Polar Convexity in finite dimensional Euclidean spaces

Shubhankar Bhatt

(Joint work with Hristo Sendov) Department of Mathematics

Western University, Ontario, Canada

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Outline



2 Polar Convexity in \mathbb{R}^n

Ontivation

- Ouality theorem and its consequences
- 5 Theorems of the alternative
- 6 Polar convexity with multiple poles

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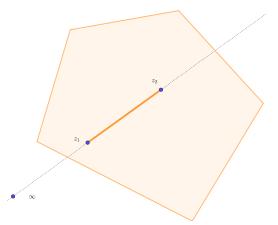
• Convex sets contain the line segment between any two points in them

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- Convex sets contain the line segment between any two points in them
- $\,\bullet\,$ Lines between two points are circles passing through $\infty\,$

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- Convex sets contain the line segment between any two points in them
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 $\bullet~\mbox{Replace}~\infty$ with a finite point ${\bf u}$

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- $\bullet~$ Replace ∞ with a finite point ${\bf u}$
- Consider the circle passing through $\mathbf{u}, \mathbf{z}_1, \mathbf{z}_2$

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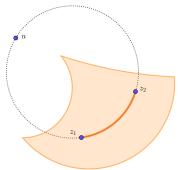
- $\bullet~$ Replace ∞ with a finite point u
- Consider the circle passing through $\mathbf{u}, \mathbf{z}_1, \mathbf{z}_2$
- $\bullet~ \mathsf{Let}~\mathrm{arc}_u[z_1,z_2]$ be the arc on that circle between z_1 and z_2 which does not contain u

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- $\bullet~$ Replace ∞ with a finite point u
- Consider the circle passing through **u**, **z**₁, **z**₂
- Let $\operatorname{arc}_u[z_1, z_2]$ be the arc on that circle between z_1 and z_2 which does not contain u
- Define a u-convex set to contain $\operatorname{arc}_u[z_1, z_2]$ for any two points z_1, z_2 in it

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Polar convexity in \mathbb{R}^n

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• Let $\hat{\mathbb{R}}^n$ be the one point compactification $\mathbb{R}^n \cup \{\infty\}$

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- Let $\hat{\mathbb{R}}^n$ be the one point compactification $\mathbb{R}^n \cup \{\infty\}$
- How to parametrize a circular arc in $\hat{\mathbb{R}}^n$?

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- How to parametrize a circular arc in $\hat{\mathbb{R}}^n$?

• Let
$$\mathbf{z}^* := \frac{\mathbf{z}}{\|\mathbf{z}\|^2}$$
 ($\mathbf{0}^* = \infty$ and $\infty^* = \mathbf{0}$) and for $\mathbf{u} \in \mathbb{R}^n$

$$\mathcal{T}_{u}(z) := \begin{cases} u + (z - u)^{*} & \text{ if } z \neq u \\ \\ \infty & \text{ if } z = u \end{cases}$$

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• Define $T_{\infty} := Id_{\hat{\mathbb{R}}^n}$

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- T_u is a Möbius transformation and an involution
- T_u sends a sphere (or hyperplane) to a sphere (or hyperplane)
- T_u sends circles (or lines) to circles (or lines)
- \bullet Any sphere passing through u is sent to a hyperplane

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Definition

For $\mathbf{z}_1, \mathbf{z}_2, \mathbf{u} \in \mathbb{R}^n$ distinct, define

$$\operatorname{arc}_{\mathbf{u}} [\mathbf{z}_{1}, \mathbf{z}_{2}] := \left\{ \mathbf{u} + \left(t(\mathbf{z}_{1} - \mathbf{u})^{*} + (1 - t)(\mathbf{z}_{2} - \mathbf{u})^{*} \right)^{*} : t \in [0, 1] \right\}$$

$$= \left\{ T_{\mathbf{u}} \left(t T_{\mathbf{u}}(\mathbf{z}_{1}) + (1 - t) T_{\mathbf{u}}(\mathbf{z}_{2}) \right) : t \in [0, 1] \right\}$$

$$(1)$$

If $\mathbf{z}_1 = \mathbf{u}$ or $\mathbf{z}_2 = \mathbf{u}$, define $\operatorname{arc}_{\mathbf{u}}[\mathbf{z}_1, \mathbf{z}_2] := \{\mathbf{z}_1, \mathbf{z}_2\}$

