Weakly differentiable mappings and their compactness properties

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Abstract

In this talk we explore the generalized sequential normal compactness of weakly differentiable mappings in variational analysis. We present complete characterizations of these compactness properties of such mappings between general Banach spaces, as well as calculus rules of sequential normal compactness involving this kind of mappings.

The talk is based on the joint work of B. Wang, X. Yang, and P. Long.

Fréchet differentiability

Let X, Y be Banach spaces. A function $f: X \to Y$ is said *strictly Fréchet differentiable* at $\bar{x}$ with derivative $\nabla f(\bar{x}) \in \mathscr{B}(X, Y)$ if

$$\lim_{x,u \to \bar{x}, x \neq u} \frac{f(u) - f(x) - \nabla f(\bar{x})(u - x)}{\|u - x\|} = 0.$$

When x is fixed as $\bar{x}$, we say that f is *Fréchet differentiable* at $\bar{x}$.

Weak differentiability

- We say that $f : X \to Y$ is *weakly* (resp. *weakly strictly*) *Fréchet differentiable* at $\bar{x}$ if $\langle y^*, f \rangle$ is Fréchet (resp. strictly Fréchet) differentiable at $\bar{x}$ for all $y^* \in Y^*$, where $\langle y^*, f \rangle(x) := \langle y^*, f(x) \rangle$ for all $x \in X$.

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- We use $\nabla^* f(\bar{x})\colon Y^* \to X^*$ to denote the operator with $\nabla^* f(\bar{x})(y^*) = \nabla\langle y^*, f\rangle(\bar{x})$, and call it the *linear coderivative* of f at $\bar{x}$.

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- If there is a linear operator $A\colon X \to Y$ (not necessarily continuous) with $A^* = \nabla^* f(\bar{x})$, then A is unique, and $A \in \mathscr{B}(X, Y)$ when $\nabla^* f(\bar{x}) \in \mathscr{B}(Y^*, X^*)$; in this case we say that f admits a weak derivative A. This weak derivative coincides with the derivative $\nabla f(\bar{x})$ if f is Fréchet or strictly Fréchet differentiable at $\bar{x}$; so we use the same notation $\nabla f(\bar{x})$ to denote the weak derivative if it exists.

Uniform weak differentiability

Let $f: X \to Y$ be weakly Fréchet differentiable at $\bar{x} \in X$. We say that f is *w^*-uniformly weakly strictly Fréchet differentiable* at $\bar{x}$ if for any w^*-convergent sequence $\{y_k^*\} \subset Y^*$, any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left|\langle y_k^*, f(x) - f(u)\rangle - \langle \nabla^* f(\bar{x})(y_k^*), x - u\rangle\right| \leq \varepsilon \|x - u\| \quad \forall x, u \in B(\bar{x}; \delta), \quad k \in \mathbb{N}.$$

When u is fixed as $\bar{x}$, we say that f is *w^*-uniformly weakly Fréchet differentiable* at $\bar{x}$.

Normal cones

- **The pre-normal cone:**
  For a fixed $\varepsilon \geq 0$, the set of ε-normals to Ω at $\bar{x} \in \Omega$ is given by

$$\widehat{N}_{\varepsilon}(\bar{x};\Omega) := \left\{ x^* \in X^* \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq \varepsilon \right\},$$

where $x \xrightarrow{\Omega} \bar{x}$ denotes $x \to \bar{x}$ with $x \in \Omega$, and when $\bar{x}$ is an isolated point of Ω, we set $\widehat{N}_{\varepsilon}(\bar{x};\Omega) := X^*$, and when $\bar{x} \notin \Omega$, we set $\widehat{N}_{\varepsilon}(\bar{x};\Omega) = \emptyset$ for all $\varepsilon \geq 0$.

