Single Element Error Correction in a Euclidean Distance Matrix

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Abstract

We consider the *exact* error correction of a noisy Euclidean distance matrix, **EDM**, where the elements are the squared distances between n points in $\mathbb{R}^d$. For our problem we are given two facts: (i) the embedding dimension, $d = \text{edim}(D)$, (ii) *exactly one* distance in the data is corrupted by *nonzero noise*. But we do not know the magnitude nor position of the noise. Thus there is a combinatorial element to the problem. We present three solution techniques. These use three divide and conquer strategies in combination with three versions of facial reduction that use: exposing vectors, facial vectors, and Gale transforms. This sheds light on the connections between the various forms of facial reduction related to Gale transforms. Our highly successful empirics confirm the success of these approaches as we can solve huge problems of the order of $100,000$ nodes in approximately one minute to machine precision.

Our algorithm depends on identifying whether a principal submatrix of the **EDM** contains the corrupted element. We provide a theorem for doing this that is related to the existing results for identifying *yielding* elements, i.e., we provide a characterization for guaranteeing the perturbed **EDM** remains an **EDM** with embedding dimension d. The characterization is particularly simple in the $d = 2$ case.

In addition, we characterize when the intuitive approach of the nearest **EDM** problem, **NEDM**, solves our problem. In fact, we show that this happens if, and only if, the original distance element is 0, degenerate, and the perturbation is negative.

Keywords: distance geometry, Euclidean distance matrices, Lindenstrauss operator, yielding interval, error correction, facial reduction.

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1 Introduction

We consider error correction for a given *Euclidean distance matrix,* ***EDM***, D, where the elements are the squared distances between n points in $\mathbb{R}^d$ with d the embedding dimension. For our problem, we know the embedding dimension, and it is given that *exactly one* distance is corrupted with nonzero noise. But we do not know the magnitude or position of the noisy distance. Thus our problem is a **EDM** completion problem with known embedding dimension where the position of the unknown element is not given. This means that there is a hard combinatorial element to the problem. We use three different *divide and conquer,* ***D&C***, approaches in combination with three different versions of *facial reduction,* ***FR***, one of which involves Gale

transforms. These methods solve this problem efficiently and accurately in all but two hard cases. The results are related to *yielding* in an **EDM** studied in [2], i.e., characterizing when a perturbation of one element yields an **EDM**. We also show that the intuitive approach of finding the nearest **EDM** only works in the trivial case where the original distance is 0, degenerate, and the perturbation is negative.

The first **D&C** approach divides the matrix into two overlapping principal submatrix blocks. We then either identify a block that contains the corrupted element and continue with the smaller problem for that block; otherwise we use **EDM** completion to identify the correct position and magnitude of the noise. The second approach divides the matrix into the largest number of overlapping principal submatrix blocks with minimal overlap that allows for **EDM** completion. Again, we either identify one block to further subdivide the problem or we complete the **EDM** to locate the correct position and identify the noise. The third approach divides the matrix into principal submatrix blocks of smallest size that can be completed and we again apply the divide and conquer approach.

The three **D&C** approaches are used in combination with three different types of facial reduction: (i) finding exposing vectors Z_i for each block and then using the exposing vector $Z = \sum_i Z_i$ to identify the face $f = \mathbb{S}^n_+ \cap Z^\perp \trianglelefteq \mathbb{S}^n_+$; (ii) finding a *facial vector,* $\boldsymbol{FV}$, denoted V_i, for each block, and then finding the intersection $\text{range}(V) = \cap_i \text{range}(V_i)$ efficiently so as to identify the face $f = V\mathbb{S}^d_+V^T \trianglelefteq \mathbb{S}^n_+$, of the semidefinite cone $\mathbb{S}^n_+$; (iii) a combination of exposing and facial vectors first finds a Gale matrix N so that $Z = NN^T$ is an exposing vector and yields a facial vector V to identify the face f. Identifying the correct face allows one to find the correct full rank factorization of the Gram matrix to complete the **EDM**.

Included are several interesting results on: the commutativity of some operators for this distance geometry problem; and properties of having exactly one noisy element. Our main result is three successful algorithms that efficiently find the position and magnitude of the corrupted distance. The algorithms scale well. The best algorithm can solve problems of the order of $100,000$ points in approximately one minute, and to machine precision. With embedding dimension d, the data is then size $100,000d$ with resulting $D \in \mathbf{EDM}$ of size $n = 100,000$. Theoretical difficulties arise when the points are not in *general position*. In fact, we prove that difficulties arise only when all but at most $d+1$ points are in a linear manifold of dimension $d-2$.

In the process of this study, we review the relationships between three methods of facial reduction and Gale transforms. In addition, we show that the only time that the intuitive approach, the nearest **EDM** problem, solves our problem is the case where the original distance is 0, and the perturbation is negative, i.e., a trivial case.

1.1 Related Literature

The literature for **EDM** and more generally *distance geometry* is vast and includes many surveys and books with many applications in multiple areas of mathematics, engineering, health sciences. Classical results on **EDM** completion problems appear in e.g., [6, 7, 19, 20, 23] and relate to applications in protein folding and molecular conformations, e.g., [9,30], as well as sensor network localization, e.g., [9, 22].

Completing a partial **EDM** using facial reduction, **FR**, and exploiting cliques and facial vectors was presented in [22]. This approach was extended to the noisy case in [12] by exploiting the exposing vector representation for **FR**. Other approaches that do not necessarily use **FR** are discussed in e.g., [3, 10, 21] and the more recent surveys [11, 24] and the references therein. The paper [17] discusses perturbation analysis for **EDM** completion and convergence of algorithms under perturbations. Matrix completion with noise and *outliers* is discussed in [31]. Further applications of **EDM** for finding the so-called *kissing number* are found in sphere packing that has further applications to error correcting codes from the fields of communications. A recent

discussion on **EDM** completion with noisy data is in [26].

An application to finding a single node in the graph where the edges to the node are known but can be noisy was done in [28]. This is closely related to our problem and the paper includes applications to real world problems, e.g., when the unknown node is a cellular phone and the other nodes are cellular towers. Our application would consider the case when one tower is partially obstructed and provides a noisy distance to the cellular phone. Further results related to our problem is [1, Section 7.1] that deals with the completion problem for *one* missing entry of an **EDM**. See also [6].

Further, we relate **FR** to Gale transforms. Details for Gale transorms are presented in [15] and discussed further in e.g., [5].

1.2 Outline and Main Results

In Section 2 we present the basic properties of **EDM** and the main problem description. This includes the relationship with semidefinite programming, **SDP** and the Gram matrix. Further definitions and results are presented as needed.

In addition, in Section 2.2 we show that the only case when the nearest **EDM** problem solves our problem is when the original element (distance) is zero and the perturbation is negative.

In Section 2.3 we present the background on **FR** and results on commutativity involving the Lindenstrauss operator

In Section 3 we present the three-by-three algorithms: Section 3.1 presents the bisection **D&C** with facial reduction using exposing vectors, **BIEV**; Section 3.2 presents the multi-block case with **FV**s, **MBFV**, and is based on an efficient technique for finding the intersection of faces. A proof of finite convergence for this algorithm can be given based on the characterization of so-called *restricted yielding* in Theorem 5.2. Finite convergence for the other approaches can be proved similarly. Section 3.3 presents an equivalent approach that uses small blocks with Gale transforms, **SBGT**. This section also highlights the interesting relationships, equivalences, between facial reduction and Gale transforms.

In Section 4 we present a complexity analysis of the methods as well as present the empirical results. We solve small, large, huge problems.

In Section 5 we discuss the two hard cases where the algorithms can fail. We present a characterization in Theorem 5.2 for detecting one noisy distance. Here we extend the results on *yielding* in [1, Section 7.2], i.e., characterizing when the **EDM** property is preserved under a single perturbation. We provide a characterization when a perturbation yields (preserves) not only an **EDM** but also maintains embedding dimension d. The result states that all but two points are on a linear manifold of dimension $d-2$; and this is particularly elegant for $d=2$. We include empirics for the hard case.

Our concluding remarks are in Section 6.

2 Problem Description, EDM and FR, Theoretical Results

We consider matrices $S \in \mathbb{S}^n$ the space of $n \times n$ symmetric matrices equipped with the *trace inner product* $\langle S, T \rangle = \operatorname{tr} ST$; we use $\operatorname{diag}(S) \in \mathbb{R}^n$ to denote the diagonal of S; the adjoint mapping is $\operatorname{diag}^*(v) = \operatorname{Diag}(v) \in \mathbb{S}^n$. For positive integers j, k, we let $[k] = 1, 2, \ldots, k$; $[j, k] = j, j+1, \ldots, k$. Moreover, we denote $E_{jk} = e_j e_k^T + e_k e_j^T$, *unit matrix*. For index subset $\alpha \subset [n]$ we let D_α denote the corresponding principal submatrix of D, and

$$\mathcal{P}_\alpha : \mathbb{S}^n \to \mathbb{S}^\alpha, \ \mathcal{P}_\alpha(D) = D_\alpha,$$

is the projection, or coordinate shadow, corresponding to α. We abuse notation by using $\mathbb{S}^\alpha$ rather than $\mathbb{S}^{|\alpha|}$ to emphasize the actual coordinates used. The inverse image is

$$\mathcal{P}_\alpha^{-1}(\bar{D}) = \{D \in \mathbb{S}^n : D_\alpha = \bar{D}\}.$$

The cone of positive semidefinite matrices is denoted $\mathbb{S}^n_+ \subset \mathbb{S}^n$, and we use $X \succeq 0$ for $X \in \mathbb{S}^n_+$. Similarly, for positive definite matrices we use $\mathbb{S}^n_{++}, X \succ 0$.

For a set of n points $p_i \in \mathbb{R}^d, i = 1, \ldots, n$, we denote the matrix of points, the configuration matrix, $P = \begin{bmatrix} p_1 & p_2 & \ldots & p_n \end{bmatrix}^T \in \mathbb{R}^{n \times d}$. The Euclidean distance matrix (of squared distances) is $D = \left(\|p_i - p_j\|^2\right) \in \mathcal{E}^n \subset \mathbb{S}^n$. We denote the closed convex *cone of* ***EDM***, $\mathcal{E}^n$. Here $0 < d = \operatorname{edim}(D)$ is the embedding dimension of the **EDM**. Without loss of generality, we can translate the points so that they are centered, i.e., with *vector of ones*, e, we have $P^T e = 0$. Note that $v := \frac{1}{n} P^T e$ is the barycenter of the points. The translation is then given by

$$P^T \leftarrow P^T - v e^T. \tag{2.1}$$

Our approach works well under the assumption that the points are in *general position*, i.e., no $d+1$ of them lie in a proper hyperplane; or, equivalently, that every subset of $d+1$ points are affinely independent, e.g., [2]. This guarantees that our algorithms succeed as it is based on identifying whether a Gram matrix corresponding to chosen principal submatrices has the correct rank d. However, we have modified our algorithm to allow for the case when the general position assumption fails.

We denote the corresponding *Gram matrix,* $G = PP^T$. Then the classical result of Schoenberg [29] relates the matrix of squared distances, the *Euclidean distance matrix,* ***EDM***, with a Gram matrix by applying the *Lindenstrauss operator,* $\mathcal{K}(G)$,

$$D = \mathcal{K}(G) = \operatorname{diag}(G) e^T + e \operatorname{diag}(G)^T - 2G \in \mathcal{E}^n. \tag{2.2}$$

Moreover, this mapping is one-one and onto between the *centered subspace,* $\mathcal{S}^n_C$, and *hollow subspace,* $\mathcal{S}^n_H$,

$$\mathcal{S}^n_C = \{X \in \mathbb{S}^n \,:\, Xe = 0\}, \quad \mathcal{S}^n_H = \{X \in \mathbb{S}^n \,:\, \operatorname{diag}(X) = 0\}.$$

We ignore the dimension n when the meaning is clear. Note that the centered assumption $P^T e = 0 \implies G \in \mathcal{S}^n_C$. Let

$$J = I - \frac{1}{n} e e^T \tag{2.3}$$

be the orthogonal projection onto the orthogonal complement $e^\perp$. Note that the translation in (2.1) is equivalent to

$$P \leftarrow JP = P - \frac{1}{n} e e^T P.$$

We let offDiag: $\mathbb{S}^n \to \mathbb{S}^n$ denote the orthogonal projection onto $\mathcal{S}^n_H$, the hollow matrices. Then the *Moore-Penrose generalized inverse,* $\mathcal{K}^\dagger$, is

$$\mathcal{K}^\dagger(D) = -\frac{1}{2} J \operatorname{offDiag}(D) J.$$

Lemma 2.1. *For $D \in \mathcal{E}^n$ with $d = \operatorname{edim}(D)$, the general position assumption is equivalent to*

$$\alpha \subset [n], |\alpha| = k \geq d+1 \implies \operatorname{rank}(\mathcal{K}^\dagger(D_\alpha)) = d. \tag{2.4}$$

Moreover, we find the orthogonal projection $\mathcal{P}_{\operatorname{range}(K^\dagger)}$ as

$$\mathcal{K}^\dagger \mathcal{K}(S) = JSJ. \tag{2.5}$$

Note that the orthogonal projection satisfies $\mathcal{P}_{\operatorname{range}(K^\dagger)} = \mathcal{P}_{\mathcal{S}^n_C}$. Throughout we abuse notation and use $\mathcal{K}, \mathcal{P}, J$ without indicating the dimension of the space that is involved.

