Lipschitz stability of least-squares problems regularized by functions with $\mathcal{C}^2$-cone reducible conjugates

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(Based on the joint work with Y. Cui (UC), T. Hoheisel (MU), and D. Sun (HKPU))

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A particular situation in different areas of engineering and science is that one has the observation $y_0 = \Phi x_0$ via a known (or random) linear operator $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ $(m << n)$ and an unknown vector $x_0 \in \mathbb{R}^n$. To recover x_0, we have to solve

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Let's generate some random data:

$$x_0 \in [-1, 1]^{250} \quad \text{with} \quad \text{nnz}(x_0) = 25 \quad \text{and} \quad \Phi \in \mathbb{R}^{100\times 250}.$$

Solving (1) by the least-square method or the projection method:

$$\text{(LS)} \quad \min_{x\in\mathbb{R}^{250}} \|\Phi x - y_0\|^2 \qquad \text{or} \qquad \text{(PM)} \quad \min_{x\in\mathbb{R}^{250}} \|x\|_2 \quad \text{subject to} \quad \Phi x = y_0$$

gives us the same solution

$$x_{LS} = \Phi^\dagger y_0 \quad \text{with} \quad \text{nnz}(x_{LS}) = 250 \;\; \text{and} \;\; \|x_{LS} - x_0\| \approx 2.9.$$

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Solving the (ℓ_1) problem

$$(\ell_1) \qquad \min_{x\in\mathbb{R}^{250}} \quad \|x\|_1 \quad \text{subject to} \quad \Phi x = y_0$$

gives us $\text{nnz}\,(\bar{x}) = 25$ and $\|\bar{x} - x_0\| \approx 10^{-14}$, and recovers exactly x_0.

When there is some error/noise in the observation y_0, i.e., our observation y satisfies

$$\|\Phi x_0 - y\| \leq \delta = 0.001 \quad \text{for some} \quad \delta > 0.$$

Solving the least square problem gives us x_δ with

$$\text{nnz}\,(x_\delta) = 250 \quad \text{and} \quad \|x_\delta - x_0\| \approx 2.6.$$

As we want to recover a sparse vector, adding the ℓ_1 norm to the least square course is helpful:

$$\min_{x \in \mathbb{R}^n} \quad \frac{1}{2}\|Ax - b\|^2 + \mu\|x\|_1. \tag{2}$$

With $\mu = \delta$, its optimal solution $\bar{x}_\mu$ satisfies

$$\text{nnz}\,(\bar{x}_\mu) = 31 \quad \text{and} \quad \|\bar{x}_\mu - x_0\| \approx 0.009.$$

If x_0 has low-complexity (e.g., sparsity, group-sparsity, or low-rank) and $y \approx \Phi x_0$, we solve

$$(P(y,\mu)) \qquad \min_{x \in \mathbb{R}^n} \quad \frac{1}{2\mu}\|\Phi x - y\|^2 + g(x), \tag{3}$$

where $\mu > 0$, $y \in \mathbb{R}^m$, and $g : \mathbb{R}^n \to \mathbb{R}$ is a convex regularizer (ℓ_1 norm, ℓ_1/ℓ_2 norm, nuclear norm, etc).

- μ is called the tuning parameter.
- y is also another parameter.

We are interested in studying stability and sensitivity of the solution mapping:

$$S(y,\mu) := \operatorname{argmin}\left\{\frac{1}{2\mu}\|\Phi x - y\|^2 + g(x) \,\middle|\, x \in \mathbb{R}^n\right\}. \tag{4}$$

- [Vaiter et al 2017] showed that $S(y,\mu)$ is a *differentiable* function around $(\bar{y},\bar{\mu}) \in \mathbb{R}^m \times \mathbb{R}_{++}$ under the nondegeneracy condition (ND):

$$-\frac{1}{\mu}\Phi^*(\Phi\bar{x}-\bar{y}) \in \operatorname{ri}\partial g(\bar{x}) \quad \text{with} \quad \bar{x} = S(\bar{y},\mu) \tag{5}$$

  together with another second-order condition.