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 or $\mathbf{z}_2 = \mathbf{u}$, define $\operatorname{arc}_{\mathbf{u}}[\mathbf{z}_1, \mathbf{z}_2] := \{\mathbf{z}_1, \mathbf{z}_2\}$

If $z_1, z_2, u \in \mathbb{C}$ (identified with \mathbb{R}^2), then (1) simplifies to

$$\operatorname{arc}_{u}[z_{1}, z_{2}] = \left\{ u + \frac{1}{\frac{t}{z_{1}-u} + \frac{1-t}{z_{2}-u}} : t \in [0, 1] \right\}$$

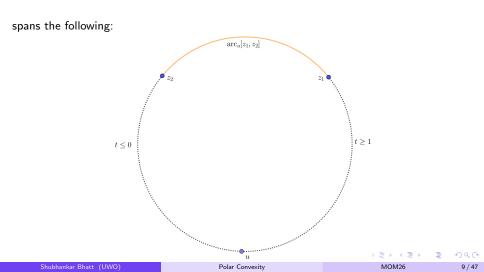
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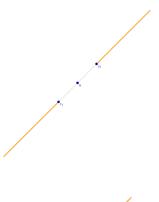
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Varying the parameter t through $\mathbb{R} \cup \{\infty\}$

$$u + (t(z_1 - u)^* + (1 - t)(z_2 - u)^*)^*$$



When \boldsymbol{u} is between \boldsymbol{z}_1 and \boldsymbol{z}_2



When $\mathbf{z}_2 = \infty$

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Definition

A is said to be **u**-convex if for any $z_1, z_2 \in A$, $\operatorname{arc}_u[z_1, z_2] \subseteq A$

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Define $conv_u(A)$ to be the smallest with respect to inclusion **u**-convex set containing A

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Definition

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Define $\operatorname{conv}_{\mathbf{u}}(A)$ to be the smallest with respect to inclusion **u**-convex set containing ADefine the pole set, $\mathcal{P}(A)$, as the set of all points $\mathbf{u} \in \mathbb{R}^n$ such that A is **u**-convex

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Definition

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Definition

Given points $\mathbf{z}_1, \ldots, \mathbf{z}_k \in \hat{\mathbb{R}}^n$ and a $\mathbf{u} \in \hat{\mathbb{R}}^n$ distinct from them, define

$$\operatorname{conv}_{\mathbf{u}}\{\mathbf{z}_1,\ldots,\mathbf{z}_k\} := \left\{\mathbf{u} + \left(\sum_{i=1}^k t_i(\mathbf{z}_i-\mathbf{u})^*\right)^* : t_i \ge 0 \text{ with } \sum_{i=1}^k t_i = 1\right\}$$

 $\mathsf{If}\; u \in \{z_1, \dots, z_k\} \; \mathsf{define}\; \mathrm{conv}_u\{z_1, \dots, z_k\} := \mathrm{conv}_u\{z_i: z_i \neq u, i = 1, \dots, k\} \cup \{u\}$

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Given $\mathbf{u}, \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^n$ and $t \in [0, 1]$

$$\mathbf{u} + \left(t(\mathbf{z}_1 - \mathbf{u})^* + (1 - t)(\mathbf{z}_2 - \mathbf{u})^*\right)^* \longrightarrow t\mathbf{z}_1 + (1 - t)\mathbf{z}_2$$

as $\|\mathbf{u}\| \to \infty$

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Given $\mathbf{u}, \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^n$ and $t \in [0, 1]$

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The transformation T_u maps u-convex sets to convex sets

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The transformation T_u maps u-convex sets to convex sets

All of the classical results can be translated

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Given $\mathbf{u}, \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^n$ and $t \in [0, 1]$

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as $\|\mathbf{u}\| \to \infty$