2.1 Main Problem Description and Model

We begin with a description of the problem.

Problem 2.2 (Find magnitude/location of noise in **EDM**). *Let $\hat{D} \in \mathcal{E}^n$, the cone of **EDMs**, $d = \text{edim}(\hat{D})$. Let $1 \le i < j \le n$ be indices and $\alpha \in \mathbb{R}$ and define the data $D = \hat{D} + \alpha E_{ij}$ with the unit matrix $E_{ij} = e_i e_j^T + e_j e_i^T$. Then:*

> *Find the position i,j and the value of the noise α from the given **EDM** D with one noisy element.*

Remark 2.3 (Naive Discrete Model). *The fact that we have a single noisy element and a known embedding dimension leads to the following rank constrained mixed integer **MIP** and **SDP**:*

$$
\begin{array}{rccl}
 & p^* = & \min & \|H \circ (\mathcal{K}(G) - D)\|_F^2 \\
 & & s.t. & G \succeq 0,\ \text{rank}(G) = d \\
(\textbf{\textit{MIP}}, \textbf{\textit{SDP}}) & & & \text{diag}(H) = 0 \\
 & & & H \in \mathbb{S}^n \cap \{0,1\}^{n \times n} \\
 & & & e^T H e = n^2 - n - 2,
\end{array}
\tag{2.6}
$$

where $H \circ D$ denotes the Hadamard product and H is the adjacency matrix of the complete graph where one edge is missing as characterized by the last constraint. Since exactly one element is noisy, we know that the optimal value $p^* = 0$ when the correct position i, j is chosen for H. This problem looks extremely difficult as we deal with binary variables as well as semidefinite and rank constraints. Our approach in this paper takes advantage of the structure to get extremely accurate solutions very quickly, while avoiding the difficult nature of the hard discrete problem (2.6).

2.2 Nearest EDM, NEDM, Formulation and Solving Main Problem 2.2

It is natural to try and solve Problem 2.2 by finding the nearest **EDM**, (**NEDM**, see (2.8) below) to the given noisy D_n, i.e., relax the MIP formulation (2.6). This emphasizes the difficulty of Problem 2.2 when the noisy data $D_n \in \mathcal{E}^n$. We now see that the **NEDM** approach works only in the case where the corrupted element arises from a zero distance in the original $D_0 \in \mathcal{E}^n$, i.e., the degenerate case of one point on top of another.

Let V be chosen so that $[V\ e]$ is an orthogonal matrix and for $X \in \mathbb{S}^{n-1}$ let $\mathcal{K}_V(X) := \mathcal{K}(VXV^T)$, where $\mathcal{K}$ is given in (2.2). Then

$$
\mathcal{K}^*(D) = 2(\text{Diag}(De) - D),\ \mathcal{K}_V^*(D) = V^T \mathcal{K}^*(D) V,
$$

Moreover,

$$
0 = VXV^T \iff 0 = V^T V X V^T V \iff X = 0,
$$

and therefore, for $D = \mathcal{K}(VXV^T)$ we get that VXV^T is a centered matrix by definition of V, and so D is a hollow matrix and

$$
X \neq 0 \implies \langle X, \mathcal{K}_V^*(\mathcal{K}_V(X)) \rangle = \|\mathcal{K}_V(X)\|_F^2 = \|D\|_F^2 \neq 0 \implies \mathcal{K}_V^* \mathcal{K}_V \succ 0, \tag{2.7}
$$

i.e., $\mathcal{K}_V^*(\mathcal{K}_V(\cdot)) \succ 0$, is a positive definite linear operator. The facially reduced **NEDM** then has a unique solution and is:

$$
\bar{X}(D_n) = \bar{X} = \text{argmin}_{X \succeq 0} \frac{1}{2} \|\mathcal{K}_V(X) - D_n\|_F^2. \tag{2.8}
$$

Lemma 2.4 ($\mathcal{N}_{\mathbb{S}^n_+}(\bar{X})$, *normal cone*). *Let*

$$\bar{X} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} \bar{\Lambda} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}^T \in \mathbb{S}^n_+, \, \bar{\Lambda} \in \mathbb{S}^r_{++},$$

with $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ *orthogonal. Let* $\lambda \in \mathbb{R}^n_+$ *be the vector of eigenvalues of* $\bar{X}$. *Then the normal cone*

$$\begin{array}{rcl} \mathcal{N}_{\mathbb{S}^n_+}(\bar{X}) & = & Q \operatorname{Diag}\left(\mathcal{N}_{\mathbb{R}^n_+}(\lambda)\right) Q^T \\ & = & \{X \in -\mathbb{S}^n_+ : X\bar{X} = 0\} \\ & = & \left\{ X \in -\mathbb{S}^n_+ : X = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \hat{\Lambda} \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}^T, \hat{\Lambda} \in -\mathbb{S}^{n-r}_+ \right\} \\ & = & -\left(\mathbb{S}^n_+ - \bar{X}\right)^+ \\ & = & -\operatorname{face}(\bar{X})^c, \quad (\textit{the conjugate face}). \end{array}$$

Proof. The proof follows from using the Eckart-Young-Mirsky, e.g., [14], and the Moreau decomposition, e.g., [25], theorems. That is $\bar{X}$ is the nearest point in $\mathbb{S}^n_+$ to a given $Y \in \mathbb{S}^n$ if, and only if, $Y - \bar{X} \in \mathcal{N}_{\mathbb{S}^n_+}(\bar{X})$, and Y has an orthogonal decomposition

$$Y = Y_+ + Y_-, \; Y_+, -Y_- \in \mathbb{S}^n_+, \; Y_+Y_- = 0,$$

using the projections Y_+, Y_- onto the nonnegative and nonpositive semidefinite cones, respectively. Therefore $Y - Y_+ = Y_-, Y_+ = \bar{X}$ yields the result. Note that the Eckart-Young-Mirsky theorem characterizes the projections as being the same as using the Moreau decomposition. □

We now present a characterization for when the nearest point finds the original **EDM** after a single element is corrupted. Note that for our problem the i, j are not known. We first introduce the notion of a face and some its properties. For a given convex cone K, a convex cone $f \subset K$ is a *face of* K, denoted $f \trianglelefteq K$, if

$$X, Y \in K, X + Y \in f \implies X, Y \in f.$$

Given $f \trianglelefteq K$, the *conjugate face*, $f^c = K^+ \cap f^\perp$. For $X \in K$, we denote $f = \operatorname{face}(X)$, *minimal face of* K, i.e., the intersection of all faces that contain X. For $X \in \mathbb{S}^n_+$ with spectral decomposition $X = \begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V & U \end{bmatrix}^T$, $\Lambda \in \mathbb{S}^r_{++}$, we get

$$\operatorname{face}(X) = V\mathbb{S}^r_+V^T, \; \operatorname{face}(X)^c = U\mathbb{S}^{n-r}_+U^T.$$

See e.g., [13].

Theorem 2.5. *Suppose that* $D_0 \in \mathcal{E}^n$, *the perturbation matrix is* $E_{ij} \in \mathbb{S}^n, i < j$, *and* $D_n = D_0 + \alpha E_{ij} \notin \mathcal{E}^n$. *Let*

$$\begin{array}{rcl} Y := \mathcal{K}^*_V(E_{ij}) & = & V^T K^*(E_{ij}) V \\ & = & 2\left[v_i^T v_i + v_j^T v_j - v_i^T v_j - v_j^T v_i\right], \end{array}$$

where v_i *is the* i*-th row of* V. *Then* $\alpha \neq 0$, $0 \neq Y \succeq 0$, *and*

$$\begin{array}{rcll} D_0 \textit{ is the nearest } \textbf{\textit{EDM}} \textit{ to } D_n & \iff & D_0 = \mathcal{K}_V(\bar{X}), \textit{ for } \bar{X} \textit{ from (2.8)}, & \\ & \iff & X_0 := \mathcal{K}^\dagger_V(D_0) \in Y^c := \mathbb{S}^{n-1}_+ \cap Y^\perp, \textit{ and } \alpha < 0, & \\ & \iff & X_0 := \mathcal{K}^\dagger_V(D_0) \in Y^c := \mathbb{S}^{n-1}_+ \cap Y^\perp, \; \forall \alpha < 0, & \\ & \iff & (D_0)_{ij} = 0, \quad \textit{and } \alpha < 0. & \end{array} \tag{2.9}$$

Proof. We consider the nearest **EDM** problem in (2.8) with $X_0 = \mathcal{K}_V^\dagger(D_0)$ as the original data. We want to characterize D_0, or equivalently X_0, so that the optimum $\bar{X}$ in (2.8) equals X_0, i.e, the **NEDM** problem solves our Problem 2.2.

First we note from (2.7) that $\mathcal{K}_V^* \mathcal{K}_V \succ 0$ so that the objective in (2.8) is strongly convex and we have a unique optimum. The optimality conditions for X_0 for the nearest point problem are that the gradient of the objective function is in the polar cone (negative normal cone) of the **SDP** cone at X, i.e., we have the equivalences

$$\begin{array}{rcl} \mathcal{K}_V^*(\mathcal{K}_V(X_0) - D_n) \in (\mathbb{S}_+^{n-1} - X_0)^+ & \iff & \mathcal{K}_V^*(\mathcal{K}_V \mathcal{K}_V^\dagger(D_0) - D_n) \in (\mathbb{S}_+^{n-1} - X_0)^+ \\ & \iff & \mathcal{K}_V^*(D_0 - D_n) \in (\mathbb{S}_+^{n-1} - X_0)^+ \\ & \iff & \alpha\, \mathcal{K}_V^*(-E_{ij}) \in (\mathbb{S}_+^{n-1} - \mathcal{K}_V^\dagger(D_0))^+ \\ & \iff & -\alpha\, \mathcal{K}_V^*(E_{ij}) \in (\mathbb{S}_+^{n-1} - \mathcal{K}_V^\dagger(D_0))^+. \end{array}$$

Note that $\operatorname{range}(\mathcal{K}) = \operatorname{range}(\mathcal{K}_V) = \mathcal{S}_H^n$. Therefore $\operatorname{null}(\mathcal{K}_V^*) = (\mathcal{S}_H^n)^\perp = \operatorname{range}(\operatorname{Diag})$. As E_{ij} is not diagonal, we get $\mathcal{K}_V^*(E_{ij}) \neq 0$. Alternatively, we recall that the choice of V implies $\bar{V} := \begin{bmatrix} V & \frac{1}{\sqrt{n}} e \end{bmatrix}$ is orthogonal. Therefore, the rows satisfy

$$v_i^T v_j + \frac{1}{n} = 0, v_i^T v_i + \frac{1}{n} = 1 \implies v_i^T v_i + v_j^T v_j - v_i^T v_j - v_j^T v_i = 2\left(1 - \frac{1}{n} + \frac{1}{n}\right) = 2 \neq 0,$$

i.e., we see again that $\mathcal{K}_V^*(E_{ij}) \neq 0$.

We now observe that necessarily $\alpha \leq 0$. Without loss of generality, we assume $\alpha < 0$. To have the nearest matrix be the original data D_0, X_0, we need to have

$$\begin{array}{rcl} \mathcal{K}_V^*(E_{ij}) & \in & (\mathbb{S}_+^{n-1} - \mathcal{K}_V^\dagger(D_0))^+ \\ & = & (\mathbb{S}_+^{n-1} - X_0)^+ \\ & = & \{X \in \mathbb{S}_+^{n-1} : XX_0 = 0\} \\ & = & \operatorname{face}(X_0)^c, \end{array}$$

i.e., the conjugate face to $\operatorname{face}(X_0)$.