- When $g = \|\cdot\|_1$, without the (ND) condition, [Bolte et al 2021] showed that if $S(y,\mu)$ is Lipschitz continuous around $(\bar{y},\bar{\mu})$, then it is path differentiable and its conservative Jacobian at $(\bar{y},\bar{\mu})$ is computable.

- When $g = \|\cdot\|_1$, [Berk-Brugiapaglia-Hoheisel 2023] shows that the Tibshirani's condition [Tibshirani 2013] is a sufficient condition for Lipschitz stability of $S(y, b)$.
- [N. 2024] obtained full characterizations for Lipschitz stability of this solution mapping for different classes of $g(x)$.

The common point of these papers is: they are heavy in second-order analysis!

Lipschitz stability and second-order analysis

The solution mapping S is Lipschitz continuous around $(\bar{y}, \bar{\mu})$ provided that

$$\|\Phi w\|^2 + \inf\{\langle z, w\rangle \,|\, z \in \partial^2 g(\bar{x}|\bar{v})(w)\} > 0 \quad \text{for any} \quad w \neq 0, \tag{6}$$

where $\bar{x} \in S(\bar{y}, \bar{\mu})$, $\bar{v} := -\frac{1}{\bar{\mu}}\Phi^*(\Phi\bar{x} - \bar{y}) \in \partial g(\bar{x})$, and $\partial^2 g(\bar{x}|\bar{v})(w)$ is the generalized Hessian of g [Mordukhovich 1992] due to the theory of full stability [Levy-Poliquin-Rockafellar 2000].

$\partial^2 g(\bar{x}|\bar{v}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a second-order structure and it could be very hard to compute it, e.g., g is the nuclear norm.

Our question was: “Can we surpass the computation of second-order structures and obtain simple conditions for Lipschitz stability?”

Our approach via the dual problem

The dual problem of (3) reads

$$\min_{z\in\mathbb{R}^m} \quad \frac{\mu}{2}\|z\|^2 - \langle z, y\rangle + g^*(\Phi^* z),$$

or equivalently

$$(D(y,\mu)) \qquad \min_{z\in\mathbb{R}^m} \quad \frac{\mu}{2}\|z\|^2 - \langle z, y\rangle + t \quad \text{subject to} \quad (\Phi^* z, t) \in \operatorname{epi} g^*. \qquad (7)$$

The key observation: This problem has a unique solution and TFAE:

- x solves problem $P(y,\mu)$;
- $(z, g^*(\Phi^* z))$ solves problem $(D(y,\mu))$ and $(x, -1)$ is the corresponding Lagrange multiplier of the constrained optimization problem (7).

Lipschitz stability of the Lagrange system reminds us about Robinson's strong regularity [Robinson 1980]!

Constrained Optimization Problems and Robinson's Strong Regularity

Consider the parametric optimization problem

$$\mathrm{OPT}(p): \quad \min_{x\in\mathbb{R}^n} \quad \varphi(x,p) \quad \text{subject to} \quad G(x,p)\in\Theta, \tag{8}$$

where $\Theta\subset\mathbb{R}^m$ is a closed and convex set, $\varphi:\mathbb{R}^n\times\mathbb{R}^d\to\mathbb{R}$ and $G:\mathbb{R}^n\times\mathbb{R}^d\to\mathbb{R}^m$ are twice continuously differentiable functions.

The *Lagrangian* of problem $\mathrm{OPT}(p)$ is $\mathscr{L}:\mathbb{R}^n\times\mathbb{R}^d\times\mathbb{R}^m\to\mathbb{R}$ given by

$$\mathscr{L}(x,p,\lambda):=\varphi(x,p)+\langle\lambda,G(x,p)\rangle \quad \text{for} \quad (x,p,\lambda)\in\mathbb{R}^n\times\mathbb{R}^d\times\mathbb{R}^m.$$

A feasible point $x\in\mathbb{R}^n$ is called a stationary point of $\mathrm{OPT}(p)$ if there exists a *Lagrange multiplier* $\lambda\in\mathbb{R}^m$ satisfying

$$0=\nabla_x\mathscr{L}(x,p,\lambda) \quad \text{and} \quad \lambda\in N_\Theta(G(x,p)).$$