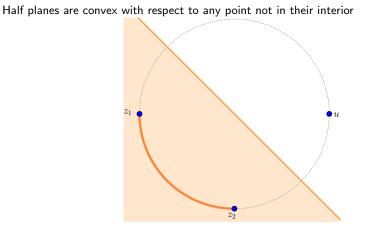
The transformation T_u maps u-convex sets to convex sets

All of the classical results can be translated

Let's see some examples

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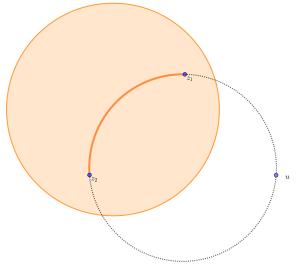
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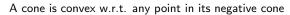
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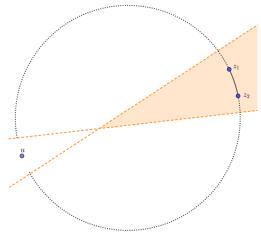
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Circular domains are convex w.r.t. any point not in their interior



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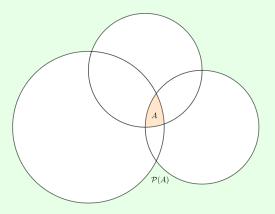




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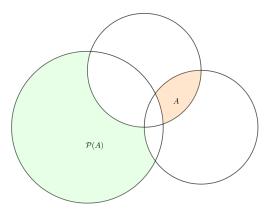
The intersection of three circular domains and its pole set



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The intersection of three circular domains and its pole set

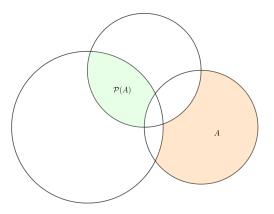
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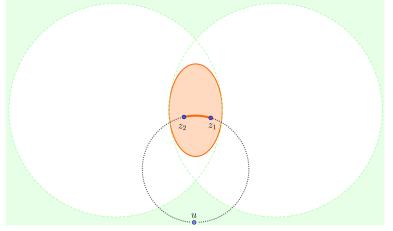
The intersection of three circular domains and its pole set

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The inside of an ellipse is convex w.r.t. any point outside the two osculating circles



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As one may notice, spherical domains are central to the theory

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As one may notice, spherical domains are central to the theory Let $A \subseteq \hat{\mathbb{R}}^n$ be a set not containing **u**

As one may notice, spherical domains are central to the theory

Let $A \subseteq \hat{\mathbb{R}}^n$ be a set not containing **u**

Proposition

Then $\operatorname{conv}_{\mathbf{u}}(A)$ is the intersection of all spherical domains that contain A and have \mathbf{u} on their boundary, with \mathbf{u} omitted

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Motivation

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Take a polynomial
$$p(z) = (z-z_1)^{r_1}\cdots(z-z_k)^{r_k}, \;\;$$
 where $\sum_{j=1}^k r_j = n$

with distinct zeros z_1, \ldots, z_k having respective multiplicities r_1, \ldots, r_k

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Motivation

Take a polynomial $p(z) = (z - z_1)^{r_1} \cdots (z - z_k)^{r_k}$, where $\sum_{j=1}^k r_j = n$ with distinct zeros z_1, \ldots, z_k having respective multiplicities r_1, \ldots, r_k For all $i, j \in \{1, \ldots, k\}$, define the points

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$$g_{i,j} := \begin{cases} (r_i z_j + (n - r_i) z_i)/n & \text{if } i \neq j \\ \infty & \text{if } i = j \end{cases}$$

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$$g_{i,j} := \begin{cases} (r_i z_j + (n - r_i) z_i)/n & \text{if } i \neq j \\ \infty & \text{if } i = j \end{cases}$$

Lemma (Specht [1959])

Every non-trivial critical points of p(z) lies in

 $\operatorname{conv}\{g_{i,j}: 1 \leqslant i, j \leqslant k, i \neq j\}$

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Theorem (Sendov [2021])

