Polar Convexity in finite dimensional Euclidean spaces

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Outline

The idea

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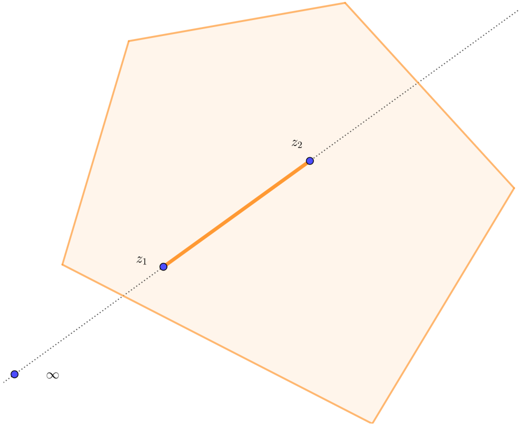

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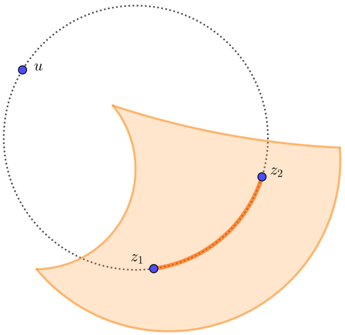

Polar convexity in $\mathbb{R}^n$

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$$T_{\mathbf{u}}(\mathbf{z}) := \begin{cases} \mathbf{u} + (\mathbf{z} - \mathbf{u})^* & \text{if } \mathbf{z} \neq \mathbf{u} \\ \infty & \text{if } \mathbf{z} = \mathbf{u} \end{cases}$$

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- Any sphere passing through $\mathbf{u}$ is sent to a hyperplane

Definition

For $\mathbf{z}_1, \mathbf{z}_2, \mathbf{u} \in \mathbb{R}^n$ distinct, define

$$\begin{aligned} \mathrm{arc}_{\mathbf{u}}\,[\mathbf{z}_1, \mathbf{z}_2] &:= \left\{\mathbf{u} + \left(t(\mathbf{z}_1 - \mathbf{u})^* + (1-t)(\mathbf{z}_2 - \mathbf{u})^*\right)^* : t \in [0,1]\right\} \\ &= \left\{ T_{\mathbf{u}}\left(t T_{\mathbf{u}}(\mathbf{z}_1) + (1-t) T_{\mathbf{u}}(\mathbf{z}_2)\right) : t \in [0,1]\right\} \end{aligned} \quad (1)$$

If $\mathbf{z}_1 = \mathbf{u}$ or $\mathbf{z}_2 = \mathbf{u}$, define $\mathrm{arc}_{\mathbf{u}}[\mathbf{z}_1, \mathbf{z}_2] := \{\mathbf{z}_1, \mathbf{z}_2\}$

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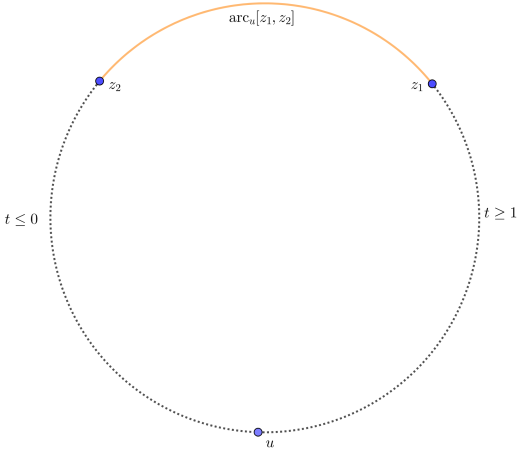
If $z_1, z_2, u \in \mathbb{C}$ (identified with $\mathbb{R}^2$), then (1) simplifies to

$$\mathrm{arc}_u[z_1, z_2] = \left\{ u + \frac{1}{\frac{t}{z_1 - u} + \frac{1-t}{z_2 - u}} : t \in [0,1] \right\}$$