The Lagrange system can be written as a generalized equation

$$\mathrm{GE}(p): \quad 0 \in \begin{pmatrix} \nabla_x \varphi(x,p) + \mathcal{J}_x G(x,p)^* \lambda \\ -G(x,p) \end{pmatrix} + \begin{pmatrix} 0 \\ N_\Theta^{-1}(\lambda) \end{pmatrix}. \tag{9}$$

Let $H : \mathbb{R}^d \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ be its *solution mapping*. [Robinson 1980] showed that this mapping has a single-valued and Lipschitz continuous localization around some $\bar{p}$ for some $(\bar{x}, \bar{\lambda}) \in H(\bar{p})$ if its linearized system

$$\delta \in \begin{pmatrix} 0 \\ -G(\bar{x},\bar{p}) \end{pmatrix} + \begin{pmatrix} \nabla^2_{xx}\mathscr{L}(\bar{x},\bar{p},\bar{\lambda})(x-\bar{x}) + \mathcal{J}_x G(\bar{x},\bar{p})^*(\lambda - \bar{\lambda}) \\ -\mathcal{J}_x G(\bar{x},\bar{p})(x-\bar{x}) \end{pmatrix} + \begin{pmatrix} 0 \\ N_\Theta^{-1}(\lambda) \end{pmatrix} \tag{10}$$

does around $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$ for $(\bar{x}, \bar{\lambda})$. Lipschitz stability of this linearized system is known as Robinson's strong regularity at $(\bar{x}, \bar{p})$ for the corresponding Lagrange multiplier $\bar{\lambda}$.

Characterizations of Robinson's strong regularity

- For nonlinear programming (NP)

$$\min \quad \varphi(x,p) \qquad \text{subject to} \quad G(x,p) \in -\mathbb{R}^m_+$$

[Robinson 1980] showed that strong regularity occurs at $(\bar{x},\bar{p})$ for $\bar{\lambda}$ if

  - The linear independence constraint qualification holds at $(\bar{x},\bar{p})$, i.e., $\{\nabla_x G_i(\bar{x},\bar{p})\}_{i\in I}$ are linearly independent with $I := \{i|\, G_i(\bar{x},\bar{p}) = 0\}$.
  - The strong second-order sufficient condition (SSOSC) holds

$$\langle \nabla^2_{xx}\mathscr{L}(\bar{x},\bar{p},\bar{\lambda})w,w\rangle > 0 \quad \text{whenever} \quad \langle \nabla_x G_i(\bar{x},\bar{p}),w\rangle = 0 \text{ with } \bar{\lambda}_i > 0. \qquad (11)$$

- [Bonnans-Ramirez 2005] extended it to second-order cone programming (SOCP).
- [Sun 2006] generalized the result to semidefinite programming when $\Theta = -\mathbb{S}^n_+$.

$\mathcal{C}^2$-reducible cone programming

Back to our dual problem:

$$(D(y,\mu)) \qquad \min_{z\in\mathbb{R}^m} \quad \frac{\mu}{2}\|z\|^2 - \langle z, y\rangle + t \quad \text{subject to} \quad (\Phi^* z, t) \in \Theta := \operatorname{epi} g^*.$$

This is a constrained problem, but it is neither NP, SOCP, nor SDP!

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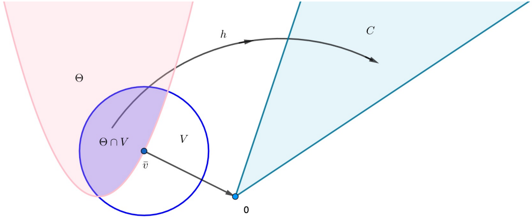
[Mordukhovich-N.-Rockafellar 2017] obtained full characterization of Robinson's strong regularity for $\mathcal{C}^2$-cone reducible programming [Bonnans-Shapiro 2000].

Definition 1 ($\mathcal{C}^2$-cone reducible sets)

A closed convex set $\Theta \subset \mathbb{R}^m$ is said to be $\mathcal{C}^2$-cone reducible at $\bar{v} \in \Theta$ if there exist a neighborhood V of $\bar{v}$, a pointed, closed, convex cone C in $\mathbb{R}^q$, and a $\mathcal{C}^2$-smooth mapping $h : V \to \mathbb{R}^q$ such that $h(\bar{v}) = 0$, $\mathcal{J}h(\bar{v})$ is surjective, and that

$\Theta \cap V = \{v \in V | \ h(v) \in C\}$.