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- **The Mordukhovich normal cone:**

$$N(\bar{x};\Omega) := \{x^* \in X^* \mid \exists \varepsilon_k \downarrow 0, x_k \xrightarrow{\Omega} \bar{x}, x_k^* \xrightarrow{w^*} x^*$$

$$\text{with } x_k^* \in \widehat{N}_{\varepsilon_k}(x_k;\Omega)\}$$

Coderivatives

Let $F\colon X \rightrightarrows Y$ be a set-valued mapping.

- **The pre-coderivative:**

$$\widehat{D}^* F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in \widehat{N}_0(\bar{x}; \operatorname{gph} F)\}$$

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- **The Mordukhovich mixed coderivative:**

$$D^* F_M(\bar{x},\bar{y})(y^*) := \{x^* \in X^* \mid \exists \varepsilon_k \downarrow 0,$$

$$(x_k, y_k) \stackrel{\operatorname{gph} F}{\to} (\bar{x},\bar{y}), x_k^* \stackrel{w^*}{\to} x^*, y_k^* \stackrel{\|\cdot\|}{\to} y^*$$

$$\text{with } (x_k^*, -y_k^*) \in \widehat{N}_{\varepsilon_k}((x_k, y_k); \operatorname{gph} F)\}$$

Weak differentiability and generalized differentiations

- If f is weakly Fréchet differentiable at $\bar{x}$, then for all $y^* \in Y^*$,

$$\widehat{D}^*(F+f)(\bar{x},\bar{y})(y^*) = \nabla^* f(\bar{x})y^* + \widehat{D}^* F(\bar{x},\bar{y}-f(\bar{x}))(y^*).$$

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- If f is weakly strictly Fréchet differentiable at $\bar{x}$, then for all $y^* \in Y^*$,

$$D_M^*(F+f)(\bar{x},\bar{y})(y^*) = \nabla^* f(\bar{x})y^* + D_M^* F(\bar{x},\bar{y}-f(\bar{x}))(y^*).$$

- If f is w^*-uniformly weakly strictly Fréchet differentiable at $\bar{x}$, then for all $y^* \in Y^*$,

$$D_N^*(F+f)(\bar{x},\bar{y})(y^*) = \nabla^* f(\bar{x})y^* + D_N^* F(\bar{x},\bar{y}-f(\bar{x}))(y^*).$$

Generalized sequential normal compactness (GSNC)

- Let $X = \prod_{j=1}^{m} X_j$, $J_1, J_2 \subset J_X := \{1, 2, \ldots, m\}$, $\Omega \subset X$ be a nonempty set with $\bar{x} \in \Omega$. Then we say that Ω is *generalized sequentially normally compact* (GSNC) at $\bar{x} \in \Omega$ with respect to $\{X_j \mid j \in J_1\}$ through $\{X_j \mid j \in J_2\}$ if for all $x_k^* = (x_{1k}^*, \ldots, x_{mk}^*) \in \widehat{N}_{\varepsilon_k}(x_k; \Omega)$ with $x_k \xrightarrow{\Omega} \bar{x}$, $\varepsilon_k \downarrow 0$, the following holds:

$$\begin{bmatrix} x_{jk}^* \xrightarrow{w^*} 0 \ (j \notin J_2) \\ \|x_{jk}^*\| \to 0 \ (j \in J_2) \end{bmatrix} \Longrightarrow [\|x_{jk}^*\| \to 0 \ (j \in J_1)];$$

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- $J_1 = J_X$, $J_2 = \emptyset$: *the sequential normal compactness*;
- $J_2 = J_X \backslash J_1$: *the partial sequential normal compactness (PSNC)* of Ω at $\bar{x}$ with respect to J_1;

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$$\begin{bmatrix} x_{jk}^* \stackrel{w^*}{\to} 0 \ (j \notin J_2) \\ \|x_{jk}^*\| \to 0 \ (j \in J_2) \end{bmatrix} \Longrightarrow [\|x_{jk}^*\| \to 0 \ (j \in J_1)];$$

- $J_1 = J_X$, $J_2 = \emptyset$: *the sequential normal compactness*;
- $J_2 = J_X \backslash J_1$: *the partial sequential normal compactness (PSNC)* of Ω at $\bar{x}$ with respect to J_1;
- $J_2 = \emptyset$: *the strong partial sequential normal compactness (strong PSNC, or SPSNC)* of Ω at $\bar{x}$ with respect to J_1.