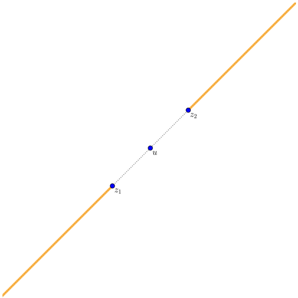
Varying the parameter t through $\mathbb{R} \cup \{\infty\}$

$$\mathbf{u} + \left(t(\mathbf{z}_1 - \mathbf{u})^* + (1-t)(\mathbf{z}_2 - \mathbf{u})^*\right)^*$$

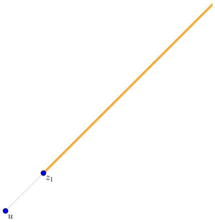
spans the following:



When $\mathbf{u}$ is between $\mathbf{z}_1$ and $\mathbf{z}_2$



When $\mathbf{z}_2 = \infty$



Let $A \subset \hat{\mathbb{R}}^n$ and $\mathbf{u} \in \hat{\mathbb{R}}^n$

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A is said to be **u**-*convex* if for any $\mathbf{z}_1, \mathbf{z}_2 \in A$, $\mathrm{arc}_{\mathbf{u}}[\mathbf{z}_1, \mathbf{z}_2] \subseteq A$

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If $\mathbf{u} \in \{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$ define $\mathrm{conv}_{\mathbf{u}}\{\mathbf{z}_1, \ldots, \mathbf{z}_k\} := \mathrm{conv}_{\mathbf{u}}\{\mathbf{z}_i : \mathbf{z}_i \neq \mathbf{u}, i = 1, \ldots, k\} \cup \{\mathbf{u}\}$

Lemma

Given $\mathbf{u}, \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^n$ *and* $t \in [0,1]$

$$\mathbf{u} + \left(t(\mathbf{z}_1 - \mathbf{u})^* + (1-t)(\mathbf{z}_2 - \mathbf{u})^*\right)^* \longrightarrow t\mathbf{z}_1 + (1-t)\mathbf{z}_2$$

as $\|\mathbf{u}\| \to \infty$

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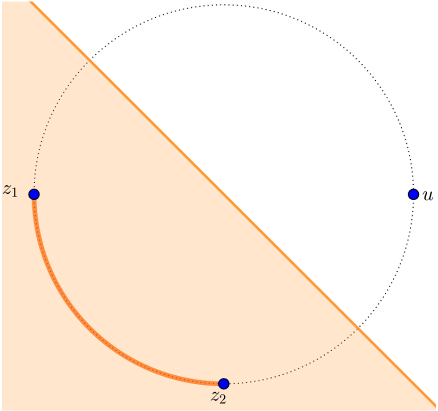
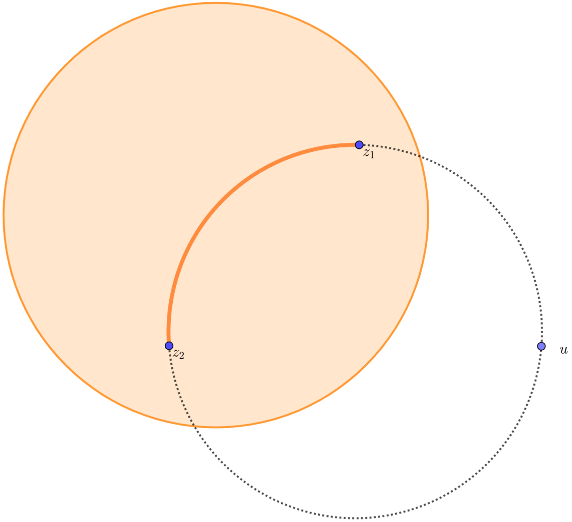
Let's see some examples

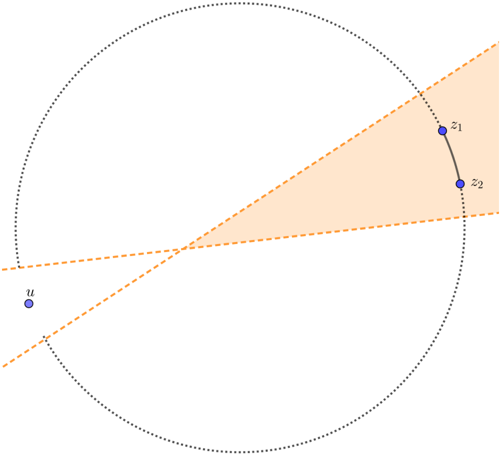
Half planes are convex with respect to any point not in their interior