Every non-trivial critical points of p(z) lies in

$$\operatorname{conv}\{z_1,\ldots,z_k\}\cap \bigcap_{i=1}^k \operatorname{conv}_{z_i}\{g_{i,1},\ldots,g_{i,k}\}$$

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Theorem (Sendov [2021])

Every non-trivial critical points of p(z) lies in

$$\operatorname{conv}\{z_1,\ldots,z_k\}\cap \bigcap_{i=1}^k \operatorname{conv}_{z_i}\{g_{i,1},\ldots,g_{i,k}\}$$

Define the polar derivative of p(z) w.r.t. a pole u as

$$\mathcal{D}_u(p;z) := \begin{cases} np(z) - (z-u)p'(z) & \text{if } u \in \mathbb{C} \\ p'(z) & \text{if } u = \infty \end{cases}$$

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Motivation

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Theorem (Sendov, Sendov, Wang [2018])

Let p(z) be a polynomial of degree n with zeroes $z_1, \ldots, z_n \in \mathbb{C}$

For any $u \in \mathbb{C}$ if $\mathcal{D}_u(p; z) \neq 0$, then all its zeros are in $\operatorname{conv}_u\{z_1, \ldots, z_n\}$

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For example, take the polynomial

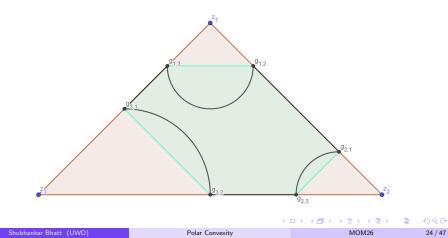
$$(z-i)(z-1)(z+1)^2$$

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Motivatio

For example, take the polynomial

$$(z-i)(z-1)(z+1)^2$$



Duality theorem and its consequences

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Theorem

Let $\mathbf{u}, \mathbf{v}, \mathbf{z}_1, \dots, \mathbf{z}_k$ be distinct points in $\hat{\mathbb{R}}^n$ then

 $\mathbf{v} \in \operatorname{conv}_{\mathbf{u}} \{ \mathbf{z}_1, \dots, \mathbf{z}_k \}$ if and only if $\mathbf{u} \in \operatorname{conv}_{\mathbf{v}} \{ \mathbf{z}_1, \dots, \mathbf{z}_k \}$

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Theorem

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In fact, if

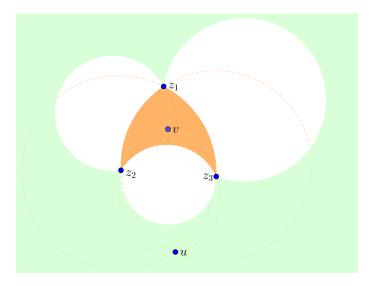
$$\mathbf{v} = \mathbf{u} + \left(\sum_{i=1}^{k} t_i (\mathbf{z}_i - \mathbf{u})^*\right)^*$$
 for some $t_i \ge 0$, $1 \le i \le k$ and $\sum_{i=1}^{k} t_i = 1$

then

$$\mathbf{u} = \mathbf{v} + \left(\sum_{i=1}^{k} \mu_i (\mathbf{z}_i - \mathbf{v})^*\right)^* \text{ for } \mu_i = \frac{t_i \|\mathbf{z}_i - \mathbf{v}\|^2 \|\mathbf{z}_i - \mathbf{u}\|^{-2}}{\sum_{j=1}^{k} t_j \|\mathbf{z}_j - \mathbf{v}\|^2 \|\mathbf{z}_j - \mathbf{u}\|^{-2}}$$

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Definition

A point $v \in \mathrm{conv}_u\{z_1,\ldots,z_k\}$ is u-extreme point if it cannot be written as a non-trivial

u-convex combination of any two distinct points in $conv_u \{z_1, \ldots, z_k\}$

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Definition

A point $\mathbf{v} \in \operatorname{conv}_{\mathbf{u}}\{\mathbf{z}_1, \dots, \mathbf{z}_k\}$ is \mathbf{u} -extreme point if it cannot be written as a non-trivial \mathbf{u} -convex combination of any two distinct points in $\operatorname{conv}_{\mathbf{u}}\{\mathbf{z}_1, \dots, \mathbf{z}_k\}$

