Convex optimization in negatively curved geodesic spaces

Adrian Lewis

Joint work with A. Goodwin, G. Lopez, and A. Nicolae

Midwest Optimization at Waterloo November 2024

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- Cutting planes for optimization on CAT(0) cubical complexes

PART I

Negative curvature

Optimization over a **Hadamard** space **X**:

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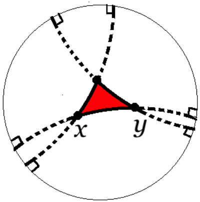

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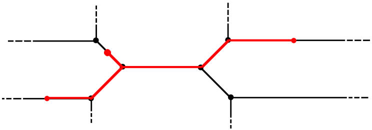

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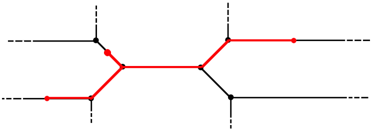

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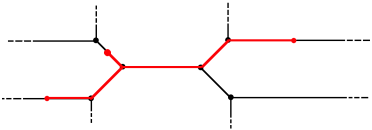

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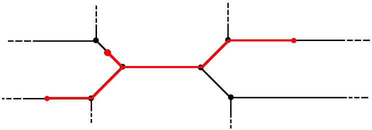

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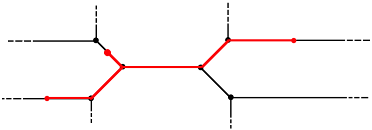

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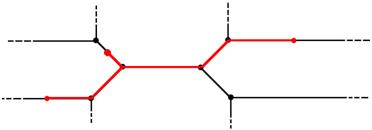

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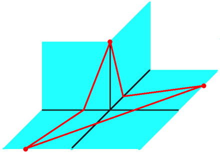

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Question   Are circumcenters in Hadamard space computable?

PART II

The subgradient method: a fresh look

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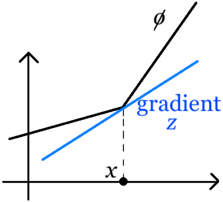

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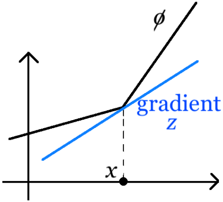

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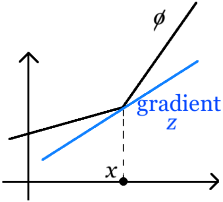

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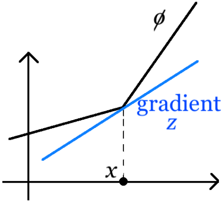

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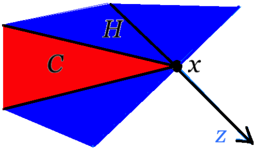

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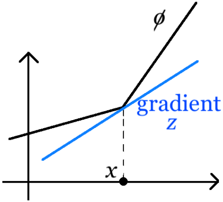

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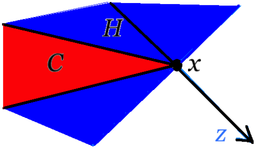

Subgradients are **dual** objects. Any analogue in Hadamard space?

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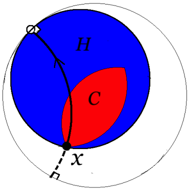

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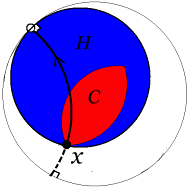

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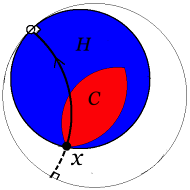

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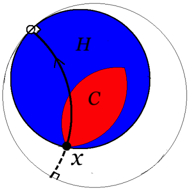

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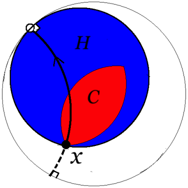

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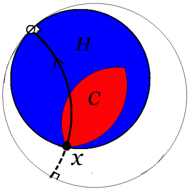

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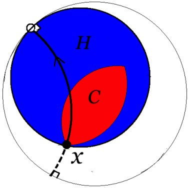

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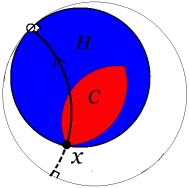

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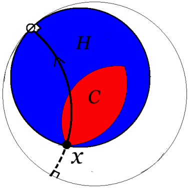

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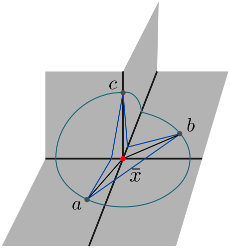

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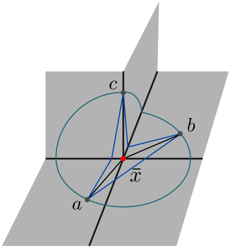

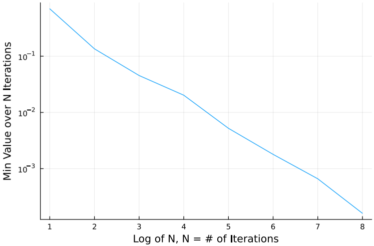

But finding supporting rays for composite objectives is hard, so...