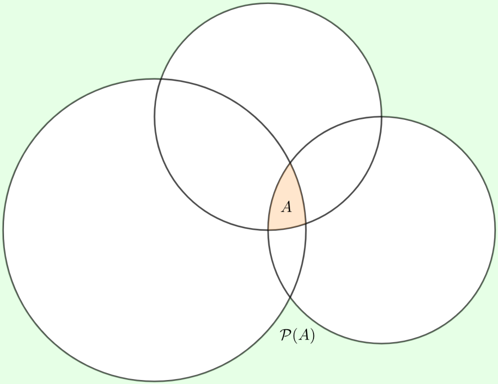
Circular domains are convex w.r.t. any point not in their interior



A cone is convex w.r.t. any point in its negative cone



The intersection of three circular domains and its pole set



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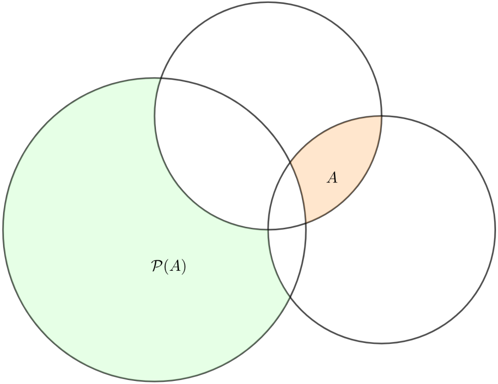

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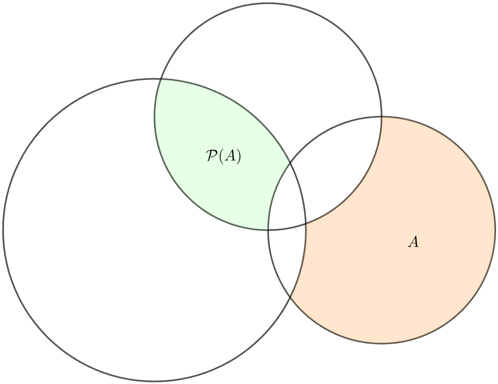
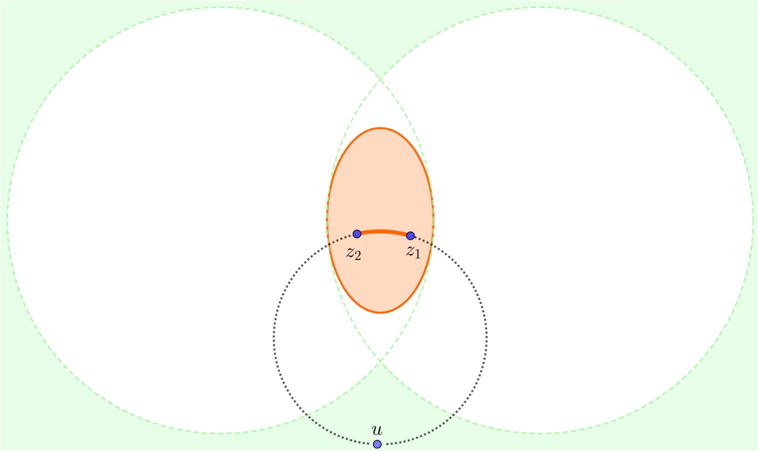


The inside of an ellipse is convex w.r.t. any point outside the two osculating circles



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Proposition

Then $\mathrm{conv}_{\mathbf{u}}(A)$ *is the intersection of all spherical domains that contain* A *and have* $\mathbf{u}$ *on their boundary, with* $\mathbf{u}$ *omitted*

Motivation

Take a polynomial $p(z) = (z - z_1)^{r_1} \cdots (z - z_k)^{r_k}$, where $\sum_{j=1}^{k} r_j = n$

with distinct zeros $z_1, \ldots, z_k$ having respective multiplicities $r_1, \ldots, r_k$