We say Θ is $\mathcal{C}^2$-cone reducible if it is $\mathcal{C}^2$-cone reducible at any $\bar{v} \in \Theta$.



- Polyhedral sets, the positive semi-definite cone, the second-order cone are $\mathcal{C}^2$-cone reducible.

[Mordukhovich-N.-Rockafellar 2017] showed that Robinson's strong regularity occurs around $(\bar{x}, \bar{p})$ for $\bar{\lambda}$ if and only if

- The following constraint nondegeneracy condition holds

$$\mathrm{Ker}\,\mathcal{J}_x G(\bar{x},\bar{p})^* \bigcap \mathrm{span} N_\Theta(G(\bar{x},\bar{p})) = \{0\}. \tag{12}$$

- A Generalized Strong Second-order Sufficient Condition (GSSOSC) holds.

Back to the dual problem again:

$$(D(y,\mu)) \qquad \min_{z\in\mathbb{R}^m} \quad \frac{\mu}{2}\|z\|^2 - \langle z, y\rangle + t \quad \text{subject to} \quad (\Phi^* z, t) \in \Theta := \operatorname{epi} g^*.$$

To apply [Mordukhovich-N.-Rockafellar 2017], we need $\Theta := \operatorname{epi} g^*$ to be $\mathcal{C}^2$-cone reducible.

- All convex piecewise linear functions g have $\mathcal{C}^2$-cone reducible $\operatorname{epi} g^*$, e.g., ℓ_1 norm, ℓ_∞ norm, 1-D total variation (semi)norm $\|Dx\|_1$, 2-D anisotropic total variation (semi)norm, ...
- The ℓ_1/ℓ_2 norm.
- Support function $g(x) = \sigma_C(x) = \sup\{\langle v, x\rangle \,|\, v \in C\}$ with closed convex set $C \subset \mathbb{R}^n$, when C is $\mathcal{C}^2$-cone reducible.
- The nuclear norm and many spectral functions.

When applying [Mordukhovich-N.-Rockafellar 2017] to the dual problems

$$(D(y,\mu)) \qquad \min_{z\in\mathbb{R}^m} \quad \frac{\mu}{2}\|z\|^2 - \langle z, y\rangle + t \quad \text{subject to} \quad (\Phi^* z, t) \in \Theta := \operatorname{epi} g^*,$$

we showed that

- The GSSOSC is free!

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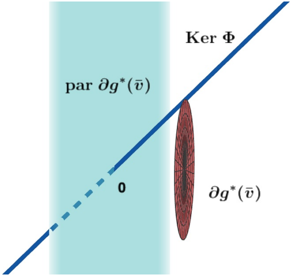
we showed that

- The GSSOSC is free!
- The constraint nondegeneracy condition turns to

$$\operatorname{Ker}\Phi \cap \boxed{\operatorname{par}\partial g^*(\bar{v})} = \{0\} \quad \text{with} \quad \bar{v} := \frac{1}{\bar{\mu}}\Phi^*(\Phi\bar{x} - \bar{y}), \tag{13}$$

where $\operatorname{par}\partial g^*(\bar{v})$ is the parallel space of $\partial g^*(\bar{v})$.



Theorem 2 (Sufficient condition for Lipschitz stability)

Let $\bar{x}$ be a solution of $P(\bar{y},\bar{\mu})$ with $(\bar{y},\bar{\mu}) \in \mathbb{R}^m \times \mathbb{R}_{++}$ and suppose that $\operatorname{epi} g^*$ is $\mathcal{C}^2$-cone reducible. If

$$\operatorname{Ker}\Phi \cap \operatorname{par}\partial g^*(\bar{v}) = \{0\},$$

then $S(y,\mu)$ is single-valued and Lipschitz continuous around $(\bar{y},\bar{\mu})$.