The duality theorem gives a nice criteria to decide whether a point is \mathbf{u} -extreme or not

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 $z_i \in \operatorname{conv}_u\{z_1, \dots, z_k\} \text{ is u-extreme } \iff u \notin \operatorname{conv}_{z_i}\{z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_k\}$

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Shubhankar Bhatt (UWO)	Polar Convexity		MOM	26		28 / 47

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```

Corollary

For $\mathbf{u} \notin A \subseteq \hat{\mathbb{R}}^n$, $\mathbf{v} \in \operatorname{conv}_{\mathbf{u}}(A)$ is \mathbf{u} -extreme $\iff \mathbf{u} \notin \operatorname{conv}_{\mathbf{v}}(A)$

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Theorems of the alternative

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A spherical domain $S \subseteq \hat{\mathbb{R}}^n$ is said to *separate* two sets $A, B \subseteq \hat{\mathbb{R}}^n$ if

$$A \subseteq S$$
 and $B \subseteq cl(S^c)$ or vice-versa

Such a spherical domain (or boundary of the spherical domain) is called a *separating spherical domain* (or *separating sphere*) for the pair A, B

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Such a spherical domain (or boundary of the spherical domain) is called a *separating spherical domain* (or *separating sphere*) for the pair A, BWe say that *S strongly separates A and B*, if in addition

$$A \cap \partial S = \emptyset = B \cap \partial S$$

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Lemma (Spherical Separation)

Let $\mathbf{u} \in \hat{\mathbb{R}}^n$ and A, B be non-intersecting **u**-convex sets in $\hat{\mathbb{R}}^n$

Then there exists a spherical domain S, with \mathbf{u} on its boundary, separating A and B

Moreover, if $\mathbf{u} \notin A \cup B$ and one of the following holds

- A is closed in $\hat{\mathbb{R}}^n$ and B is closed in $\hat{\mathbb{R}}^n \setminus \{\mathbf{u}\}$
- A and B are both open

then S can be chosen to strongly separate A and B

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Lemma (Gordan's Lemma)

Let $\mathbf{u}, \mathbf{z}_1, \dots, \mathbf{z}_k \in \hat{\mathbb{R}}^n$ be distinct, such that $\mathbf{u} \neq \mathbf{0}, \infty$

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Either there are numbers $t_1, \ldots, t_k \in [0, 1]$ with $\sum_{i=1}^k t_i = 1$, such that

$$(\mathbf{0}-\mathbf{u})^* = \sum_{i=1}^k t_i (\mathbf{z}_i - \mathbf{u})^*$$

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or there exist some $\mathbf{a} \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$, with $\beta < 0$, such that

$$\alpha \langle \mathbf{z}_i, \mathbf{z}_i \rangle + \langle \mathbf{z}_i, \mathbf{a} \rangle + \beta > 0, \text{ for all } i = 1, \dots, k$$

 $\alpha \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{a} \rangle + \beta = 0$

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Lemma

Let $\mathbf{z}_1, \ldots, \mathbf{z}_k \in \hat{\mathbb{R}}^n$ be distinct and let $\mathbf{u} := -\sum_{i=1}^k t_i \mathbf{z}_i$ for some $t_1, \ldots, t_k \in [0, \infty)$ For any $\mathbf{v} \in \mathbb{R}^n \setminus {\mathbf{u}}$

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$$lpha\langle \mathbf{z}_i, \mathbf{z}_i
angle + \langle \mathbf{z}_i, \mathbf{a}
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Let $\mathbf{z}_1, \ldots, \mathbf{z}_k, \mathbf{u} \in \hat{\mathbb{R}}^n$ be distinct, $\mathbf{u} \neq \infty$ and define

$$\operatorname{cone}_{\mathbf{u}}\{\mathbf{z}_1,\ldots,\mathbf{z}_k\} := \left\{\mathbf{u} + \left(\sum_{i=1}^k t_i(\mathbf{u} + (\mathbf{z}_i - \mathbf{u})^*) - \mathbf{u}\right)^* : t_j \in [0,\infty)\right\} \cup \{\mathbf{u}\}$$

