	$\mathcal{K}$	Invariance
reordering / eigenvalue map	$\mathbb{R}^n_{\downarrow}$	permutation inv.
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- In the FTvN system, "invariance" is defined via another spectral map γ on $\mathbb{R}^n$ that is compatible to $\mathcal{K}$, but γ *may not exist* for any closed convex cone $\mathcal{K}$.

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  - This characterization sometimes has a simpler description than $\lambda^{-1}(\mathsf{clconv}\,\mathcal{C})$.
  - Our result unifies and extends the results in Kim et al. [KTR22] developed for special cases of λ and $\mathcal{C}$.

First Main Result

Theorem 1 (Closed and Convex $\mathcal{C}$)

Let $\lambda : \mathbb{E} \to \mathcal{K}$ be a spectral map, and $\mathcal{C}$ be closed and convex such that $\mathcal{C} \cap \mathsf{ri}\,\mathcal{K}$ is nonempty and bounded.

If $\mathcal{C} \cap \mathsf{RS}(\mathcal{K}) \neq \emptyset$, then

$$\mathsf{clconv}\,\mathcal{S} = \{x \in \mathbb{E} :\ \exists\, \mu \in \mathcal{C} \cap \mathcal{K}\ \ \text{s.t.}\ \lambda(x) - \mu \in \mathcal{K}^\circ\}. \tag{PC$_0$}$$

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- Checking if $\mu \in \mathsf{clconv}\,\mathcal{S}$ is a convex feasibility problem, and can be solved in polynomial time under some assumptions.
- (PC$_0$) is derived using convex dualities, and the assumptions on $\mathcal{C} \cap \mathsf{ri}\,\mathcal{K}$ mainly ensure strong dualities hold.

Corollary

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Let $\lambda : \mathbb{E} \to \mathcal{K}$ *be a spectral map, and* $\mathcal{D} := \mathsf{clconv}\,(\mathcal{C} \cap \mathcal{K})$ *satisfy that* $\mathcal{D} \cap \mathsf{ri}\,\mathcal{K}$ *is nonempty and bounded.*

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Moreover, if $\mathcal{K}$ *is polyhedral, then the assumptions on* $\mathcal{D} \cap \mathsf{ri}\,\mathcal{K}$ *can be dropped.*

Proof. Define $\mathcal{S}' := \lambda^{-1}(\mathcal{D})$. Then $\mathsf{clconv}\,\mathcal{S}' = \mathsf{clconv}\,\mathcal{S}$. □

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- When λ is the absolute-value map $|\cdot|$ on $\mathbb{R}^n$: $\mathcal{K} = \mathbb{R}^n_+$ and

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Proof Sketch

Lemma 1 (Linear optimization over spectral sets)

For any $c \in \mathbb{R}^n$ and any nonempty set $\mathcal{U} \subseteq \mathbb{R}^n$, we have

$$\sup\nolimits_{x\in\mathbb{E}} \{\langle y, x\rangle + \langle c, \lambda(x)\rangle : \lambda(x) \in \mathcal{U}\} = \sup\nolimits_{\mu\in\mathcal{U}\cap\mathcal{K}} \langle \lambda(y) + c, \mu\rangle$$

For any set $\mathcal{U} \neq \emptyset$, define its support function $\sigma_{\mathcal{U}} : y \mapsto \sup_{x\in\mathcal{U}} \langle y, x\rangle$ and

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Lemma 2 (Characterizing $\mathcal{S}^\circ$)

Let $\mathcal{C}$ *be closed and convex, and define* $\mathcal{D} := \mathcal{C} \cap \mathcal{K} \neq \emptyset$*. If* $\mathcal{K}$ *is polyhedral or* $\mathcal{C} \cap \mathsf{ri}\,\mathcal{K} \neq \emptyset$*, then*

$$\mathcal{S}^\circ = \{y \in \mathbb{E} : \exists\, z \in \mathcal{D}^\circ \text{ s.t. } \lambda(y) - z \in \mathcal{K}^\circ\}$$

Since $\mathcal{S}^{\circ\circ} = \{x \in \mathbb{E} : \sigma_{\mathcal{S}^\circ}(x) \leq 1\}$

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i) If $0 \in \mathcal{C}$, $0 \in \mathcal{D}\,(= \mathcal{C}\cap\mathcal{K})$ and $0 \in \mathcal{S}$, and we have

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Note that $\mathsf{clconv}\,\mathcal{S} = \mathcal{S}^{\circ\circ}$.

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$$\begin{aligned}\sigma_{\mathcal{S}^\circ}(x) &= \sup_{y\in\mathbb{E},\, z\in\mathcal{D}^\circ} \{\langle x, y\rangle :\ \lambda(y) - z \in \mathcal{K}^\circ\} \\ &= \inf_{\mu\in\mathcal{K}} \{\sigma_{\mathcal{D}^\circ}(\mu) :\ \lambda(x) - \mu \in \mathcal{K}^\circ\}\end{aligned}$$

i) If $0 \in \mathcal{C}$, $0 \in \mathcal{D}\,(= \mathcal{C}\cap\mathcal{K})$ and $0 \in \mathcal{S}$, and we have

$$\begin{aligned}\mathcal{S}^{\circ\circ} &= \{x \in \mathbb{E} : \sigma_{\mathcal{S}^\circ}(x) \le 1\} \\ &= \{x \in \mathbb{E} : \exists\, \mu \in \mathcal{K} \ \ \text{s.t.} \quad \sigma_{\mathcal{D}^\circ}(\mu) \le 1,\ \lambda(x) - \mu \in \mathcal{K}^\circ\} \\ &= \{x \in \mathbb{E} : \exists\, \mu \in \mathcal{D} \ \ \text{s.t.} \quad \lambda(x) - \mu \in \mathcal{K}^\circ\}\end{aligned}$$