- Although our approach is based on second-order analysis, this is a first-order condition for Lipschitz stability. It is very simple to check.
- It does NOT need the nondegeneracy condition $\bar{v} \in \operatorname{ri}\partial g(\bar{x})$.

Is it a necessary condition?

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Theorem 3 (Necessary condition for single-valuedness)

Let $\bar{x}$ be a solution of $P(\bar{y},\bar{\mu})$ with $(\bar{y},\bar{\mu}) \in \mathbb{R}^m \times \mathbb{R}_{++}$. If $S(y,\mu)$ is single-valued around $(\bar{y},\bar{\mu})$, then

$$\operatorname{Ker}\Phi \cap \operatorname{par}\partial g^*(\bar{v}) = \{0\}.$$

Corollary 4 (Full characterization of Lipschitz stability)

Let $\bar{x}$ be a solution of $P(\bar{y},\bar{\mu})$ with $(\bar{y},\bar{\mu}) \in \mathbb{R}^m \times \mathbb{R}_{++}$ and suppose that $\operatorname{epi} g^*$ is $\mathcal{C}^2$-cone reducible. TFAE:

- **(i)** *$S(y,\mu)$ is single-valued around $(\bar{y},\bar{\mu})$.*
- **(ii)** $\operatorname{Ker}\Phi \cap \operatorname{par}\partial g^*(\bar{v}) = \{0\}$.
- **(iii)** *$S(y,\mu)$ is single-valued and Lipschitz continuous around $(\bar{y},\bar{\mu})$.*

Proof. [(i)⇒(ii)⇒(iii)⇒(i)]. □

- Recover quickly the results in [Berk-Brugiapaglia-Hoheisel 2023] for ℓ_1 norm and ℓ_1/ℓ_2 and nuclear norms in [N. 2024].

No $\mathcal{C}^2$-cone reducibility?

Example 5 (No Lipschitz-stability without $\mathcal{C}^2$-cone reducibility)

Consider the following problem

$$(P(y)) \qquad \min_{x\in\mathbb{R}^2} \quad \frac{1}{2}\left[(x_1 - x_2 - y_1)^2 + (x_1 - x_2 - y_2)^2\right] + \frac{1}{4}\left(x_1^4 + x_2^4\right).$$

With $\bar{y} = (0,0)$, $\bar{x} = (0,0)$ is an optimal solution. Moreover, $\left(y_1^{\frac{1}{3}}, y_1^{\frac{1}{3}}\right) \in S(y_1, y_1)$ is NOT Lipschitz continuous around $(0,0)$. With $g(x) = \dfrac{1}{4}(x_1^4 + x_2^4)$, we have

$$g^*(v) = \frac{3}{4}\left(v_1^{\frac{4}{3}} + v_2^{\frac{4}{3}}\right) \quad \text{and} \quad \partial g^*(\bar{v}) = \{(0,0)\} \quad \text{with} \quad \bar{v} = (0,0).$$

Obviously, $\operatorname{Ker}\Phi \cap \operatorname{par}\partial g^*(\bar{v}) = \{0\}$ and $\operatorname{epi} g^*$ is not $\mathcal{C}^2$-cone reducible.

Some important functions may be not $\mathcal{C}^2$-cone reducible:

- Convex piecewise linear-quadratic functions.
- Composite functions such as the 2-D anisotropic total variation (semi)norm $\|D^*x\|_{1,2}$ and the lifted nuclear norm $\|D^*X\|_*$ for some linear operators D^*.

With Nondegeneracy Condition

Corollary 6 (Lipschitz stability under Nondegeneracy Condition)

Let $\bar{x}$ be a solution of $P(\bar{y}, \bar{\mu})$ with $(\bar{y}, \bar{\mu}) \in \mathbb{R}^m \times \mathbb{R}_{++}$ and suppose that $\operatorname{epi} g^*$ is $\mathcal{C}^2$-cone reducible. Suppose further that the dual nondegeneracy condition

$$\bar{x} \in \operatorname{ri} \partial g^*(\bar{v}). \tag{14}$$

TFAE:

(i) $\bar{x}$ *is the unique solution of* $P(\bar{y}, \bar{\mu})$

(ii) $S(y, \mu)$ *is single-valued around* $(\bar{y}, \bar{\mu})$.