PSNC with respect to the domain space

If $f\colon X \to Y$ is weakly strictly Fréchet differentiable at $\bar{x} \in X$, then $\operatorname{gph} f$ is PSNC with respect to X.

PSNC with respect to the image/range space

Let $f\colon X \to Y$ with $\bar{x} \in X$. Then the following assertions hold:

(i) Suppose that f is weakly Fréchet differentiable at $\bar{x}$ that admits a weak derivative $\nabla f(\bar{x})$. Then

$$\operatorname{codim} \nabla f(\bar{x})(X) < \infty$$

if $\operatorname{gph} f$ is PSNC at $(\bar{x}, f(\bar{x}))$ with respect to Y.

PSNC with respect to the image/range space

(ii) Suppose that f is w^*-uniformly weakly Fréchet differentiable at $\bar{x}$ that admits a weak derivative $\nabla f(\bar{x})$, and $\nabla f(\bar{x})^*(X)$ is of finite codimension in Y. Then $\operatorname{gph} f$ is PSNC at $(\bar{x}, f(\bar{x}))$ with respect to Y.

PSNC with respect to the image/range space

(iii) Suppose that f is w^*-uniformly weakly strictly Fréchet differentiable at $\bar{x}$, $\left[\nabla^* f(\bar{x})\right]^*(\phi_X(X))$ is closed in Y^{**}, and $\left[\nabla^* f(\bar{x})\right]^*(\phi_X(X)) \cap \phi_Y(Y)$ is of finite codimension in $\phi_Y(Y)$. Then $\operatorname{gph} f$ is PSNC at $(\bar{x}, f(\bar{x}))$ with respect to Y.

The canonical embedding

The canonical embedding $\phi_X : X \to X^{**}$:

$$\langle \phi_X(x), x^* \rangle = \langle x^*, x \rangle \quad \forall x \in X, x^* \in X^*.$$

Strong PSNC with respect to the domain space

Let $f: X \to Y$ with $\bar{x} \in X$. Then the following assertions hold:

(i) Suppose that f is weakly Fréchet differentiable at $\bar{x}$ that admits a weak derivative $\nabla f(\bar{x})$ at $\bar{x}$, $\nabla f(\bar{x})(X)$ is closed and w^*-extensible in Y. Then

$$\dim \nabla f(\bar{x})(X) < \infty$$

if gph f is strongly PSNC at $(\bar{x}, f(\bar{x}))$ with respect to X.

Strong PSNC with respect to the domain space

(i) Suppose that f is w^*-uniformly weakly strictly Fréchet differentiable at $\bar{x}$ with a weak derivative $\nabla f(\bar{x})$, and

$$\dim \nabla f(\bar{x})(X) < \infty.$$

Then $\operatorname{gph} f$ is strongly PSNC at $(\bar{x}, f(\bar{x}))$ with respect to X.

Strong PSNC with respect to the domain space

(iii) Suppose that f is weakly Fréchet differentiable at $\bar{x}$ with $\nabla^* f(\bar{x})$ satisfying

$$[u_k^* \stackrel{w^*}{\to} 0] \Longrightarrow [\nabla^* f(\bar{x})(u_k^*) \stackrel{w^*}{\to} 0]$$

for all sequence $\{u_k^*\} \subset Y^*$, $\left[\nabla^* f(\bar{x})\right]^*(\phi_X(X))$ is closed in Y^{**}, and $\left[\nabla^* f(\bar{x})\right]^*(\phi_X(X)) \cap \phi_Y(Y)$ is w^*-extensible in $\phi_Y(Y)$. Then

$$\dim \left[\nabla^* f(\bar{x})\right]^*(\phi_X(X)) \cap \phi_Y(Y) < \infty$$

if gph f is strongly PSNC at $(\bar{x}, f(\bar{x}))$ with respect to X.