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This is the image under ${\mathcal T}_u$ of $\operatorname{cone}\{{\mathcal T}_u(z_1),\ldots,{\mathcal T}_u(z_k)\}\cup\{\infty\}$

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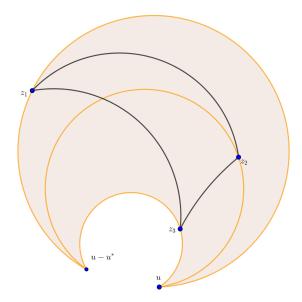
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This is the image under T_u of $\operatorname{cone}\{T_u(z_1), \ldots, T_u(z_k)\} \cup \{\infty\}$

It is the union of all circular arcs through $u-u^*,\,u,$ and some $z\in \mathrm{conv}_u\{z_1,\ldots,z_k\}$

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hubhankar Bhatt (UWO)	Polar Convexity	MOM26	35 / 47

Lemma (Farkas' Lemma)

Let $\mathbf{u}, \mathbf{z}_1, \dots, \mathbf{z}_k \in \hat{\mathbb{R}}^n$ be distinct, $\mathbf{u} \neq \infty$ and let $\mathbf{v} \in \hat{\mathbb{R}}^n \smallsetminus \{\mathbf{u}\}$

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Lemma (Farkas' Lemma)

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$$\begin{aligned} \alpha \langle \mathbf{z}_i, \mathbf{z}_i \rangle + \langle \mathbf{z}_i, \mathbf{a} \rangle + \beta &\leq 0, \text{ for all } i = 1, \dots, k \\ \alpha \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{a} \rangle + \beta > 0 \\ \alpha \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{a} \rangle + \beta = 0 \\ \alpha \langle \mathbf{u}, \mathbf{u} \rangle - \alpha - \beta = 0 \end{aligned}$$

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Polar convexity with multiple poles

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However a set can be convex with respect to multiple poles

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However a set can be convex with respect to multiple poles

Definition

Given $U, Z \subseteq \hat{\mathbb{R}}^n$ define the convex hull of Z with respect to U, denoted by $\operatorname{conv}_U(Z)$, to be the smallest set in $\hat{\mathbb{R}}^n$ containing Z and convex with respect to each $\mathbf{u} \in U$

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However a set can be convex with respect to multiple poles

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If $U = \emptyset$ then $\operatorname{conv}_U(Z) = Z$

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However a set can be convex with respect to multiple poles

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- If $U = \emptyset$ then $\operatorname{conv}_U(Z) = Z$
- If $Z = \emptyset$ then $\operatorname{conv}_U(Z) = \emptyset$

It is hard to determine $\operatorname{conv}_U(Z)$ in general

However a set can be convex with respect to multiple poles

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It is hard to determine $\operatorname{conv}_U(Z)$ in general

However, things are easier when U and Z are finite

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For any $Z \subset \hat{\mathbb{R}}^n$

Lemma

Given distinct points $\mathbf{u}_1, \mathbf{u}_2 \in \hat{\mathbb{R}}^n$, we have

$$\operatorname{conv}_{\{\boldsymbol{u}_1,\boldsymbol{u}_2\}}(Z) = \operatorname{conv}_{\boldsymbol{u}_1}(\operatorname{conv}_{\boldsymbol{u}_2}(Z)) = \operatorname{conv}_{\boldsymbol{u}_2}(\operatorname{conv}_{\boldsymbol{u}_1}(Z))$$

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Lemma

Given distinct points $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \hat{\mathbb{R}}^n$, we have

$$\operatorname{conv}_{\{\mathbf{u}_1,\ldots,\mathbf{u}_m\}}(Z) = \operatorname{conv}_{\mathbf{u}_m}(\operatorname{conv}_{\{\mathbf{u}_1,\ldots,\mathbf{u}_{m-1}\}}(Z))$$