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For all $i, j \in \{1, \ldots, k\}$, define the points

$$g_{i,j} := \begin{cases} (r_i z_j + (n - r_i) z_i)/n & \text{if } i \neq j \\ \infty & \text{if } i = j \end{cases}$$

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Lemma (Specht [1959])

Every non-trivial critical points of $p(z)$ lies in

$$\operatorname{conv}\{g_{i,j} : 1 \leqslant i, j \leqslant k, i \neq j\}$$

Theorem (Sendov [2021])

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$$\mathrm{conv}\{z_1, \ldots, z_k\} \cap \bigcap_{i=1}^{k} \mathrm{conv}_{z_i}\{g_{i,1}, \ldots, g_{i,k}\}$$

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Define the polar derivative of $p(z)$ w.r.t. a pole u as

$$\mathcal{D}_u(p; z) := \begin{cases} np(z) - (z-u)p'(z) & \text{if } u \in \mathbb{C} \\ p'(z) & \text{if } u = \infty \end{cases}$$

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Theorem (Sendov, Sendov, Wang [2018])

Let $p(z)$ be a polynomial of degree n with zeroes $z_1, \dots, z_n \in \mathbb{C}$

For any $u \in \mathbb{C}$ if $\mathcal{D}_u(p; z) \not\equiv 0$, then all its zeros are in $\operatorname{conv}_u\{z_1, \dots, z_n\}$

For example, take the polynomial

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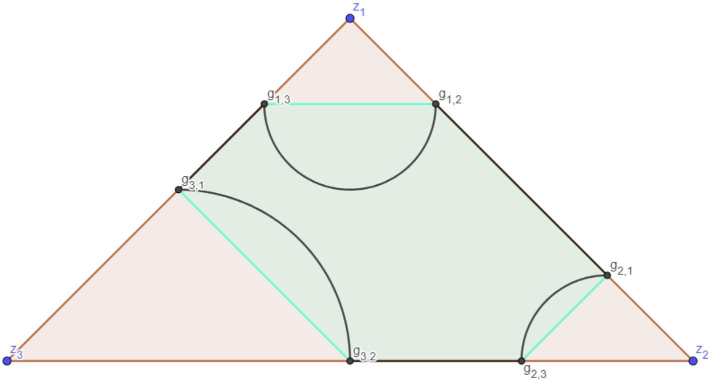

Duality theorem and its consequences

Theorem

Let $\mathbf{u}, \mathbf{v}, \mathbf{z}_1, \ldots, \mathbf{z}_k$ *be distinct points in* $\hat{\mathbb{R}}^n$ *then*

$$\mathbf{v} \in \mathrm{conv}_{\mathbf{u}}\{\mathbf{z}_1, \ldots, \mathbf{z}_k\} \quad \textit{if and only if} \quad \mathbf{u} \in \mathrm{conv}_{\mathbf{v}}\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$$

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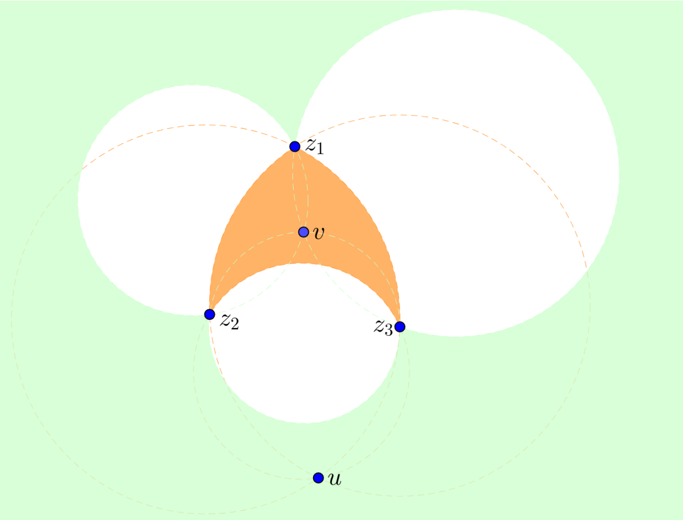
In fact, if