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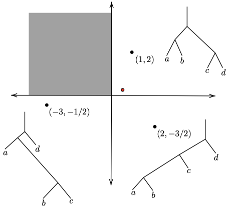

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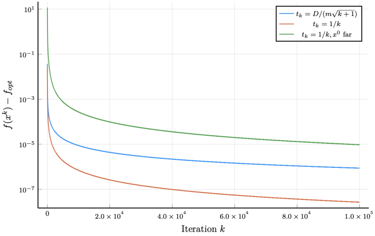

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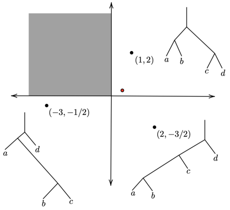

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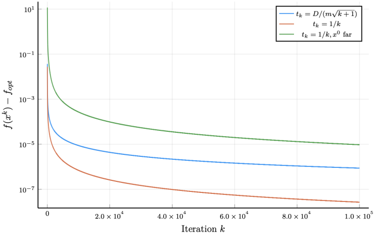

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(See Ariel Goodwin's poster: "Incremental minimization. . .")

PART III

Weighted means in Hadamard spaces

Averaging finite sets

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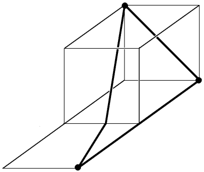

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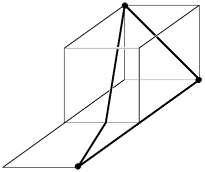

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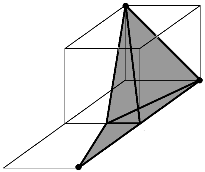

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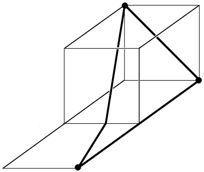

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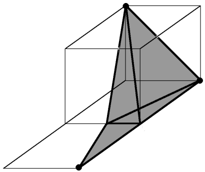

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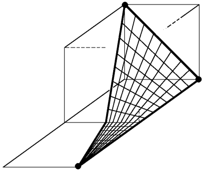

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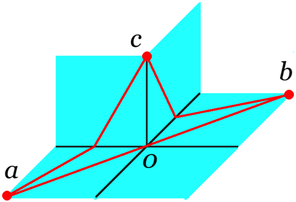

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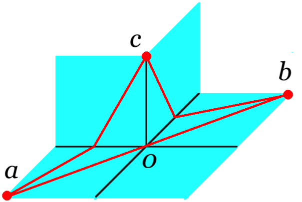

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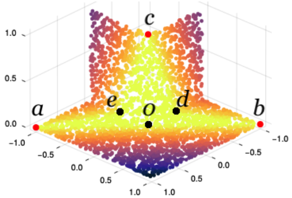

Heat map of test function slope

PART IV

Convex optimization on CAT(0) cubical complexes

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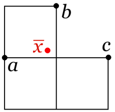

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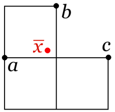

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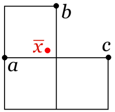

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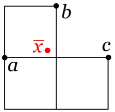

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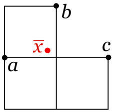

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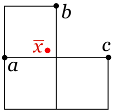

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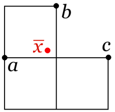

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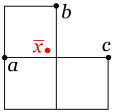

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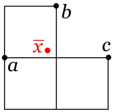

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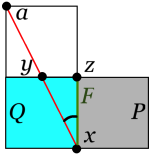

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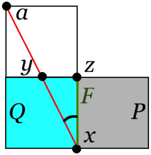

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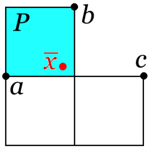

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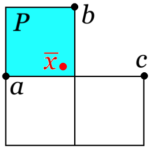

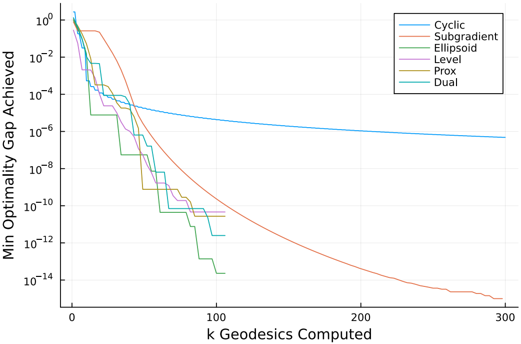

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- Horoballs and the subgradient method
  `arXiv:2403.15749`
- Recognizing weighted means in geodesic spaces
  `arXiv:2406.03913`
- Convex optimization on CAT(0) cubical complexes
  `arXiv:2405.01968`
- Incremental minimization in spaces of nonpositive curvature
  `arielgoodwin.github.io/talks`