Note that $\mathsf{clconv}\,\mathcal{S} = \mathcal{S}^{\circ\circ}$.

ii) If $\mathcal{C} \cap \mathsf{RS}(\mathcal{K}) \neq \emptyset$, since λ satisfies (Res), define

$$\mathcal{S}' := \mathcal{S} - d = \{x \in \mathbb{E} :\ \lambda(x + d) \in \mathcal{D}\} = \{x \in \mathbb{E} :\ \lambda(x) \in \mathcal{D} - \omega\}$$

and note that $\mathsf{clconv}\,(\mathcal{S}) = \mathsf{clconv}\,(\mathcal{S}') + d$.

Invariance in the FTvN System

Definition 1 (λ-Compatible Spectral Map)

Let $\gamma : \mathbb{R}^n \to \mathcal{K}$ be a spectral map. If $\gamma \circ \lambda = \lambda$ on $\mathbb{E}$, then γ is called λ-compatible.

Definition 2 (γ-Invariant Set)

Let $\gamma : \mathbb{R}^n \to \mathcal{K}$ be a spectral map. A set $\emptyset \neq \mathcal{U} \subseteq \mathbb{R}^n$ is called γ-invariant if for any $\mu \in \mathcal{U}$, $[\mu] \subseteq \mathcal{U}$, where

$$[\mu] := \{\nu \in \mathbb{R}^n : \gamma(\nu) = \gamma(\mu)\}.$$

The Second Main Result

Theorem 2 (Projection-Based Characterization for Invariant $\mathcal{C}$)

Given a spectral map $\lambda : \mathbb{E} \to \mathcal{K}$, *let* $\gamma : \mathbb{R}^n \to \mathcal{K}$ *be a* λ*-compatible spectral map, and* $\mathcal{C}$ *be a* γ*-invariant set.*

Then for any $\mathcal{D}$ *satisfying that*

$$\mathsf{conv}\,(\mathcal{C} \cap \mathcal{K}) \subseteq \mathcal{D} \subseteq (\mathsf{clconv}\,\mathcal{C}) \cap \mathcal{K},$$

we have $\mathsf{clconv}\,\mathcal{S} = \mathsf{cl}\,\mathcal{P}_{\mathcal{D}}$, *where*

$$\mathcal{P}_{\mathcal{D}} := \{x \in \mathbb{E} : \ \exists\, \mu \in \mathcal{D} \ \text{ s.t. } \ \lambda(x) - \mu \in \mathcal{K}^\circ\}.$$

An Example

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- Let $\mathcal{K} := \mathbb{R}^n_{\downarrow} \cap \mathbb{R}^n_+$, $\gamma(\mu) := |\mu|^{\downarrow}$ and $\mathcal{C} := \{\mu \in \mathbb{R}^n : \|\mu\|_0 \leq k, \|\mu\|_2 \leq 1\}$ for some $1 < k < n$.

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- Note that $\mathsf{clconv}\,\mathcal{C}$ can be rather complicated to describe, but

$$\mathcal{C} \cap \mathcal{K} = \{\mu \in \mathbb{R}^n_\downarrow : \mu \ge 0,\ \mu_{k+1} \le 0,\ \|\mu\|_2 \le 1\},$$

  which is convex and compact. We let $\mathcal{D} = \mathcal{C} \cap \mathcal{K}$.

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which is convex and compact. We let $\mathcal{D} = \mathcal{C} \cap \mathcal{K}$.

- Since $\mathcal{D}$ is bounded, we have
$$\mathsf{clconv}\, \mathcal{S} = \{x \in \mathbb{E} :\ \exists\, \mu \in \mathcal{D}\ \text{ s.t. } \lambda(x) \prec_{\mathrm{w}} \mu\}.$$

Corollary

Corollary 2 (Projection-Based Characterization for Any Feasible $\mathcal{C}$)

Given a spectral map $\lambda : \mathbb{E} \to \mathcal{K}$, *let* $\gamma : \mathbb{R}^n \mapsto \mathcal{K}$ *be a* λ*-compatible spectral map, and* $\mathcal{C}$ *be any feasible set (namely,* $\mathcal{C} \cap \mathcal{K} \neq \emptyset$*).*

Then for any $\mathcal{D}$ *satisfying that*

$$\mathsf{conv}\,(\mathcal{C} \cap \mathcal{K}) \subseteq \mathcal{D} \subseteq \mathsf{clconv}\,(\mathcal{C} \cap \mathcal{K}),$$

we have $\mathsf{clconv}\,\mathcal{S} = \mathsf{cl}\,\mathcal{P}_{\mathcal{D}}$, *where*

$$\mathcal{P}_{\mathcal{D}} := \{x \in \mathbb{E} : \exists\, \mu \in \mathcal{D} \;\; \text{s.t.} \;\; \lambda(x) - \mu \in \mathcal{K}^{\circ}\}.$$

Proof. Define $\tilde{\mathcal{C}} := \cup_{\mu \in \mathcal{C} \cap \mathcal{K}}\,[\mu]$. Note that $\tilde{\mathcal{C}} \cap \mathcal{K} = \mathcal{C} \cap \mathcal{K}$ and $\mathcal{S} = \lambda^{-1}(\tilde{\mathcal{C}})$. Then apply Theorem 2 to $\lambda^{-1}(\tilde{\mathcal{C}})$. □

Thank you!

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