$$\mathbf{v} = \mathbf{u} + \left(\sum_{i=1}^{k} t_i (\mathbf{z}_i - \mathbf{u})^* \right)^* \text{ for some } t_i \geqslant 0,\ 1 \leqslant i \leqslant k \text{ and } \sum_{i=1}^{k} t_i = 1$$

then

$$\mathbf{u} = \mathbf{v} + \left(\sum_{i=1}^{k} \mu_i (\mathbf{z}_i - \mathbf{v})^* \right)^* \text{ for } \mu_i = \frac{t_i \|\mathbf{z}_i - \mathbf{v}\|^2 \|\mathbf{z}_i - \mathbf{u}\|^{-2}}{\sum_{j=1}^{k} t_j \|\mathbf{z}_j - \mathbf{v}\|^2 \|\mathbf{z}_j - \mathbf{u}\|^{-2}}$$


z_1
v
z_2
z_3
u

Let $\mathbf{u}, \mathbf{z}_1, \ldots, \mathbf{z}_k \in \hat{\mathbb{R}}^n$ be distinct points

Definition

A point $\mathbf{v} \in \mathrm{conv}_{\mathbf{u}}\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$ is $\mathbf{u}$-*extreme point* if it cannot be written as a non-trivial $\mathbf{u}$-convex combination of any two distinct points in $\mathrm{conv}_{\mathbf{u}}\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$

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Corollary

$\mathbf{z}_i \in \mathrm{conv}_{\mathbf{u}}\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$ *is* $\mathbf{u}$-*extreme* $\iff$ $\mathbf{u} \notin \mathrm{conv}_{\mathbf{z}_i}\{\mathbf{z}_1, \ldots, \mathbf{z}_{i-1}, \mathbf{z}_{i+1}, \ldots, \mathbf{z}_k\}$

Let $\mathbf{u}, \mathbf{z}_1, \ldots, \mathbf{z}_k \in \hat{\mathbb{R}}^n$ be distinct points

Definition

A point $\mathbf{v} \in \mathrm{conv}_{\mathbf{u}}\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$ is $\mathbf{u}$-*extreme point* if it cannot be written as a non-trivial $\mathbf{u}$-convex combination of any two distinct points in $\mathrm{conv}_{\mathbf{u}}\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$

The duality theorem gives a nice criteria to decide whether a point is $\mathbf{u}$-extreme or not

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Corollary

For $\mathbf{u} \notin A \subseteq \hat{\mathbb{R}}^n$, $\mathbf{v} \in \mathrm{conv}_{\mathbf{u}}(A)$ *is* $\mathbf{u}$-*extreme* $\iff$ $\mathbf{u} \notin \mathrm{conv}_{\mathbf{v}}(A)$

Theorems of the alternative

Definition

A spherical domain $S \subseteq \hat{\mathbb{R}}^n$ is said to *separate* two sets $A, B \subseteq \hat{\mathbb{R}}^n$ if

$$A \subseteq S \text{ and } B \subseteq \mathrm{cl}(S^c) \text{ or vice-versa}$$

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We say that S *strongly separates* A *and* B, if in addition

$$A \cap \partial S = \varnothing = B \cap \partial S$$

Lemma (Spherical Separation)

Let $\mathbf{u} \in \hat{\mathbb{R}}^n$ *and* A, B *be non-intersecting* $\mathbf{u}$*-convex sets in* $\hat{\mathbb{R}}^n$

Then there exists a spherical domain S*, with* $\mathbf{u}$ *on its boundary, separating* A *and* B

Moreover, if $\mathbf{u} \notin A \cup B$ *and one of the following holds*

1. A *is closed in* $\hat{\mathbb{R}}^n$ *and* B *is closed in* $\hat{\mathbb{R}}^n \setminus \{\mathbf{u}\}$
2. A *and* B *are both open*

then S *can be chosen to strongly separate* A *and* B

Lemma (Gordan's Lemma)

Let $\mathbf{u}, \mathbf{z}_1, \ldots, \mathbf{z}_k \in \hat{\mathbb{R}}^n$ *be distinct, such that* $\mathbf{u} \neq \mathbf{0}, \infty$