Next we see the equivalence using v_i, the rows of V:

$$\begin{array}{rcll} Y = \mathcal{K}_V^*(E_{ij}) & = & V^T K^*(E_{ij}) V & \\ & = & 2V^T(\operatorname{Diag}(E_{ij}e) - E_{ij})V, & (\succeq 0, \text{ since congruence of a Laplacian matrix}) \\ & = & 2V^T(e_i e_i^T + e_j e_j^T - e_i e_j^T + e_j e_i^T)V & \\ & = & 2\left[v_i^T v_i + v_j^T v_j - v_i^T v_j - v_j^T v_i\right]. & \end{array}$$

Since $Y, X_0 \in \mathbb{S}_+^{n-1}$, we have

$$Y \in \operatorname{face}(X_0)^c \iff X_0 \in \operatorname{face}(Y)^c \iff YX_0 = 0 \iff \operatorname{tr} YX_0 = 0.$$

Therefore, the first two equivalences in (2.9) follow as $D_0 = K_V(X_0)$.

Finally, we note that the conjugate face condition is equivalent to

$$0 = \operatorname{tr} X_0 Y = \operatorname{tr} X_0 V^T K^*(E_{ij}) V = \operatorname{tr} \mathcal{K}(VX_0V^T) E_{ij} = \operatorname{tr} D_0 E_{ij}.$$

Therefore, we can only have a negative perturbation from $(D_0)_{ij} = 0$. □

2.3 Results on EDM, FR

We now present know results for **EDM** and **FR** as well as new results related to our specific problem. In particular, Lemma 2.6 provides a useful commutativity result for $\mathcal{K}, \mathcal{P}_\alpha$, while Lemma 2.7 recalls the result that a sum of exposing vectors is an exposing vector. Theorem 2.8 shows

how to find a (centered) **FV** for the face corresponding to a given principal submatrix D_α. Then Lemma 2.9 illustrates an efficient and accurate way to find the intersection of two given faces by finding a **FV** from the corresponding two given **FV**s.

The results here use the *general position* assumption. Further results are in Theorem 5.2 that characterizes when the perturbation of a single element in an **EDM** with embedding dimension d can still be a **EDM** with embedding dimension d. This is important for our general algorithm in identifying the location of the corrupted element in the **EDM**.

2.3.1 Commutativity of $\mathcal{P}_\alpha, \mathcal{K}$

We first include a useful and very interesting observation about the commutativity of $\mathcal{K}, \mathcal{P}_\alpha, \alpha \subset [n]$.

Lemma 2.6. *Let $M \in \mathbb{R}^{n\times n}, \alpha \subset [n]$. Then, by abuse of notation on the dimensions of e and the transformations $\mathcal{K}, \mathcal{P}_\alpha$, we get*

$$\mathcal{P}_\alpha \mathcal{K}(M) = \mathcal{K}\mathcal{P}_\alpha(M).$$

Proof. We note that $\mathcal{P}_\alpha(\operatorname{diag}(M)) = \operatorname{diag}(\mathcal{P}_\alpha(M))$. Therefore,

$$\begin{aligned} \mathcal{P}_\alpha \mathcal{K}(M) &= \mathcal{P}_\alpha \left(\operatorname{diag}(M)e^T + e\operatorname{diag}(M) - 2M\right) \\ &= \operatorname{diag}(\mathcal{P}_\alpha(M))e^T + e(\operatorname{diag}(\mathcal{P}_\alpha M)) - 2\mathcal{P}_\alpha(M) \\ &= \mathcal{K}\mathcal{P}_\alpha(M). \end{aligned}$$

□

We note that symmetry for M is not needed. Moreover, the commutativity reveals information on the eigenspace of $\mathcal{K}$, for if it was an operator we would have joint diagonalization with $\mathcal{P}$.

2.3.2 Facial Reduction

We follow notation and definitions in [1, 13]. First we recall that both $\mathbb{S}^n_+, \mathcal{E}^n$ are closed convex cones in $\mathbb{S}^n$. We let W, V be two full column rank matrices that satisfy

$$\operatorname{range}(V) = \operatorname{range}(X),\ \operatorname{range}(W) = \operatorname{null}(X),\ Z = WW^T,\ d = \operatorname{rank}(X).$$

Then $X \in \operatorname{relint}(f)$ and we have two representations for f:

$$f = V\mathbb{S}^d_+V^T = \mathbb{S}^n_+ \cap Z^\perp.$$

We call V, Z a facial and exposing vector, respectively.

If we choose the facial vector to satisfy $\begin{bmatrix} V & e \end{bmatrix}$ nonsingular and $V^Te = 0$, then we can characterize the face of centered Gram matrices

$$\mathcal{S}^n_C \cap \mathbb{S}^n_+ = V\mathbb{S}^{n-1}_+V^T = \mathcal{K}^\dagger(\mathcal{E}^n) \trianglelefteq \mathbb{S}^n_+.$$

This is used in [4] to regularize the **EDM** completion problem using $\mathcal{K}_V(X) := \mathcal{K}(VXV^T)$, i.e., $\mathcal{K}_V : \mathbb{S}^{n-1}_+ \to \mathcal{E}^n$ and strict feasibility is satisfied for **EDM** completion problems. Here V is a *centered facial vector*, i.e., $V^Te = 0$. Since we work with centered Gram matrices, we often use centered facial vectors below.

Our divide and conquer methods use principal submatrices D_α of D and corresponding Gram matrices, with ordered integers $\alpha = [i, i+k], 1 \le i \le i+k \le n$. The first **FR** method finds exposing vectors for each submatrix and then adds these up to get an exposing vector for the entire Gram matrix G, i.e., we exploit Lemma 2.7. Note that if the overlap of the exposing vectors is deficient, then it is not necessarily true that Z is a maximum rank exposing vector.

Lemma 2.7 ([13]). *Let*

$$G \in \mathbb{S}^n_+,\ Z_i \in \mathbb{S}^n_+, \operatorname{tr}(GZ_i) = 0, i = 1, \ldots, k.$$

Then $Z = \sum_{i=1}^k Z_i \succeq 0$ *and* $GZ = 0$.

The second **FR** method uses adjacent pairs of principal submatrices in order to do **FR** by intersecting pairs of faces. We first consider the representation of **FR** for a single principal submatrix, i.e., the representation of the smallest face obtained using the Gram matrix corresponding to D_α. Without loss of generality we use the first, top left, principal submatrix. We modify the notation in Theorem 2.8 to match the notation herein. Recall that $\mathcal{P}_\alpha^{-1}$ denotes the projection inverse image.

Theorem 2.8 ([22, Theorem 2.3]). *Let*

$$D \in \mathcal{E}^n, d = \operatorname{edim}(D),\ \alpha = [k],\ \bar{D} = D_\alpha, t = \operatorname{edim}(\bar{D}).$$

Let

$$\bar{G} = \mathcal{K}^\dagger(\bar{D}) = \bar{U}_G S \bar{U}_G^T,\ S \in \mathbb{S}^t_{++},$$

where

$$\bar{U}_G \in \mathbb{R}^{k\times t},\ \bar{U}_G^T \bar{U}_G = I_t,\ \bar{U}_G^T e = 0.$$

Furthermore, let $U_G := \begin{bmatrix} \bar{U}_G & \frac{1}{\sqrt{k}} e \end{bmatrix}$, $U := \begin{bmatrix} U_G & 0 \\ 0 & I_{n-k} \end{bmatrix}$, *and* $\begin{bmatrix} V & \frac{1}{\|U^T e\|} U^T e \end{bmatrix} \in \mathbb{R}^{n-k+t+1 \times n-k+t+1}$ *be orthogonal. Then*

$$\operatorname{face}\left(\mathcal{K}^\dagger\left(\mathcal{P}_\alpha^{-1}(\bar{D}) \cap \mathcal{E}^n\right)\right) = \left(U \mathbb{S}^{n-k+t+1}_+ U^T\right) \cap \mathcal{S}^n_C = \left((UV)\mathbb{S}^{n-k+t}_+ (UV)^T\right).$$

The matrix UV in Theorem 2.8 provides a facial vector for the minimal face corresponding to the block $\bar{D}$. If we have two overlapping principal submatrices with the overlap having the proper embedding dimension d, then we can efficiently find a facial vector for the intersection of the two corresponding faces. This is given in [22, Lemma 2.9] and we have added that option to our code and include the details here in Lemma 2.9 for completeness.

Lemma 2.9 ([22, Lemma 2.9]). *Let*

$$U_1 := \begin{matrix} \\ s_1 \\ k \end{matrix} \overset{r+1}{\begin{bmatrix} U_1' \\ U_1'' \end{bmatrix}},\quad U_2 := \begin{matrix} \\ k \\ s_2 \end{matrix} \overset{r+1}{\begin{bmatrix} U_2'' \\ U_2' \end{bmatrix}},\quad \hat{U}_1 := \begin{matrix} \\ s_1 \\ k \\ s_2 \end{matrix} \overset{\begin{matrix} r+1 & s_2 \end{matrix}}{\begin{bmatrix} U_1' & 0 \\ U_1'' & 0 \\ 0 & I \end{bmatrix}},\quad \hat{U}_2 := \begin{matrix} \\ s_1 \\ k \\ s_2 \end{matrix} \overset{\begin{matrix} s_1 & r+1 \end{matrix}}{\begin{bmatrix} I & 0 \\ 0 & U_2'' \\ 0 & U_2' \end{bmatrix}}$$

be appropriately blocked with $U_1'', U_2'' \in \mathcal{M}^{k\times(r+1)}$ *full column rank and* $\operatorname{range}(U_1'') = \operatorname{range}(U_2'')$. *Furthermore, let*

$$\bar{U}_1 := \begin{matrix} \\ s_1 \\ k \\ s_2 \end{matrix} \overset{r+1}{\begin{bmatrix} U_1' \\ U_1'' \\ U_2'(U_2'')^\dagger U_1'' \end{bmatrix}},\quad \bar{U}_2 := \begin{matrix} \\ s_1 \\ k \\ s_2 \end{matrix} \overset{r+1}{\begin{bmatrix} U_1'(U_1'')^\dagger U_2'' \\ U_2'' \\ U_2' \end{bmatrix}}. \tag{2.10}$$

Then $\bar{U}_1$ *and* $\bar{U}_2$ *are full column rank and satisfy*

$$\operatorname{range}(\hat{U}_1) \cap \operatorname{range}(\hat{U}_2) = \operatorname{range}\left(\bar{U}_1\right) = \operatorname{range}\left(\bar{U}_2\right).$$

Moreover, if $e_{r+1} \in \mathbb{R}^{r+1}$ *is the* $(r+1)^{\text{st}}$ *standard unit vector, and* $U_i e_{r+1} = \alpha_i e$, *for some* $\alpha_i \neq 0$, *for* $i = 1, 2$, *then* $\bar{U}_i e_{r+1} = \alpha_i e$, *for* $i = 1, 2$.

Remark 2.10. *In Lemma 2.9, if we have two overlapping blocks $D_{s_1\cup k}, D_{k\cup s_2}$ we can get facial vectors from bases of the corresponding Gram matrices in U_1, U_2, respectively. These are matched up in $\hat{U}_1, \hat{U}_2$. Then a facial vector for the intersection of the minimal faces containing these blocks is given in either of the formulae in (2.10). In our implementation, we choose the one for which U_1'', U_2'' is better conditioned. Thus we use the two facial vectors and find a new facial vector for the intersection of the overlapping ranges.*

This emphasizes the importance of the conditioning of the overlap in U_1'', U_2''. In fact, it is essential that the rank of each overlap is eventually d, the embedding dimension of the problem. But there is no reason that the ordering of the nodes (rows) of D cannot be changed. Therefore, if we find a well-conditioned block of correct rank d, we can include that in the overlap for further iterations, i.e., we can always save and use the best conditioned block with the largest rank in the overlap.

3 Three D&C and Three FR Methods

We have three different **D&C** methods: bisection; minimum block overlap; and minimum block size. And we have three different **FR** methods: exposing vector; facial vector; and Gale transform. The result is nine possible algorithms. We now pair each **D&C** method with exactly one **FR** method and describe and implement these three methods. For simplicity, we assume that the *general position* assumption holds, i.e., we can easily find a principal submatrix of the Gram matrix with rank d, see Remark 3.1.

3.1 Bisection D&C with Exposing Vector for FR

The first algorithm combines the bisection approach for **D&C** with the exposing vector approach for **FR**, denoted **BIEV**. We divide the data matrix D into 2 properly overlapping blocks corresponding to principal submatrices. We first identify whether the noisy element is within one of these two principal blocks and reduce the problem to that block. Or if it is outside both chosen principal blocks, then we apply **FR** explicitly to solve the problem and find the position ij and noise δ.

Recall that d is the embedding dimension. We assume that $n \gg d$, i.e., is sufficiently larger than d. Then we use the two blocks indexed by columns (rows)

$$I_1 = \{1, \ldots, \lceil (n+d+2)/2 \rceil\}\,, \quad I_2 = \{\lfloor (n-d-2)/2 \rfloor, \ldots, n\}\,, \tag{3.1}$$

i.e., we have the two principal submatrices $D_1 = D_{I_1}, D_2 = D_{I_2}$ which overlap in the block of size at least $\frac{1}{2}((n+d+2)-(n-d-2)) = d+2$. In addition, the assumption $n \gg d$ implies each block is at least size $d+1$.