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Let $n \ge 2$ and take distinct points $\mathbf{z}_1, \ldots, \mathbf{z}_k \in \mathbb{R}^n$ Let the points $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \mathbb{R}^n$, $m \ge 2$, be distinct (But not necessarily distinct from $\mathbf{z}_1, \ldots, \mathbf{z}_k$)

Let

 $Z := \{\mathbf{z}_1, \dots, \mathbf{z}_k\}$ $U := \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$

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$$U := \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$$

Consider the following family for $i \in \{1, \ldots, m\}$

$$\mathcal{L}_i := \left\{ S \subset \hat{\mathbb{R}}^n : S \text{ closed spherical domain, } Z \subset S, \mathbf{u}_i \in \partial S \text{ and} \\ U \subset \operatorname{cl}(S^c) \text{ and } S \text{ is determined by } Z \cup U \right\}$$

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Let $n \ge 2$ and take distinct points $\mathbf{z}_1, \ldots, \mathbf{z}_k \in \hat{\mathbb{R}}^n$ Let the points $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \hat{\mathbb{R}}^n$, $m \ge 2$, be distinct (But not necessarily distinct from $\mathbf{z}_1, \ldots, \mathbf{z}_k$) Let

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If $\operatorname{conv}_U(Z)$ has non-empty interior, these are finite

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Jilu	unan	Nai L	matt j	0000	,

If $\operatorname{conv}_U(Z)$ has non-empty interior then the boundary of $\operatorname{conv}_U(Z)$ is made up of pieces of the boundaries of closed spherical domains *S* with the following properties:

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If $\operatorname{conv}_U(Z)$ has non-empty interior then the boundary of $\operatorname{conv}_U(Z)$ is made up of pieces of the boundaries of closed spherical domains S with the following properties:

• Each S lies in \mathcal{L}_i , for some $i = 1, \ldots, m$

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- Each S lies in \mathcal{L}_i , for some $i = 1, \ldots, m$
- Each piece of the boundary is of the form $\operatorname{conv}_{\partial S \cap U}(\partial S \cap Z)$

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- Each S lies in \mathcal{L}_i , for some $i = 1, \ldots, m$
- Each piece of the boundary is of the form $\operatorname{conv}_{\partial S \cap U}(\partial S \cap Z)$
- $\operatorname{conv}_U(Z) = \bigcap_{i=1}^m \bigcap_{S \in \mathcal{L}_i} S$

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If $\operatorname{conv}_U(Z)$ has non-empty interior then the boundary of $\operatorname{conv}_U(Z)$ is made up of pieces of the boundaries of closed spherical domains S with the following properties:

- Each S lies in \mathcal{L}_i , for some $i = 1, \ldots, m$
- Each piece of the boundary is of the form $\operatorname{conv}_{\partial S \cap U}(\partial S \cap Z)$

•
$$\operatorname{conv}_U(Z) = \bigcap_{i=1}^m \bigcap_{S \in \mathcal{L}_i} S$$

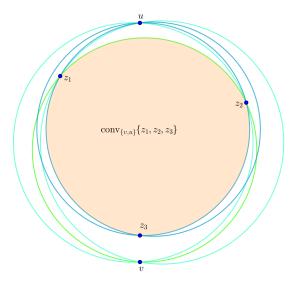
In other words, given a point $\mathbf{z} \notin \operatorname{conv}_U(Z)$, there exists a spherical domain $S \in \mathcal{L}_i$

such that $\mathbf{z} \notin S$, for some $i = 1, \ldots, m$

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Recall

$$\operatorname{conv}_{u}(\operatorname{conv}_{\infty}(Z)) = \operatorname{conv}_{\infty}(\operatorname{conv}_{u}(Z))$$

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Recall

$$\operatorname{conv}_{\boldsymbol{\mathsf{u}}}(\operatorname{conv}_{\boldsymbol{\infty}}(Z)) = \operatorname{conv}_{\boldsymbol{\infty}}(\operatorname{conv}_{\boldsymbol{\mathsf{u}}}(Z))$$