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or there exist some $\mathbf{a} \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$*, with* $\beta < 0$*, such that*

$$\alpha\langle \mathbf{z}_i, \mathbf{z}_i\rangle + \langle \mathbf{z}_i, \mathbf{a}\rangle + \beta > 0, \text{ for all } i = 1, \ldots, k$$

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Lemma

Let $\mathbf{z}_1, \ldots, \mathbf{z}_k \in \hat{\mathbb{R}}^n$ *be distinct and let* $\mathbf{u} := -\sum_{i=1}^{k} t_i \mathbf{z}_i$ *for some* $t_1, \ldots, t_k \in [0, \infty)$

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Definition

Let $\mathbf{z}_1, \ldots, \mathbf{z}_k, \mathbf{u} \in \hat{\mathbb{R}}^n$ be distinct, $\mathbf{u} \neq \infty$ and define

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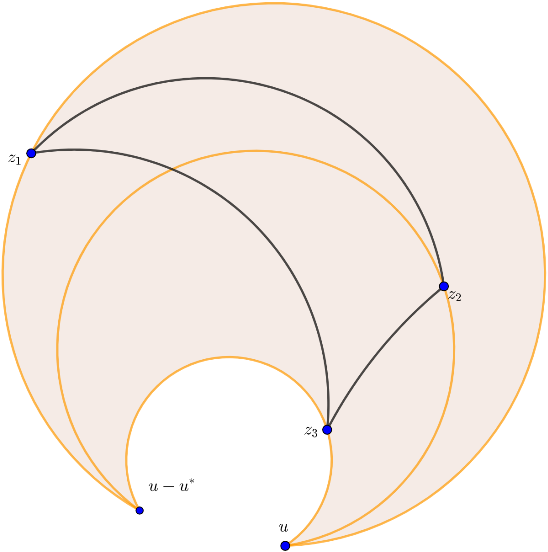
It is the union of all circular arcs through $\mathbf{u} - \mathbf{u}^*$, $\mathbf{u}$, and some $\mathbf{z} \in \mathrm{conv}_{\mathbf{u}}\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$



z_1
z_2
z_3
$u - u^*$
u

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Polar convexity with multiple poles

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It is hard to determine $\mathrm{conv}_U(Z)$ in general

However, things are easier when U and Z are finite

For any $Z \subset \hat{\mathbb{R}}^n$

Lemma

Given distinct points $\mathbf{u}_1, \mathbf{u}_2 \in \hat{\mathbb{R}}^n$, we have

$$\mathrm{conv}_{\{\mathbf{u}_1, \mathbf{u}_2\}}(Z) = \mathrm{conv}_{\mathbf{u}_1}(\mathrm{conv}_{\mathbf{u}_2}(Z)) = \mathrm{conv}_{\mathbf{u}_2}(\mathrm{conv}_{\mathbf{u}_1}(Z))$$

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Given distinct points $\mathbf{u}_1, \dots, \mathbf{u}_m \in \hat{\mathbb{R}}^n$*, we have*

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Let $n \geqslant 2$ and take distinct points $\mathbf{z}_1, \ldots, \mathbf{z}_k \in \hat{\mathbb{R}}^n$

Let the points $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \hat{\mathbb{R}}^n$, $m \geqslant 2$, be distinct

(But not necessarily distinct from $\mathbf{z}_1, \ldots, \mathbf{z}_k$)

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Consider the following family for $i \in \{1, \ldots, m\}$

$$\mathcal{L}_i := \Big\{ S \subset \hat{\mathbb{R}}^n : S \text{ closed spherical domain, } Z \subset S,\ \mathbf{u}_i \in \partial S \text{ and } U \subset \mathrm{cl}(S^c) \text{ and } S \text{ is determined by } Z \cup U \Big\}$$

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If $\mathrm{conv}_U(Z)$ has non-empty interior, these are finite