Remark 3.1. *The overlap corresponds to the points*

$$I_1 \cap I_2 = \{\lfloor (n-d-2)/2 \rfloor, \ldots, \lceil (n+d+2)/2 \rceil\}\,.$$

If the corresponding (centered) Gram matrix has rank d, then we know that the overlap of the graph is rigid. If this is not the case, then we need to permute the columns of D in order to obtain a rigid overlap.

This raises the hard *question of how to find the best overlap. The simple case occurs if the overlap is in general position, i.e., each principal submatrix $D_\alpha, \alpha \subset [n], |I| \geq d+1$ corresponds to a Gram matrix with rank d.*

We now find the *supposed* Gram matrices $G_i = \mathcal{K}^\dagger(D_i), i = 1, 2$, i.e., these are indeed *centered* Gram matrices with rank d if D_i is an **EDM**. There are now three cases to consider for this current approach.

3.1.1 Case 1: Reducing Size of Problem

Suppose that one of $G_i, i = 1, 2$, is not positive semidefinite of rank d, i.e., as a result of the noise in the data, the corresponding submatrix D_i is not an **EDM** or does not have embedding dimension d. Then we can continue on the **D&C** approach and reduce our problem to that submatrix. And then we continue the division, i.e., we have reduced the problem by a factor of roughly 2. We then redefine D and n appropriately and return to dividing the indices in (3.1).

3.1.2 Case 2: EDM Completion using FR

We first recall that for $G \succeq 0$, face(G) denotes the smallest face containing G. Let V, N denote matrices with columns that are orthonormal basis for range(G), null(G), respectively, e.g., made up of an orthonormal set of eigenvectors. Let $d = \text{rank}(G)$. Then as stated above,

$$\text{face}(G) = V\mathbb{S}^d_+V^T = \mathbb{S}^n_+ \cap (NN^T)^\perp. \tag{3.2}$$

V is a *facial vector*, [18], while $Z = NN^T$ is an *exposing vector*, e.g., [12]. And moreover, the sum of exposing vectors is an exposing vector, Lemma 2.7:

$$Z_i \succeq 0, \text{tr}\, Z_iG = 0, i = 1, \ldots, k \implies \sum_i Z_i \succeq 0, \text{tr}\left(\left(\sum_i Z_i\right) G\right) = 0. \tag{3.3}$$

Suppose both $G_i, i = 1, 2$, are *centered* Gram matrices with rank d. Then we know the corrupted element/distance is outside the principal blocks. We now continue to completely solve the problem using **FR** as we can now complete the partial **EDM** formed from the two blocks. We now give the details.[1]

1. Using the spectral decomposition of $G_i, i = 1, 2$, given above, we obtain orthonormal bases of *centered* null vectors for the nullspaces of the Gram matrices $G_i, i = 1, 2$,

$$G_iN_i = 0,\ N_i^Te = 0,\ N_i^TN_i = I,\ i = 1, 2. \tag{3.4}$$

   Let $Z_i = N_iN_i^T \succeq 0, i = 1, 2$, be the corresponding *centered* exposing vectors, i.e., we have

$$Z_iG_i = 0, G_ie = 0,\ Z_ie = 0, G_i + Z_i + ee^T \succ 0,\ i = 1, 2. \tag{3.5}$$

2. Let W_i be zero matrices of order n and set

$$(W_i)_{I_i} = Z_i,\ i = 1, 2,$$

   i.e., we place the *centered* exposing vectors Z_i into the correct blocks. Now each $W_i, i = 1, 2$, is a *centered* exposing vector for the true centered Gram matrix G. As the sum of exposing vectors is an exposing vector, we form the *centered* exposing vector of the true centered Gram matrix G,

$$Z = W_1 + W_2. \tag{3.6}$$

   This yields a *maximum rank exposing vector*

$$GZ = 0,\ Ge = 0,\ Ze = 0,\ Z \succeq 0,\ G + Z + ee^T \succ 0. \tag{3.7}$$

3. We choose $V, V^TV = I$, to be full column rank and to span null$([Z\ e]^T)$.[2] This completes the **FR** as we have

$$G = VRV^T,\ R \in \mathbb{S}^d_{++},\ Ge = 0. \tag{3.8}$$

[1] See e.g., [12, Algor. 1, Pg. 2308] for more details.

[2] As noted following (3.2), this is called a *facial vector* [18]. In fact, this is a centered facial vector.

4. We denote the *adjacency matrix,* H_α to be the matrix of zeros with ones in the positions indexed by I. Recall that $H \circ D$ denotes the Hadamard product. We solve the cone least squares problem that does not include the noisy element:

$$\min_{R \succeq 0} \|H_{I_1 \cup I_2} \circ \mathcal{K}(VRV^T) - H_{I_1 \cup I_2} \circ D\|_F^2. \tag{3.9}$$

Since R is order d and at the start $d << n$, this can be a very overdetermined problem and can be helped by using a *sketch matrix*, see e.g. [12].

5. In the case of random data, we expect all non-noisy blocks of size at least $d+1$ to be proper **EDM**s. Therefore, the optimal solution R in (3.9) is unique and in $\mathbb{S}^d_{++}$. Therefore, we can solve this least squares problem as an unconstrained problem and improve on the accuracy and speed. There is one constraint, linear equation, for each distinct pair in the set

$$\mathcal{I}_R := \{(i,j) \,:\, (i,j) \in (I_1 \times I_1) \cup (I_2 \times I_2),\, i < j\}\,.$$

We let $\text{svec} : \mathbb{S}^n \to \mathbb{R}^{t(n)}$ denote the isometric mapping that vectorizes a symmetric matrix columnwise with multiplying the off-diagonal elements by $\sqrt{2}$. The inverse (and adjoint) is $\text{sMat} = \text{svec}^*$. Similarly, $\text{usvec} : \mathbb{S}^n \to \mathbb{R}^{t(n-1)}$ is for the strict triangular part with $\text{usMat} = \text{usvec}^*$. Let the unknown variable be $r = \text{svec}(R) \in \mathbb{R}^{t(d)}$, where we denote $t(d)$, *triangular number*. The $m_R \times n_R$ system, $m_R = t(|I_1|-1) + t(|I_2|-1) - t(|I_1 \cap I_2|-1) \times (n_R := t(d))$ linear system to solve for r is

$$\text{usvec}\, H_{I_1 \cup I_2} \circ \mathcal{K}(V\, \text{sMat}(r) V^T) = \text{usvec}\, H_{I_1 \cup I_2} \circ D, \quad (: \mathbb{R}^{t(d)} \to \mathbb{R}^{t(n-1)}). \tag{3.10}$$

The columns of the matrix representation are obtained by replacing r with unit vectors $e_i, i = 1, \ldots t(d)$. We note that the rows corresponding to i, j not in the union of the two blocks are zero and can be discarded.
Alternatively, we could find the transpose of the matrix representation by taking the adjoint of the left-hand side in (3.10) and working on $\mathbb{R}^{t(n)} \to \mathbb{R}^{t(d)}$. We use $g = e_\iota \in \mathbb{R}^{t(n-1)}$, $\text{usMat}(g)$. The adjoint is given by

$$\text{svec}\left(V^T\, \mathcal{K}^*\left[H_{I_1 \cup I_2} \circ \text{usMat}(g)\right] V\right). \tag{3.11}$$

We first simplify $H_{I_1 \cup I_2} \circ \text{usMat}(e_\iota)$ to get the row of the matrix representation. For each $\iota \in [t(n-1)]$, we can find $i, j \in [n]$ such that $\iota = i + \sum_{\ell=0}^{j-2} \ell$ and observe

$$H_{I_1 \cup I_2} \circ \text{usMat}(e_\iota) = \begin{cases} \frac{1}{\sqrt{2}}\left(e_i e_j^T + e_j e_i^T\right) & \text{if } H_{ij} = 1; \\ 0 & \text{otherwise,} \end{cases}$$

where H_{ij} denotes the i,j-th entry of $H_{I_1 \cup I_2}$. Restricting g to the case when $H_{ij} = 1$, gives

$$\begin{array}{rcl} \mathcal{K}^*\left[H_{I_1 \cup I_2} \circ \text{usMat}(e_\iota)\right] & = & \mathcal{K}^*\left[\frac{1}{\sqrt{2}}\left(e_i e_j^T + e_j e_i^T\right)\right] \\ & = & \sqrt{2}\left[\text{Diag}\left((e_i e_j^T + e_j e_i^T)e\right) - (e_i e_j^T + e_j e_i^T)\right] \\ & = & \sqrt{2}\left[\text{Diag}\,(e_i + e_j) - e_i e_j^T - e_j e_i^T\right] \\ & = & \sqrt{2}\left(e_i e_i^T + e_j e_j^T - e_i e_j^T - e_j e_i^T\right). \end{array}$$

Plugging this into (3.11), we obtain

$$\begin{array}{rcl} \sqrt{2}\,\text{svec}\left(v_i^T v_i + v_j^T v_j - v_i^T v_j - v_j^T v_i\right) & = & \sqrt{2}\,\text{svec}\left(v_i^T(v_i - v_j) - v_j^T(v_i - v_j)\right) \\ & = & \sqrt{2}\,\text{svec}\left((v_i - v_j)^T(v_i - v_j)\right) \end{array} \tag{3.12}$$

as a column of the matrix representation of the adjoint. Here v_i is the i-th row vector of V. The advantage here is that we can use unit vectors for g restricted to the indices corresponding to $I_1 \cup I_2$ when finding the columns.

In our implementations, the calculation of the matrix representation was the most expensive step. We now show how to avoid this step.

This provides a simplification for solving the least squares problem in (3.9).

Corollary 3.2. *Let*

$$D \in \mathcal{E}^n,\ G = \mathcal{K}^\dagger(D),$$

and let $\alpha \subset [n], |I| \geq d+1$. *Define*

$$D_\alpha := \mathcal{P}_\alpha(D),\ G_\alpha := \mathcal{K}^\dagger(D_\alpha).$$

Suppose that $\operatorname{rank}(G_\alpha) = \operatorname{rank}(G) = d$. *Then*

$$\begin{aligned} \mathcal{K}^\dagger \mathcal{P}_\alpha \mathcal{K}(G) &= \mathcal{K}^\dagger \mathcal{K}(\mathcal{P}_\alpha(G)) \\ &= J(\mathcal{P}_\alpha(G))J. \end{aligned}$$

Moreover, if $G = WW^T, G_\alpha = W_\alpha W_\alpha^T$ *are full rank factorizations, and* V, V_α *are centered facial vectors with*

$$\operatorname{range}(V) = \operatorname{range}(G),\ \operatorname{range}(V_\alpha) = \operatorname{range}(G_\alpha),$$

then

$$Q = V^\dagger W = (JV_\alpha)^\dagger W_\alpha, \quad W = W_\alpha,$$

and the ***EDM*** *can be recovered with* $D = \mathcal{K}(VQQ^TV^T)$.

Proof. Let $G \in \mathbb{S}_+^n \cap \mathcal{S}_C^n$, $face(G) = V\mathbb{S}_+^d V^T \trianglelefteq \mathbb{S}_+^n$, V a given centered, $V^T e = 0$, facial vector of full rank. The first part of the corollary is a direct consequence from Lemma 2.6. Let $G = WW^T$ be a full rank factorization and let Q be a solution of

$$J\mathcal{P}_\alpha VQ = W_\alpha.$$

Then,

$$\begin{aligned} \mathcal{K}^\dagger \mathcal{P}_\alpha \mathcal{K}(VQQ^TV^T) &= J(\mathcal{P}_\alpha(VQQ^TV^T))J \\ &= JV_\alpha QQ^TV_\alpha^T J \\ &= JV_\alpha Q(JV_\alpha Q)^T \\ &= W_\alpha W_\alpha^T = G_\alpha. \end{aligned}$$

Therefore,

$$\mathcal{P}_\alpha \mathcal{K}(VQQ^TV^T) = \mathcal{K}(G_\alpha) = D_\alpha$$

and thus

$$\mathcal{P}_\alpha\big(\mathcal{K}(VQQ^TV^T) - D\big) = 0.$$

□

Using the above lemma we can avoid finding the matrix representation and find the full rank factoriazation of the small R instead.