Given distinct points $\mathbf{u}, \mathbf{z}_1, \dots, \mathbf{z}_k \in \mathbb{R}^n$, and $t, \alpha_j, \beta_j \in [0, 1]$, for $1 \leq j \leq k$, such that

$$\sum_{j=1}^{k} \alpha_j = \sum_{j=1}^{k} \beta_j = 1$$

there exist $\gamma_i, \delta_{i,j} \in [0,1]$, for $1 \leq i \leq n+1$ and $1 \leq j \leq k$, such that

$$\sum\limits_{i=1}^{n+1} \gamma_i = \sum\limits_{j=1}^k \delta_{i,j} = 1$$
 for all $1 \leqslant i \leqslant n+1$

and satisfying

$$\left(t\left(\sum_{i=1}^{k}\alpha_{i}(\mathbf{z}_{i}-\mathbf{u})\right)^{*}+(1-t)\left(\sum_{i=1}^{k}\beta_{i}(\mathbf{z}_{i}-\mathbf{u})\right)^{*}\right)^{*}=\sum_{i=1}^{n+1}\gamma_{i}\left(\sum_{j=1}^{k}\delta_{i,j}(\mathbf{z}_{j}-\mathbf{u})^{*}\right)^{*}$$

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Restricting it to \mathbb{C} , we get

Proposition

Given distinct points $\mathbf{u}, \mathbf{z}_1, \ldots, \mathbf{z}_k \in \mathbb{C}$, and $t, \alpha_j, \beta_j \in [0, 1]$, for $1 \leq j \leq k$, such that

$$\sum_{j=1}^k \alpha_j = \sum_{j=1}^k \beta_j = 1$$

there exist $\gamma_i, \delta_{i,j} \in [0,1]$, for $1 \leq i \leq 3$ and $1 \leq j \leq k$, such that

$$\sum\limits_{i=1}^{3}\gamma_{i}=\sum\limits_{j=1}^{k}\delta_{i,j}=1$$
 for all $1\leqslant i\leqslant 3$

and satisfying

$$\frac{1}{\frac{t}{\sum_{i=1}^{k} \alpha_i(\mathbf{z}_i - \mathbf{u})} + \frac{1 - t}{\sum_{i=1}^{k} \beta_i(\mathbf{z}_i - \mathbf{u})}} = \frac{\gamma_1}{\sum_{j=1}^{k} \frac{\delta_{1,j}}{\mathbf{z}_j - \mathbf{u}}} + \frac{\gamma_2}{\sum_{j=1}^{k} \frac{\delta_{2,j}}{\mathbf{z}_j - \mathbf{u}}} + \frac{\gamma_3}{\sum_{j=1}^{k} \frac{\delta_{3,j}}{\mathbf{z}_j - \mathbf{u}}}$$

For $A \subseteq \hat{\mathbb{R}}^n$ we have $A \subseteq \mathcal{P}(\mathcal{P}(A))$

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For $A \subseteq \hat{\mathbb{R}}^n$ we have $A \subseteq \mathcal{P}(\mathcal{P}(A))$

As a consequence, we get two increasing chains

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For $A \subseteq \hat{\mathbb{R}}^n$ we have $A \subseteq \mathcal{P}(\mathcal{P}(A))$

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 $A \subseteq \mathcal{P}(\mathcal{P}(A)) \subseteq \mathcal{P}(\mathcal{P}(\mathcal{P}(A)))) \subseteq \cdots$

 $\mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(\mathcal{P}(A))) \subseteq \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(A))))) \subseteq \cdots$

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$$\mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(\mathcal{P}(A))) \subseteq \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(A))))) \subseteq \cdots$$

What are the sets A, B such that $\mathcal{P}(A) \supseteq B$ and $\mathcal{P}(B) \supseteq A$?

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For $A \subseteq \hat{\mathbb{R}}^n$ we have $A \subseteq \mathcal{P}(\mathcal{P}(A))$

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What are the sets A, B such that $\mathcal{P}(A) \supseteq B$ and $\mathcal{P}(B) \supseteq A$?

Moreover, what are the pairs of sets A, B such that $\mathcal{P}(A) = B$ and $\mathcal{P}(B) = A$?

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Thank You!

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