Theorem

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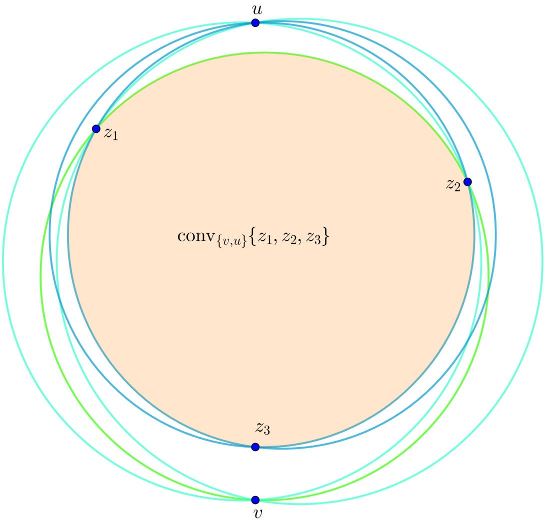
- $\mathrm{conv}_U(Z) = \bigcap_{i=1}^{m} \bigcap_{S \in \mathcal{L}_i} S$

In other words, given a point $\mathbf{z} \notin \mathrm{conv}_U(Z)$*, there exists a spherical domain* $S \in \mathcal{L}_i$ *such that* $\mathbf{z} \notin S$*, for some* $i = 1, \ldots, m$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z_1
z_2
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z_3
v

Recall

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Given distinct points $\mathbf{u}, \mathbf{z}_1, \ldots, \mathbf{z}_k \in \mathbb{R}^n$, and $t, \alpha_j, \beta_j \in [0,1]$, for $1 \leqslant j \leqslant k$, such that

$$\sum_{j=1}^{k} \alpha_j = \sum_{j=1}^{k} \beta_j = 1$$

there exist $\gamma_i, \delta_{i,j} \in [0,1]$, for $1 \leqslant i \leqslant n+1$ and $1 \leqslant j \leqslant k$, such that

$$\sum_{i=1}^{n+1} \gamma_i = \sum_{j=1}^{k} \delta_{i,j} = 1 \text{ for all } 1 \leqslant i \leqslant n+1$$

and satisfying

$$\left(t \Big(\sum_{i=1}^{k} \alpha_i (\mathbf{z}_i - \mathbf{u}) \Big)^* + (1-t) \Big(\sum_{i=1}^{k} \beta_i (\mathbf{z}_i - \mathbf{u}) \Big)^* \right)^* = \sum_{i=1}^{n+1} \gamma_i \Big(\sum_{j=1}^{k} \delta_{i,j} (\mathbf{z}_j - \mathbf{u})^* \Big)^*$$

Restricting it to $\mathbb{C}$, we get

Proposition

Given distinct points $\mathbf{u}, \mathbf{z}_1, \ldots, \mathbf{z}_k \in \mathbb{C}$*, and* $t, \alpha_j, \beta_j \in [0,1]$*, for* $1 \leqslant j \leqslant k$*, such that*

$$\sum_{j=1}^{k} \alpha_j = \sum_{j=1}^{k} \beta_j = 1$$

there exist $\gamma_i, \delta_{i,j} \in [0,1]$*, for* $1 \leqslant i \leqslant 3$ *and* $1 \leqslant j \leqslant k$*, such that*

$$\sum_{i=1}^{3} \gamma_i = \sum_{j=1}^{k} \delta_{i,j} = 1 \textit{ for all } 1 \leqslant i \leqslant 3$$

and satisfying

$$\frac{1}{\frac{t}{\sum_{i=1}^{k} \alpha_i (\mathbf{z}_i - \mathbf{u})} + \frac{1-t}{\sum_{i=1}^{k} \beta_i (\mathbf{z}_i - \mathbf{u})}} = \frac{\gamma_1}{\sum_{j=1}^{k} \frac{\delta_{1,j}}{\mathbf{z}_j - \mathbf{u}}} + \frac{\gamma_2}{\sum_{j=1}^{k} \frac{\delta_{2,j}}{\mathbf{z}_j - \mathbf{u}}} + \frac{\gamma_3}{\sum_{j=1}^{k} \frac{\delta_{3,j}}{\mathbf{z}_j - \mathbf{u}}}$$

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Moreover, what are the pairs of sets A, B such that $\mathcal{P}(A) = B$ and $\mathcal{P}(B) = A$?

Thank You!

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