3.1.3 Case 3: Small Remaining Block

Suppose that there is a single last block I with G_α that is not a proper Gram matrix with the correct rank but is too small to divide further, e.g., $< 2d+2$. Without loss of generality, we assume $I = \{1, 2, \ldots, \ell\}$. Then there are $\ell(\ell-1)/2$ possible elements that are noisy. We can now use other distances to find the noisy one, i.e., for $i, j \in I, i \neq j$, we set

$$I_{ij} = \{i, j, \ell+1, \ell+2, \ldots \ell+t\},\ |I_{ij}| = d+1,$$

and verify whether or not $K^\dagger\left(D_{I_{ij}}\right)$ is a Gram matrix with rank d. As soon as we find the one that is not, then we have found the noisy position i, j. We then choose a *well conditioned* block $I_0, |I_0| \geq d+1$ and set

$$I_1 = I_0 \cup \{i\},\ I_2 = I_0 \cup \{j\}.$$

We then use Section 3.1.2 to find the true value of D_{ij}.

3.2 Multi-Blocks with Facial Vectors, FV, for FR, MBFV

The second approach uses overlapping principal submatrices, matrix blocks, where the overlap is minimal size $\geq d+1$ but with embedding dimension d. Therefore, we have a larger number of principal submatrices but they are significantly smaller. If at anytime we find a block that is not a **EDM**, then we stop and find the noise for this small problem by using submatrices with the correct embedding dimension. We denote this approach **MBFV**.

Rather than using exposing vectors as done above in the bisection approach Section 3.1, the **FR** is done by efficiently finding the intersection of two faces corresponding to two blocks by finding the **FV** using the approach in [22, Lemma 2.9]. We find a **FV** for the small blocks using Theorem 2.8. We then find the new **FV** that represents the intersection of faces, i.e., the union of blocks, using Lemma 2.9. If we end up with a small final block, we use the same strategy as in Section 3.1.3.

Remark 3.3. *We chose to divide the problem using principal submatrices. But it is clear that we could permute the columns and rows $D \leftarrow QDQ^T$, where $Q \in \Pi$ is a permutation matrix, before applying the subdivisions and the algorithm. In fact, we could use overlapping cliques in the graph as long as we maintain a chordal structure, as chordalty allows for **EDM** completion, see e.g., [6, 20].*

3.3 Equivalent Approach Using Gale Transforms

In this section we present an alternative approach, using Gale transforms that is equivalent to that of **FR** and exposing vectors discussed above. We use the smallest blocks along with Gale transforms and denote this as **SBGT**. We note that the notion of Gale transform [15, 16] is well known and widely used in the theory of polytopes. Our approach reveals interesting relationships between **FR** and Gale transforms. We only consider the multi-block case presented in Section 3.2.

We recall from Section 2 that we have points $p_1, \ldots, p_n$ in $\mathbb{R}^d$ and we assume that the affine hull of these points has full dimension d. (In fact after centering as in (2.1), we can assume they are centered and span $\mathbb{R}^d$.) Recall that the $n \times d$ matrix of points P with $P^T = \begin{bmatrix} p_1 & \ldots & p_n \end{bmatrix}$, is called the *configuration matrix* of these points. Note that P has full column rank, and the Gram and **EDM** matrices defined by these points are $G = PP^T$ and $D = \mathcal{K}(G)$, respectively. The *Gale space of D,* $\operatorname{gal}(D)$, is given by

$$\operatorname{gal}(D) = \operatorname{null}\left(\begin{bmatrix} P^T \\ e^T \end{bmatrix}\right) = \operatorname{null}\left(\begin{bmatrix} G \\ e^T \end{bmatrix}\right). \tag{3.13}$$

Any $n \times (n-d-1)$ matrix N such that the columns of N form a basis of $\operatorname{gal}(D)$ is called a *Gale matrix* of D. The i-th row of N is called a Gale transform of p_i. Note that N is not unique. In addition, we recall that a face $f \trianglelefteq \mathbb{S}^n_+$ is characterized by the nullspace or range space of any $X \in \operatorname{relint}(f)$ and $f = \operatorname{face}(X)$, i.e., f is the *minimal face containing X*.

Theorem 3.4. *Let $G \in \mathbb{S}^n_+ \cap \mathcal{S}^n_C$ be a centered Gram matrix of the **EDM** D. Let Z be a maximum rank centered exposing vector for G as given in (3.7). Equivalently, $Z \in \operatorname{relint}(\operatorname{face}(G)^c \cap \mathcal{S}^n_C)$, where $\cdot^c$ denotes conjugate face. Then any full rank factorization $Z = NN^T$ yields a Gale matrix N of D. Conversely, if N is a Gale matrix of D, then $Z = NN^T$ is a maximum rank centered exposing vector of G.*

Proof. This follows from the definitions in (3.7) and (3.13). □

Recall that a set of points $\{p_1, \ldots, p_k\}$ in $\mathbb{R}^d$ is said to be *affinely independent* if

$$\operatorname{null}\left(\begin{bmatrix} p_1 & \cdots & p_k \\ 1 & \cdots & 1 \end{bmatrix}\right) = \{0\}.$$

It is easy to see that the columns of the Gale matrix N encode the affine dependency of the points $p_1, \ldots, p_n$. The following Lemma 3.5 is an immediate consequence of this definition.

Lemma 3.5. *Let points $p_1, \ldots, p_k \in \mathbb{R}^d, k > d$, be in general position, and let P be their configuration matrix. Then every (square) submatrix of $[P\ e]$ of order $d+1$ is nonsingular.*

Corollary 3.6. *Let P be the configuration matrix in Lemma 3.5 with corresponding **EDM** D. Let $\alpha \subset \{1, \ldots, k\}, |\alpha| > d$, and let $D_{\mathcal{I}}$ be the corresponding principal submatrix of D. Then the rank of the centered Gram matrix*

$$\operatorname{rank}(\mathcal{K}^\dagger(D_{\mathcal{I}})) = d.$$

Given $D \in \mathcal{E}^n$ of embedding dimension d, and $j \in [n-d-1]$, let $D_j = D_{[j,j+d+1]}$ be the principal submatrix of D induced by the columns (rows) $[j, j+d+1]$. Therefore, each D_j is an **EDM** of order $d+2$. Furthermore, for $j \in [n-d-2]$, the submatrices D_j and D_{j+1} overlap in $d+1$ columns (rows).

Now suppose we have an incomplete $D \in \mathcal{E}^n$ where only the principal diagonal blocks $D_i, i \in [n-d-1]$ are known, while the entries of D outside these diagonal submatrices are not known or are noisy. The problem addressed in this paper is how to recover all the entries of D, i.e., how to complete the **EDM**. We now show how to recover D by computing P^0, a configuration matrix of D, using the Gale matrix N. Recall that we have assumed that the points $p_1, \ldots, p_n$ that generate D are in *general position*. We now build N using only information from the diagonal submatrices $D_i, i \in [n-d-1]$. We compare this to (3.4) and see the relation between the Gale matrix and exposing vectors.

Recall that $\mathcal{K}^\dagger(D_j) = -\frac{1}{2} J D_j J$. Let

$$G_j = \mathcal{K}^\dagger(D_j) \tag{3.14}$$

be the Gram matrix of D_j. Then a Gale matrix N_j corresponding to D_j is the $(d+2)$-vector which forms a basis of[3]

$$\operatorname{null}\left(\begin{bmatrix} G_j \\ e^T \end{bmatrix}\right). \tag{3.15}$$

Moreover, it follows from Lemma 3.5 that all the entries of N_j are nonzero. Now a Gale matrix N for the entire matrix D can be built one column at a time[4] as follows: the entries of the jth column of N are zeros except in the positions $i = j, \ldots j+d+1$ which are equal to those of N_j. Obviously, by construction, N is $n \times (n-d-1)$ of full column rank.

Now let V be the $n \times d$ matrix whose columns form a basis of[5]

$$\operatorname{null}\left(\begin{bmatrix} N^T \\ e^T \end{bmatrix}\right).$$

Note that a configuration matrix of D is given by

$$P^0 = VQ, \tag{3.16}$$

for some nonsingular Q of order d, since the columns of both P^0 and V are both bases of the same space. Once we find Q, both P^0 and consequently $D = \mathcal{K}(P^0 P^{0^T})$ can be recovered. Note that (3.16) is an overdetermined system. We see next how to find Q by considering only the first $d+2$ equations of (3.16) that we denote by:

$$P_1^0 = V_1 Q. \tag{3.17}$$

[3] See also (3.4) where N_i is not necessarily a single column as the size of G_i can be larger than G_j in (3.15).

[4] This step is equivalent to the summation of exposing vectors in (3.6). It is shown in [12] that the summation reduces noise if the data is random and from a (Gaussian) normal distribution.

[5] This is equivalent to the V found in (3.8).

Here both P_1^0 and V_1 are $(d+2) \times d$.[6]

Now (3.17) cannot be solved as is since both P_1^0 and Q are unknown. In order to overcome this hurdle, we multiply both sides of (3.17) from the left with J and let $V_1^0 := JP_1^0$. Thus we get

$$V_1^0 = JP_1^0 = JV_1Q. \tag{3.18}$$

Note that V_1^0 is a configuration matrix corresponding to D_1 and $(V_1^0)^T e_{d+2} = 0$. Recall that D_1 is the principal submatrix of D induced by the rows (columns): $1, \ldots, d+2$. Now let B_1 be the Gram matrix corresponding to D_1. Then V_1^0 be can be found by the full-rank factorization of B_1, i.e., $B_1 = V_1^0 {V_1^0}^T$. Thus, Equation (3.18) can be solved since Q is the only unknown. Then Lemma 3.7 shows that (3.17) and (3.18) are equivalent.

.

Lemma 3.7. *Let V_1, P_1^0 and V_1^0 be as defined above. Under the general position assumption, consider the two systems in the $d \times d$ variable matrix X*

$$\textit{System I: } V_1X = P_1^0 \qquad \textit{System II: } JV_1X = JP_1^0 = V_1^0.$$

Then both systems have the same unique solution.

Proof. First note that, by relabeling the points if necessary, we can assume without loss of generality that P_1^0 has full column rank. Moreover, it follows from (3.17) that rank (V_1) = rank (P_1^0) and rank $([V_1\ e])$ = rank $([P_1^0\ e])$ since $[P_1^0\ e] = [V_1\ e] \begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix}$. Thus, V_1 has full column rank.

Now Q is the unique solution of System I. It is easy to see that the solution of System II is $Q + Y$ where the columns of Y are in null(JV_1). Let y be a nonzero vector in null(JV_1), then $V_1 y = \alpha e$ for some scalar α. Thus rank $([V_1\ \ e])$ = rank $([P_1^0\ \ e]) = d$ which contradicts Lemma 3.5 since we assume that $p^1, \ldots, p^n$ are in general position. Thus null(JV_1) is trivial and the result follows. □

An immediate consequence of Lemma 3.7 is that Q in (3.16) can be calculated from System II, for example $Q = ({V_1}^T JV_1)^{-1} {V_1}^T V_1^0$. Consequently, $P^0 = VQ$ and $D = \mathcal{K}(P^0 {P^0}^T)$.

3.3.1 Example Using Gale Transforms

Example 3.8. *Consider the following **EDM** D of embedding dimension $d = 2$,*

$$D = \begin{bmatrix} 0 & 2 & 5 & 9 & 5 & 2 \\ & 0 & 1 & 5 & 5 & 4 \\ & & 0 & 2 & 4 & 5 \\ & & & 0 & 2 & 5 \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}.$$

*(For both the **EDM** and Gram matrices we only provide the upper triangular parts.) And assume that noise is added to the entries d_{15}, d_{16} and d_{26}. Then the Gram matrix corresponding to D_1 is $G_1 = \frac{1}{2} \begin{bmatrix} 5 & 1 & -2 & -4 \\ & 1 & 0 & -2 \\ & & 1 & 1 \\ & & & 5 \end{bmatrix}$. Therefore, V_1^0, a configuration matrix of D_1, is obtained by the*

[6] Finding Q is equivalent to solving the system for r in (3.10). In fact, we have the equivalence $R = QQ^T, G = VQQ^TV^T = VRV^T$. Using the smaller system with V_1 would be equivalent to not using the entire overdetermined system in (3.10).

full-rank factorization of G_1, *i.e.,* $V_1^0 = \frac{1}{2}\begin{bmatrix} 1 & -3 \\ -1 & -1 \\ -1 & 1 \\ 1 & 3 \end{bmatrix}$. *Furthermore, a Gale matrix for* D *is* $N = \begin{bmatrix} -1 & 0 & 0 \\ 3 & -2 & 0 \\ -3 & 3 & -1 \\ 1 & -2 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$. *Hence,* V, *the matrix whose columns form a basis of* $\text{null}\left(\begin{bmatrix} N^T \\ e^T \end{bmatrix}\right)$, *is* $V = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ -2 & 1 \\ -3 & 0 \\ 0 & -1 \\ 2 & -1 \end{bmatrix}$ *and hence* $V_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ -2 & 1 \\ -3 & 0 \end{bmatrix}$ *and* $JV_1 = \frac{1}{2}\begin{bmatrix} 7 & -1 \\ 1 & 1 \\ -3 & 1 \\ -5 & -1 \end{bmatrix}$. *Hence, solving* (3.18) $V_1^0 = JV_1Q$, *we get* $Q = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ -2 & -1 \end{bmatrix}$ *Therefore,* $P^0 = VQ = \frac{1}{2}\begin{bmatrix} 0 & -3 \\ -2 & -1 \\ -2 & 1 \\ 0 & 3 \\ 2 & 1 \\ 2 & -1 \end{bmatrix}$ *and the full* ***EDM*** D *is recovered by using* $D = \mathcal{K}(P^0 {P^0}^T)$.