(iii) $S(y, \mu)$ *is single-valued and Lipschitz continuous around* $(\bar{y}, \bar{\mu})$.

- The dual ND condition (14) and the (primal) ND condition $\bar{v} \in \operatorname{ri} \partial g(\bar{x})$ are equivalent in many frameworks.
- Under the primal ND condition, [Lewis and Zhang 13] established the equvilance between strong solutions and tilt-stable solutions.

Full stability

In this section, we study the following optimization problem

$$P(\bar{p}) \qquad \min_{x\in\mathbb{R}^n} \quad f(x,\bar{p}) + g(x), \tag{15}$$

where $f : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ are closed, proper, convex functions. Moreover, f is $\mathcal{C}^2$ differentiable.

Definition 7 (Full stability and tilt stability)

The point $\bar{x}$ is called a fully stable optimal solution of problem (15) if there exists $\gamma > 0$ such that the solution map

$$M_\gamma(v,p) := \text{argmin}\,\{f(x,p) + g(x) - \langle v, x\rangle \mid x \in \mathbb{B}_\gamma(\bar{x})\} \quad \text{for} \quad (v,p) \in \mathbb{R}^n \times \mathbb{R}^d \tag{16}$$

is Lipschitz continuous around $(0,\bar{p}) \in \mathbb{R}^n \times \mathbb{R}^d$ with $M_\gamma(0,\bar{p}) = \bar{x}$.

The point $\bar{x}$ is called a tilt stable optimal solution of problem (15) if the mapping $M_\gamma(\cdot,\bar{p})$ is Lipschitz continuous around $0 \in \mathbb{R}^n$ with $M_\gamma(0,\bar{p}) = \bar{x}$.

$\bar{x}$ is a fully stable optimal solution of problem (15) iff it is fully stable for

$$\min_{x \in \mathbb{R}^n} \quad \frac{1}{2}\langle \nabla^2_{xx} f(\bar{x},\bar{p})(x-\bar{x}), x-\bar{x}\rangle + \langle \nabla_x f(\bar{x},\bar{p}), x-\bar{x}\rangle + f(\bar{x},\bar{p}) + g(x).$$

Theorem 8 (Characterization of full stability)

Suppose that $\bar{x}$ is an optimal solution of problem (15) and that the function g^* is $\mathcal{C}^2$-cone reducible at $\bar{v} = -\nabla_x f(\bar{x},\bar{p}) \in \partial g(\bar{x})$. Then the following are equivalent:

(i) *$\bar{x}$ is a fully stable optimal solution of problem (15).*

(ii) *$\bar{x}$ is a tilt stable optimal solution of problem (15).*

(iii) $\operatorname{Ker} \nabla^2_{xx} f(\bar{x},\bar{p}) \cap \operatorname{par} \partial g^*(\bar{v}) = \{0\}$.

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Do we have

$$\operatorname{Ker} \partial^2 g(\bar{x},\bar{v}) = \operatorname{par} \partial g^*(\bar{v})?$$

So far, we have

$$\boxed{\text{span}}\left[\operatorname{Ker}\partial^2 g(\bar{x},\bar{v})\right] = \operatorname{par}\partial g^*(\bar{v}).$$

This formula is very useful in second-order variational analysis especially for the chain rule of composite function $g \circ G$ when G is a $\mathcal{C}^2$ differentiable functions:

- $\delta_\Theta \circ G$ for constrained optimization problems, where δ_Θ is the indicator function.
- convex regularizers: isotopic total variation (semi) norm $\|D^* x\|_{1,2}$ and lifted nuclear norm $\|D^* X\|_*$.

In this work, we obtain

- A first-order simple characterization for Lipschitz stability of least-square problems.

- A first-order simple characterization for full/tilt stability of convex problems.

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We are working on some directions:

- Sensitivity analysis for our optimal solution mapping when it is single-valued and Lipschitz continuous.
- Extending these results to composite/constrained problems and variational systems.
- Using $\text{span}[\text{Ker}\,\partial^2 g(\bar{x},\bar{v})] = \text{par}\,\partial g^*(\bar{v})$ to derive chain rule of second-order structures.

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