w^*-Extensibility

Let L be a closed linear subspace of a Banach space X. We say that L is w^*-extensible in X if every sequence $\{v_k^*\} \subset L^*$ with $v_k^* \xrightarrow{w^*} 0$ as $k \to \infty$ contains a subsequence $\{v_{k_j}^*\}$ such that each $v_{k_j}^*$ can be extended to $x_j^* \in X^*$ with $x_j^* \xrightarrow{w^*} 0$ as $j \to \infty$.

GSNC sum rule and weak differentiability

Let $X = X_1 \times X_2 \times X_3$, $Y = Y_1 \times Y_2$, $Z = Z_1 \times Z_2$ be products of Banach spaces, and $F\colon X \rightrightarrows Y \times Z$ be a multifunction with $(\bar{x}, \bar{y}, \bar{z}) \in \operatorname{gph} F$, $g_1 : X \to Y$, $h_1 : X \to Z_1$, $h_2 : X \to Z_2$, $g_2 = h_1 \times h_2$, $g = g_1 \times g_2$ such that g_1, h_1 are w^*-uniformly weakly strictly Fréchet differentiable at $\bar{x}$, h_2 is weakly strictly Fréchet differentiable at $\bar{x}$, and $\operatorname{gph} g_1$ is strong PSNC with respect to X at $(\bar{x}, g_1(\bar{x}))$. Then the following two statements are equivalent:

(a) $\operatorname{gph} F$ is GSNC with respect to Z_1 through Z_2, and GSNC with respect to $\{X_1, Y_1, Z_1\}$ through $\{X_2, Z_2\}$ at $(\bar{x}, \bar{y}, \bar{z})$;

(b) $\operatorname{gph}(F + g)$ is GSNC with respect to Z_1 through Z_2, and GSNC with respect to $\{X_1, Y_1, Z_1\}$ through $\{X_2, Z_2\}$ at $(\bar{x}, \bar{y} + g_1(\bar{x}), \bar{z} + g_2(\bar{x}))$.

GSNC chain rule and weak differentiability

Let $X = X_1 \times X_2 \times X_3$, $Y = Y_1 \times Y_2 \times Y_3$, $Z = Z_1 \times Z_2 \times Z_3$, $F\colon X \rightrightarrows Y$, $g_i\colon Y_i \to Z_i$ $(i = 1, 2, 3)$, $g = g_1 \oplus g_2 \oplus g_3$, and $\bar{y} = (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in F(\bar{x})$, $\bar{z} = g(\bar{y})$. Assume that g_i is w^*-uniformly weakly strictly Fréchet differentiable at $\bar{y}_i$ $(i = 1, 3)$, g_2 is weakly strictly Fréchet differentiable at $\bar{y}_2$, and $\operatorname{gph} g_1$ is PSNC with respect to in Z_1, $\operatorname{gph} F$ is GSNC at $(\bar{x}, \bar{y})$ with respect to $\{X_1, Y_1\}$ through $\{X_2, Y_2\}$, and $F \cap g^{-1}\colon X \times Z \rightrightarrows Y$ (defined by $(F \cap g^{-1})(x, z) = F(x) \cap g^{-1}(z)$ for all $x \in X$, $z \in Z$) is inner semicontinuous at $(\bar{x}, \bar{z}, \bar{y})$. Then $\operatorname{gph}(g \circ F)$ is GSNC at $(\bar{x}, \bar{z})$ with respect to $\{X_1, Z_1\}$ through $\{X_2, Z_2\}$.

References

The talk is based on the following paper:

B. Wang, X. Yang, P. Long, Generalized sequential normal compactness and weak differentiabilities, *Journal of Optimization Theory and Applications* (to appear).

Further references can be found in the paper.