4 Empirics and Complexity

We generate random problems based on: the number of points n; the embedding dimension d; the magnitude of the noise.[7] Our table compares three methods: the first uses two blocks and **FR** with exposing vectors; the second uses a minimum number of multiblocks with facial vectors; and the third uses Gale transforms with the maximum number of multiblocks. Our output indicates that all the problems were solved successfully and this follows from the fact that the general position property holds generically. The output includes: the relative error for the accurate **EDM** found; and the time in cpu seconds. We discuss the hard cases below in Section 5.

4.1 Random Problems

Tables 4.1 to 4.3 (on pages 19, 20, 20) illustrate the high efficiency of the algorithms for speed, accuracy, and size. The noise α was a normal random variable with nonzero absolute value greater than .01. Both the position and a near machine precision accurate value for the noise was found in 100% of the instances.

4.2 Complexity Estimates

We now look at theoretical complexity estimate results and compare them to the empirical output. For randomly generated problems, we have plotted dimension versus solution time in

[7] We used MATLAB version R2022b on the two servers at University of Waterloo: biglinux, cpu149.math.private Dell PowerEdge R840 four Intel Xeon Gold 6230 20-core 2.1 GHz (Cascade Lake) 768 GB; and fastlinux, cpu157.math.private Dell PowerEdge R660 Two Intel Xeon Gold 6434 8-core 3.7 GHz (Sapphire Rapids) 256 GB

Data specifications			**BIEV**		**MBFV**		**SBGT**	
n	d	noise	rel-error	time(s)	rel-error	time(s)	rel-error	time(s)
1000	5	0.289	1.86e-12	0.200	8.44e-14	0.022	3.10e-12	0.121
2000	5	-0.235	1.22e-12	0.916	1.28e-13	0.044	3.84e-13	0.364
3000	5	-0.843	2.33e-12	1.713	3.36e-13	0.130	1.28e-13	0.638
4000	5	0.570	7.01e-13	1.842	3.41e-13	0.195	2.74e-13	1.342
5000	5	0.517	1.10e-12	4.394	8.80e-14	0.292	2.07e-13	1.847
6000	5	0.659	1.30e-12	6.861	3.07e-13	0.414	1.31e-12	2.889
7000	5	0.200	2.16e-12	11.759	7.95e-14	0.631	2.01e-13	3.895
8000	5	-0.240	1.71e-12	10.993	1.12e-13	0.768	2.01e-13	5.338
9000	5	-0.294	4.63e-13	18.939	1.73e-13	0.974	1.54e-13	7.299
10000	5	0.197	2.01e-12	23.177	1.63e-13	1.179	5.51e-13	9.529
11000	5	-0.405	4.42e-12	18.598	7.43e-14	1.383	3.15e-13	11.282
12000	5	0.085	4.99e-12	20.521	3.88e-13	1.732	3.16e-13	14.150
13000	5	0.311	1.42e-12	44.017	3.35e-13	2.097	2.55e-13	17.511
14000	5	-0.390	4.50e-12	53.028	7.69e-14	2.201	2.14e-11	20.961
15000	5	-0.348	5.31e-12	54.837	3.78e-13	2.383	6.69e-12	25.517
16000	5	-0.294	6.14e-12	51.610	1.30e-13	2.780	6.97e-13	29.842
17000	5	0.063	3.08e-12	64.764	2.12e-13	3.176	3.78e-13	33.774
18000	5	-0.064	1.33e-11	92.478	2.20e-13	3.485	1.27e-12	38.898
19000	5	0.001	2.81e-11	99.526	3.89e-13	3.986	2.99e-13	43.750
20000	5	-0.004	9.56e-13	97.926	9.13e-13	4.229	4.21e-13	51.621
21000	5	0.368	1.09e-12	130.590	1.93e-13	4.749	9.99e-13	58.367
22000	5	0.035	1.21e-11	177.391	7.86e-14	5.232	1.34e-13	66.882
23000	5	0.018	5.76e-12	173.445	2.44e-13	5.786	1.06e-12	73.720
24000	5	1.000	2.69e-12	160.173	1.38e-13	5.970	2.18e-13	82.715
25000	5	0.139	4.08e-12	242.229	3.11e-13	6.804	3.07e-12	91.045
26000	5	-0.385	1.91e-12	91.091	1.28e-13	6.946	3.23e-13	102.973
27000	5	-0.131	7.17e-12	173.206	9.43e-14	7.791	2.47e-13	112.248
28000	5	0.022	1.18e-11	245.353	3.61e-13	8.199	2.10e-10	124.508
29000	5	0.109	1.05e-11	264.581	6.11e-13	8.728	3.85e-12	134.517
30000	5	0.089	2.68e-12	299.627	2.58e-13	8.793	7.09e-13	149.871

Table 4.1: Fastlinux; n = 1K to 30K; mean of 3 instances per row

Figure 4.1, page 21. This agrees with the estimates in (4.1) to (4.3) for the three methods, respectively:

$$\mathbf{BIEV}, \mathbf{MBFV}, \mathbf{SBGT}: \quad O(n^3), O(n), O(n^3).$$

Note that the expense for generating the random problems is $O(n^2)$ as the main work is the multiplication using the configuration matrix PP^T.

4.2.1 Bisection with Exposing Vectors, BIEV

At each step, we divide an $n \times n$ matrix into two blocks with size $\frac{n+d+2}{2}$ so they overlap in a block of size $d+2$. The most time consuming calculation in each iteration is calculating the spectral decomposition of the corresponding Gram matrices G_1, G_2 with runtime $2O\left(\left(\frac{n+d+2}{2}\right)^3\right) = O(n^3)$, as $n \gg d$. The number of points outside of the two principal submatrices is $2\left(\frac{n-d-2}{2}\right)^2 = \frac{(n-d-2)^2}{2}$. If the noisy element is outside of the two principal submatrices, the algorithm will terminate in this step. Thus, the probability to continue to another step is

$$1 - \frac{(n-d-2)^2}{2n^2} \approx \frac{1}{2}.$$

Data specifications				**MBFV**	
n	d	noise	gen-time	rel-error	time(s)
60000	3	-0.405	82.643	7.63e-14	54.850
65000	3	0.386	94.786	1.24e-13	64.485
70000	3	-0.203	111.036	3.53e-13	75.925
75000	3	0.436	128.879	5.53e-13	88.849
80000	3	-0.129	156.128	2.86e-13	105.870
85000	3	-0.081	190.250	3.21e-13	123.193
90000	3	1.000	213.570	4.92e-13	134.878

Table 4.2: BigLinux; Multi-block solver with gen time; mean of 3 instances per row

Data specifications				**MBFV**	
n	d	noise	gen-time	rel-error	time(s)
60000	5	0.436	90.767	1.32e-13	53.049
70000	5	0.026	109.262	3.37e-13	72.006
80000	5	0.550	155.565	3.12e-13	93.529
90000	5	0.435	196.015	8.91e-13	114.084
100000	5	0.420	243.331	9.21e-13	144.727
110000	5	0.330	315.411	3.01e-13	196.053
120000	5	0.205	385.318	7.16e-13	237.335

Table 4.3: BigLinux; Multi-block solver with gen time; mean of 3 instances per row

If the noisy element is in one of the principal submatrices, then we continue with the divide and conquer algorithm. After the division, the smaller matrix has size $\frac{n+d+2}{2}$, i.e., we reduce the size of problem by approximately half. After i divisions, the size of the matrix is $\frac{n+2^{i-1}(d+2)}{2^i}$. The total runtime of this algorithm is $O(n^3)$. We now drop the d and constants in the analysis as $n \gg d$. Let $T(n)$ be the total runtime of the algorithm, and $f(n)$ be the runtime of one iteration of the subproblem. Then, we have the recurrence

$$T(n) = \frac{1}{2}T\left(\frac{n}{2}\right) + f(n) = \frac{1}{2}T\left(\frac{n}{2}\right) + O(n^3).$$

The $O(n^3)$ term dominates the recursive relationship. We get:

$$\text{total runtime is } O(n^3). \tag{4.1}$$

4.2.2 Multi-Block with Facial Vectors, MBFV

We build up the full facial vector from overlapping small principal submatrices of size $2d + 6$. Each overlaps with the previous one with a block of size $d + 3$. Solving for the final connecting matrix Q also involves solving a small system of equations using the first block, thus costing a constant time. Finding the facial vector of each small matrix has constant runtime as well,

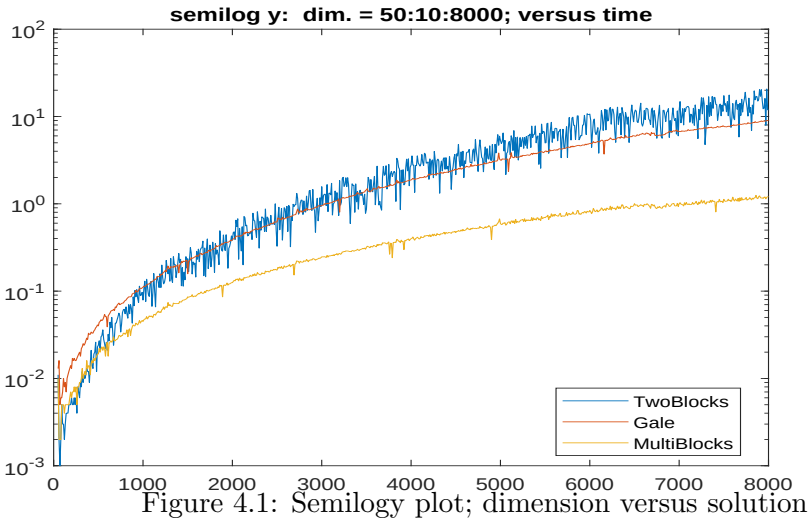

Figure 4.1: Semilogy plot; dimension versus solution time

since n is large. The number of small blocks we look at is roughly $\frac{n}{d+2}$. Therefore, we get the

$$\text{total runtime is } O(n). \tag{4.2}$$

4.2.3 Small Blocks with Gale Transform, SBGT

We calculate the Gale matrix of each small principal submatrix of size $d+2$, and then build up the Gale matrix for the entire matrix D. The most time consuming step is in finding $V \in \mathbb{R}^{n\times d}$, the **FV** , whose columns form a basis of null $\left(\begin{bmatrix} N^T \\ e^T \end{bmatrix}\right)$. This has runtime complexity of:

$$\text{total runtime is } O(n^3). \tag{4.3}$$

5 Hard Cases; No General Position Assumption

We now consider problems where the general position assumption may not hold. This can lead to hard cases for our algorithm. The hard cases are related to problems where all but 2 points are in a linear manifold of dimension $d-2$. We present examples as well as empirics for the hard cases.

Our modified algorithm for the hard case finds a block with rank d by checking all *consecutive* principal diagonal blocks $D_{[i,i+d]}$ of size $d+1$. If we can find a block with the correct embedding dimension d from the principal diagonal blocks, we can proceed with the Gale algorithm as before. Otherwise, we calculate the Gale matrix

$$N_i = \begin{bmatrix} n_i^T \\ \vdots \\ n_{i+d}^T \end{bmatrix}, \text{ whose columns form a basis for null } \left(\begin{bmatrix} G_i \\ e^T \end{bmatrix}\right),$$

for each small block $D_i := D_{[i,i+d]}$. Since the G_i does not have rank d, the points $p_i, \cdots, p_{i+d}$ are not in general position. We identify the indices where the corresponding Gale transform results in a row of zeros. If $n_j = 0$, then p_j is not in the affine hull of $\{p_i, \cdots, p_{i+d}\} \setminus \{p_j\}$ [1, Section 7.2.1]. We collect these indices in a set I, so $j \in I$ if there is some $i \in \{1, \cdots, n-d\}$ such that

$j \in \{i, \cdots, i+d\}$ and $n_j = 0$ in N_i. Additionally, let $I' = \{1, \cdots, n\} \setminus I$. We repeat the process on $D_{I \cup I'(1:d+1)}$ until we have $d+1$ points that are in general position.

There are certain cases where we can not solve the problem, so-called *hard cases*. The results in Section 5.1 show that critical to our conclusions is the difficult case when all but two points are in a linear manifold of dimension $d-2$.

Definition 5.1 (good, bad blocks)**.** *Let D_n be the noisy **EDM** as above. Let (i,j) be the corrupted index.*

1. *By a good block we mean a principal submatrix $(D_n)_\alpha$, such that the corresponding $\mathcal{K}^\dagger\left((D_n)_\alpha\right)$ is positive semidefinite with rank d.*
2. *By a bad block we mean a principal submatrix $(D_n)_\alpha$, such that the corresponding $\mathcal{K}^\dagger\left((D_n)_\alpha\right)$ is either not positive semidefinite or the rank is greater than d.*
3. *An uncorrupted good block means a principal submatrix $(D_n)_\alpha$, such that the corresponding $\mathcal{K}^\dagger\left((D_n)_\alpha\right)$ is positive semidefinite with rank d, and α does not contain either corrupted index i or j.*

If we assume that we can find an *uncorrupted good block* of size $d+1$, then we will not encounter an unsolvable hard case. Under the assumption there exists an uncorrupted good block, we will not encounter the unsolvable case. After we found a block D_α of size $d+1$ with embedding dimension d, we calculate the Gale matrix for $D_{\alpha \cup \{i\}}$ for each $i \in \{1, \cdots, n\} \setminus \alpha$. If D_α contains the corrupted entry, then by the result in Section 5.1, the only case we cannot recognize $D_{\alpha \cup \{i\}}$ to be a corrupted **EDM** is if all but two points in this set are in a linear manifold of dimension $d-2$. However, under the assumption of the data, there exists $d+1$ points: $\{p_{i_1}, \cdots, p_{i_{d+1}}\}$ in general position and is uncorrupted. Therefore, it cannot be that $D_{\alpha \cup \{i_1\}}, \cdots, D_{\alpha \cup \{i_{d+1}\}}$ all result in the case of all but two points are in a linear manifold of dimension $d-2$. Thus, if D_α contains the corrupted entry, we will be able to recognize it is not a proper **EDM**.

5.1 Characterizing Good and Bad Blocks; $\operatorname{edim}(D) = 2$ and Beyond

We know that an **EDM** D with embedding dimension d corresponds to a Gram matrix $G = \mathcal{K}^\dagger(D) \succeq 0$ with $\operatorname{rank}(G) = d$. The question that arises is whether, when a single element is perturbed in D, can this corrupted block *always* be verified correctly by checking that there are no longer exactly d nonzero and positive eigenvalues for the corresponding Gram matrix G? More precisely, does a corrupted block always have a negative eigenvalue and/or more than d positive eigenvalues. We show that indeed this can always be detected except when all but two points of a corresponding configuration matrix P lie in a linear manifold of dimension $d-2$. In addition, we show how to handle these cases by extending the notion of *yielding* in [1, Section 7.2]. Note that the 2 in the results arises from the fact that the perturbation matrix ϵE_{ij} is rank 2.

The case when the embedding dimension $d = 2$ is special. We conclude that no failures in detection can occur if the distances are positive, an easy check. For this special case $d = 2$, we do not have to assume that the problem is in general position to recognize a corrupt entry.

In our algorithms we have to identify whether the corrupted element is within a principal submatrix D_α, where for **MBFV** the cardinality $|\alpha| \geq d+2$ and is often relatively small $|\alpha| << n$. We do not assume general position, see Lemma 2.1. We present the characterization in the general d case in Theorem 5.2. This is related to the results on *yielding* in [1, Section 7.2], [2]. Also, this is related to the general question of when a matrix pencil is positive semidefinite. The interval for this is studied recently in [27]. Our case is special in that we look at rank two updates

$$D(\epsilon) := D + \epsilon E_{ij} \in \mathcal{E}^n, D \in \mathcal{E}^n, \; G(\epsilon) := G + \epsilon\, \mathcal{K}^\dagger(E_{ij}) \succeq 0, G \succeq 0.$$

The specific case of a rank two perturbation as in our case is studied in [8]. However, we have the additional condition that the rank is maintained at d.

Essentially we characterize the cases when the perturbation $D(\epsilon)$ results in a corresponding Gram matrix with the correct rank, thus fooling the algorithm. We use the MATLAB notation blkdiag to denote the block diagonal matrix formed from the arguments.

Theorem 5.2. *Let $D \in \mathcal{E}^k$ with $\operatorname{edim}(D) = d \leq k-2$. Suppose that there is a single, nonzero corrupted distance $D_{ij}, i<j$, and the noisy **EDM** is denoted by the singular matrix pencil*

$$D(\epsilon) := D + \epsilon E_{ij},\ \epsilon \in \mathbb{R} \setminus \{0\}.$$

Let $G := \mathcal{K}^\dagger(D) = Q\Lambda Q^T$ be the Gram matrix with its spectral decomposition and, without loss of generality,

$$\Lambda = \operatorname{blkdiag}(0, \Lambda_+),\ \Lambda_+ \in \mathbb{S}^d_{++}.$$

Denote $G_E := \mathcal{K}^\dagger(E_{ij}) = -\frac{1}{2} J E_{ij} J$ and let $\bar{G}_E$ be defined and appropriately blocked with size d:

$$\bar{G}_E := Q^T G_E Q = \begin{bmatrix} \bar{G}_{11} & \bar{G}_{12} \\ \bar{G}_{12}^T & \bar{G}_{22}, \end{bmatrix},\ \bar{G}_{22} \in \mathbb{S}^d.$$

Define the open interval I_ϵ for maintaining $\Lambda_+ + \epsilon \bar{G}_{22} \succ 0$, equivalently for maintaining $I_d + \epsilon \Lambda_+^{-1/2} \bar{G}_{22} \Lambda_+^{-1/2} \succ 0$, as

$$I_\epsilon = \left(-\frac{1}{\lambda_{\max}\left(\Lambda_+^{-1/2} \bar{G}_{22} \Lambda_+^{-1/2}\right)}, -\frac{1}{\lambda_{\min}\left(\Lambda_+^{-1/2} \bar{G}_{22} \Lambda_+^{-1/2}\right)} \right) \subseteq (-\infty, +\infty), \tag{5.1}$$

where a 0 in the denominator results in $-\infty, +\infty$, for the left/right bound of the interval, appropriately. Define the condition:

$$\textbf{\textit{EDM}}\ \textit{condition:}\quad \boxed{\epsilon \in I_\epsilon\ \textit{and}\ \epsilon \left(\bar{G}_{11} - \epsilon \bar{G}_{12} \left(\Lambda_+ + \epsilon \bar{G}_{22} \right)^{-1} \bar{G}_{21} \right) \succeq 0}. \tag{5.2}$$

Then:

1. *The **EDM** condition (5.2) defines the convex yielding interval for D_{ij}. More precisely, suppose that the **EDM** condition (5.2) holds. Then:*

$$D(\epsilon) \in \mathcal{E}^k \ \textit{and}\ \operatorname{edim}(D(\epsilon)) \geq d; \tag{5.3a}$$

$$\bar{G}_{11} = 0 \implies D(\epsilon) \in \mathcal{E}^k,\ \operatorname{edim}(D(\epsilon)) = d; \tag{5.3b}$$

$$\bar{G}_{11} - \epsilon \bar{G}_{12} \left(\Lambda_+ + \epsilon \bar{G}_{22} \right)^{-1} \bar{G}_{21} = 0 \implies D(\epsilon) \in \mathcal{E}^k,\ \operatorname{edim}(D(\epsilon)) = d. \tag{5.3c}$$

2. *Conversely, we have the following necessary conditions for restricted yielding. Suppose that there exists $\delta \in \mathbb{R}_{++}$ such that*

$$D(\epsilon) \in \mathcal{E}^k,\ \operatorname{edim}(D(\epsilon)) = d,\ \forall \epsilon \in (-\delta, \delta).$$

Then the configuration matrix P of D, $PP^T = G = \mathcal{K}^\dagger(D)$, has $k-2 \geq d$ points that are in a linear manifold $\mathcal{L}$ of dimension $d-2$. Necessarily, the two points i, j outside the linear manifold define the corrupted distance D_{ij}.

Proof. We define the *matrix pencil*

$$G(\epsilon) := \mathcal{K}^\dagger(D(\epsilon)) = G + \epsilon G_E.$$

Our results depend on identifying when the perturbed Gram matrix, the matrix pencil, maintains: $G(\epsilon) = G + \epsilon G_E \succeq 0$ with $\operatorname{rank} G(\epsilon) = d$. We use the spectral decompositions and the *Sylvester law of inertia*. The latter identifies when positive semidefiniteness and rank d are maintained under a congruence.

To begin, we need to consider the eigenpairs for the two nonzero eigenvalues of

$$G_E := \mathcal{K}^\dagger(E_{ij}) = -\frac{1}{2} J E_{ij} J.$$

An orthogonal pair of eigenvectors of E_{ij} is $e_i \pm e_j$. We have the properties $E_{ij} \neq 0, \operatorname{tr}(E_{ij}) = 0$ and

$$\lambda_1(E_{ij}) = 1 > 0 = \lambda_2(E_{ij}) = \ldots = \lambda_{n-1}(E_{ij}) = 0 > \lambda_n(E_{ij}) = -1.$$

Therefore, the above singular congruence $JE_{ij}J = \mathcal{P}_{\mathcal{S}^n_{\mathbb{C}}}(E_{ij})$, by (2.5), implies

$$\lambda_1(G_E) > 0 = \lambda_2(G_E) = \ldots = \lambda_{n-1}(G_E) = 0 > \lambda_n(G_E),$$

i.e., there are exactly two nonzero eigenvalues of G_E; verified as well in the following. Define the two orthogonal vectors

$$e_{ij} := e_i - e_j,\ e_{ijc} := \frac{2}{k} e - (e_i + e_j) \in e^\perp \subset \mathbb{R}^k. \tag{5.4}$$

One can verify that e_{ij} and e_{ijc} are eigenvectors of G_E and the corresponding eigenvalues are:

$$\lambda_1(G_E) = \frac{1}{2} > 0 > \lambda_k(G_E) = \frac{2-k}{2k}. \tag{5.5}$$

Notice that signs of the eigenvalues are strictly positive and strictly negative. The congruences with Q and the interlace theorem for eigenvalues yield $\lambda_{\max}(\bar{G}_{22}) \geq 0 \geq \lambda_{\min}(\bar{G}_{22})$. The congruence with $\Lambda_+^{-1/2}$ and the definition (5.1) of I_ϵ bring us to $0 \in I_\epsilon$ and we conclude that

$$\begin{array}{rcrcl} \epsilon \in I_\epsilon & \iff & I_d + \epsilon \Lambda_+^{-1/2} \bar{G}_{22} \Lambda_+^{-1/2} & \succ & 0 \\ & \iff & \Lambda_+ + \epsilon \bar{G}_{22} & \succ & 0, \end{array}$$

and that

$$G + \epsilon G_E = Q(\Lambda + \epsilon \bar{G}_E) Q^T \succeq 0 \iff \Lambda + \epsilon \bar{G}_E \succeq 0.$$

1. Assuming (5.2), we have $\Lambda_+ + \epsilon \bar{G}_E \succ 0$. By the Schur complement theorem,

$$\epsilon \left(\bar{G}_{11} - \epsilon \bar{G}_{12} \left(\Lambda_+ + \epsilon \bar{G}_{22} \right)^{-1} \bar{G}_{21} \right) \succeq 0 \iff G(\epsilon) \succeq 0.$$

   Under the assumptions and definitions on $D(\epsilon), I_\epsilon$, and by continuity of eigenvalues and of the linear transformation $\mathcal{K}^\dagger$, we get that the d positive eigenvalues of G are perturbed but stay positive in $G(\epsilon)$, i.e., recalling the assumption that $\operatorname{edim}(D) = d$, we have

$$\operatorname{rank}(G(\epsilon)) \geq \operatorname{rank}(G) = d \implies \operatorname{edim}(D(\epsilon)) \geq d.$$

   This proves (5.3a).
   (5.3b) and (5.3c) follow from a Schur complement argument. That is,

$$\begin{aligned} \Lambda + \epsilon \bar{G}_E &= \begin{bmatrix} \epsilon \bar{G}_{11} & \epsilon \bar{G}_{12} \\ \epsilon \bar{G}_{12}^T & \Lambda_+ + \epsilon \bar{G}_{22} \end{bmatrix} \\ &\cong \begin{bmatrix} \epsilon \left(\bar{G}_{11} - \epsilon \bar{G}_{12} \left(\Lambda_+ + \epsilon \bar{G}_{22} \right)^{-1} \bar{G}_{21} \right) & \epsilon \bar{G}_{12} \\ 0 & \Lambda_+ + \epsilon \bar{G}_{22} \end{bmatrix}. \end{aligned}$$

Since $\operatorname{rank}(\Lambda_+ + \epsilon\bar{G}_{22}) = d$, $\bar{G}_{11} - \epsilon\bar{G}_{12}\left(\Lambda_+ + \epsilon\bar{G}_{22}\right)^{-1}\bar{G}_{21} = 0$ implies

$$\operatorname{rank}(\Lambda + \epsilon\bar{G}_E) = \operatorname{rank}(G + \epsilon G_E) = d.$$

Similarly, $\bar{G}_{11} = 0$ leads to $\bar{G}_{12} = 0$ and thus $\operatorname{rank}(G + \epsilon G_E) = d$. Therefore (5.3) holds.

2. Suppose that there exists $\delta > 0$ such that, for all $|\epsilon| < \delta$, we have $D(\epsilon) \in \mathcal{E}^k$ and $\operatorname{edim}(D(\epsilon)) = d$. This is equivalent to

$$G + \epsilon G_E \succeq 0 \text{ and } \operatorname{rank}(G + \epsilon G_E) = d.$$

We want to show that

$$G + \epsilon G_E \succeq 0,\ \operatorname{rank}(G + \epsilon G_E) = \operatorname{rank}(G) \implies \operatorname{range}(G_E) \subset \operatorname{range}(G).$$

Recall that $G + \epsilon G_E \succeq 0 \iff \Lambda + \epsilon\bar{G}_E \succeq 0$ and that

$$\Lambda + \epsilon\bar{G}_E \cong \begin{bmatrix} \epsilon\left(\bar{G}_{11} - \epsilon\bar{G}_{12}\left(\Lambda_+ + \epsilon\bar{G}_{22}\right)^{-1}\bar{G}_{21}\right) & \epsilon\bar{G}_{12} \\ 0 & \Lambda_+ + \epsilon\bar{G}_{22} \end{bmatrix}.$$

Since $0 \in I_\epsilon$, we can find small enough $|\epsilon| > 0$, namely $\epsilon \in I_\epsilon \cap (-\delta, \delta)$, so that $\Lambda_+ + \epsilon\bar{G}_{22} \succ 0$. Hence $\operatorname{rank}(\Lambda_+ + \epsilon\bar{G}_{22}) = d$ and note that the rank of the entire matrix is the sum of the ranks of the diagonal blocks. Hence, the top left block after the elimination has to have rank 0, by $\operatorname{rank}(G + \epsilon G_E) = d$ assumption. Therefore,

$$\bar{G}_{11} - \epsilon\bar{G}_{12}\left(\Lambda_+ + \epsilon\bar{G}_{22}\right)^{-1}\bar{G}_{21} = 0. \tag{5.6}$$

(5.6) is violated for small enough $|\epsilon|$ unless $\bar{G}_{11} = 0$. Hence $\bar{G}_{11} = 0$, which also implies $\bar{G}_{12} = 0$. Hence,

$$\operatorname{range}(\bar{G}_E) \subseteq \operatorname{range}(\operatorname{blkdiag}(0, I_d)) = \operatorname{range}(\operatorname{blkdiag}(0, \Lambda_+)) = \operatorname{range}(\Lambda),$$

which is equivalent to

$$\operatorname{range}(G_E) = \operatorname{range}(Q\bar{G}_E Q^T) \subseteq \operatorname{range}(Q\Lambda Q^T) = \operatorname{range}(G).$$

This implies that the facial vector V for $G = \mathcal{K}^\dagger(D)$ must have range that contains the span of the two eigenvectors of $G_E = \mathcal{K}^\dagger(E_{ij})$. Otherwise, at least one of the following happens: (i) we lose positive semidefiniteness; (ii) the number of positive eigenvalues increases. Therefore, we must have

$$\operatorname{range}\left(\begin{bmatrix} e_{ij} & e_{ijc} \end{bmatrix}\right) \subset \operatorname{range}(V), \quad V = \begin{bmatrix} e_{ij} & e_{ijc} & \bar{V} \end{bmatrix}, \quad \begin{bmatrix} e_{ij} & e_{ijc} \end{bmatrix}^T \bar{V} = 0. \tag{5.7}$$

Thus, the ℓ-th row of V is

$$V(\ell, :) = \begin{bmatrix} 0 & \frac{2}{k} & \bar{V}(\ell, :) \end{bmatrix} \in \{0\} \times \{2/k\} \times \mathbb{R}^{d-2},$$

for all $\ell \neq i, j$. This shows that for $R = I_d$, all but two points are in a manifold of dimension $d - 2$. This means the same is true for general $R \in \mathbb{S}^d_{++}$. Without loss of generality, let $i = n - 1, j = n$. Then,

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \in \mathbb{R}^{k \times d}, V_1 \in \mathbb{R}^{(k-2)\times d}, V_2 \in \mathbb{R}^{2\times k}.$$

Note that rows of V_1 are in a $d - 2$ dimensional manifold if, and only if, $\operatorname{rank} V_1 \leq d - 2$. Since $R \succ 0$, $V_1 R^{1/2}$ maintains the same rank as V_1, i.e., the rows of $V_1 R^{1/2}$ are in a manifold of dimension $d - 2$ where

$$\begin{aligned} G &= VRV^T = [VR^{1/2}][VR^{1/2}]^T \\ &= \begin{bmatrix} V_1 R^{1/2} \\ V_2 R^{1/2} \end{bmatrix} \begin{bmatrix} V_1 R^{1/2} \\ V_2 R^{1/2} \end{bmatrix}^T. \end{aligned}$$

□

Corollary 5.3. *Let $D \in \mathcal{E}^k$ with $\operatorname{edim}(D) = 2 < k-1$. Suppose that there is a corrupted position $i < j$ and the noisy **EDM** is*

$$D_n = D + \epsilon E_{ij},\ \epsilon \neq 0. \tag{5.8}$$

If $D_n \in \mathcal{E}^k, \operatorname{edim}(D_n) \leq d$, and $|\epsilon|$ is sufficiently small, then the (centered) configuration matrix P of D, $PP^T = G = \mathcal{K}^\dagger(D)$, has (at least) $k-2 \geq 2$ points that are equal, i.e., the rows $P_{s:} = p \in \mathbb{R}^d, \forall s \in [k], s \neq i, s \neq j$. Thus

$$D_{st} = (D_n)_{st} = 0, \forall s, t \in [k], s \neq i, t \neq j.$$

Proof. We apply Theorem 5.2, Item 2 for this special $d = 2$ case with $|\epsilon|$ small to guarantee we are within the interval. □

5.2 Example of Hard Case; Multiple Solutions

The following is an example where we have determined a *bad* block but where we cannot determine the *position* of the entry that is corrupted. Consider the following corrupted **EDM** in embedding dimension 3.

$$D(18) := D_n = \begin{bmatrix} 0 & 4 & 16 & 8 & 6 & 14 \\ 4 & 0 & 4 & 4 & 6 & 6 \\ 16 & 4 & 0 & 8 & 14 & 6 \\ 8 & 4 & 8 & 0 & \boxed{18} & 14 \\ 6 & 6 & 14 & \boxed{18} & 0 & 20 \\ 14 & 6 & 6 & 14 & 20 & 0 \end{bmatrix} \quad G_n = \mathcal{K}^\dagger(D_n);\ \lambda(G(18)) = \lambda(G_n) \approx \begin{pmatrix} -0.7252 \\ 0.0000 \\ 0.0000 \\ 4.7157 \\ 7.4003 \\ 13.2759 \end{pmatrix}.$$

This is not an **EDM**, as its corresponding $G_n = \mathcal{K}^\dagger(D_n)$ is not positive semidefinite. However, there is more than one way to change only one entry of this matrix and obtain an **EDM**. We can change the 18 in the $(4,5)$ entry in two different ways, to 14 and $6/5$, respectively, and get **EDM**s $D_1 := D(14), D_2 := D(6/5)$, with corresponding positive semidefinite $G(\alpha) = \mathcal{K}^\dagger(D(\alpha))$:

$$\lambda(G(14)) \approx \begin{pmatrix} 0.000000 \\ 0.000000 \\ 0.000000 \\ 4.876894 \\ 6.000000 \\ 13.123106 \end{pmatrix}, \qquad \lambda\left(G\left(\frac{6}{5}\right)\right) \approx \begin{pmatrix} 0.000000 \\ 0.000000 \\ 0.000000 \\ 1.562857 \\ 6.189609 \\ 14.114201 \end{pmatrix}.$$

However, if we use a value in the middle of the changes $[6/5, 14]$, e.g., $D(5)$, then the corresponding $G(5)$ is indeed a Gram matrix but has $4 > d$ positive eigenvalues. This corresponds with Theorem 5.2 and means that we have the end points of a yielding interval where $D(6/5) + \epsilon E_{45} \in \mathcal{E}^6$ if, and only if, $\epsilon \in [0, 12.8]$. This also means that the two eigenvectors of $G_E = \mathcal{K}^\dagger(E_{45})$ are not in range$(G(6/5)$ or range$(G(14))$. In fact, by checking the rank of $[G\ v]$, with v one of the eigenvectors for G_E, we can see that one eigenvector is in the range while the other is not, thus explaining the finite yielding intervals. The example can be extended to higher embedding dimension using yelding intervals formed with the eigenvectors in the appropriate range.

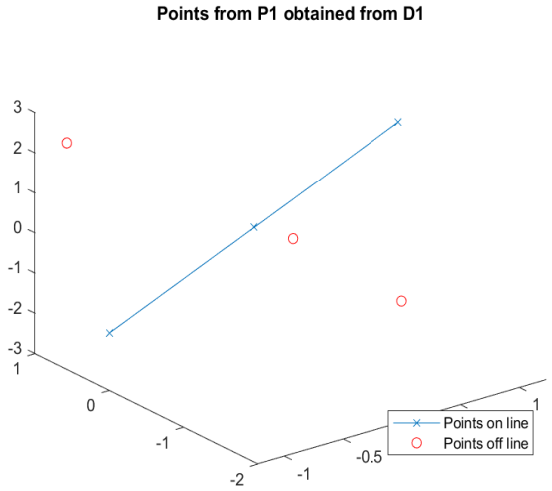

Figure 5.1: Three points off the line

Or we can change the $(4,6)$ entry in 2 different ways and obtain a proper **EDM**

$$D_3 = \begin{bmatrix} 0 & 4 & 16 & 8 & 6 & 14 \\ 4 & 0 & 4 & 4 & 6 & 6 \\ 16 & 4 & 0 & 8 & 14 & 6 \\ 8 & 4 & 8 & 0 & 18 & \boxed{\frac{42}{5}} \\ 6 & 6 & 14 & 18 & 0 & 20 \\ 14 & 6 & 6 & \boxed{\frac{42}{5}} & 20 & 0 \end{bmatrix} \quad D_4 = \begin{bmatrix} 0 & 4 & 16 & 8 & 6 & 14 \\ 4 & 0 & 4 & 4 & 6 & 6 \\ 16 & 4 & 0 & 8 & 14 & 6 \\ 8 & 4 & 8 & 0 & 18 & \boxed{2} \\ 6 & 6 & 14 & 18 & 0 & 20 \\ 14 & 6 & 6 & \boxed{2} & 20 & 0 \end{bmatrix}$$

Similarly, changing the $(5,6)$ entry can also make D_n into an **EDM**

$$D_5 = \begin{bmatrix} 0 & 4 & 16 & 8 & 6 & 14 \\ 4 & 0 & 4 & 4 & 6 & 6 \\ 16 & 4 & 0 & 8 & 14 & 6 \\ 8 & 4 & 8 & 0 & 18 & 14 \\ 6 & 6 & 14 & 18 & 0 & \boxed{14} \\ 14 & 6 & 6 & 14 & \boxed{14} & 0 \end{bmatrix} \quad D_6 = \begin{bmatrix} 0 & 4 & 16 & 8 & 6 & 14 \\ 4 & 0 & 4 & 4 & 6 & 6 \\ 16 & 4 & 0 & 8 & 14 & 6 \\ 8 & 4 & 8 & 0 & 18 & 14 \\ 6 & 6 & 14 & 18 & 0 & \boxed{6} \\ 14 & 6 & 6 & 14 & \boxed{6} & 0 \end{bmatrix}$$

This situation occurs in general when all but 3 points are in a manifold of dimension $d-2$. In this case, points 1-3 are on a line in $\mathbb{R}^3$, and point 4-6 are off the line.
For example, the configuration of D_1, see Figure 5.1, page 27, is

$$P_1 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \\ -2 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & -2 & -1 \end{bmatrix}$$

If we perturb the distance between points 4 and 5, then we no longer have a **EDM**. However, if we ignore the existence of point 6, then all but 2 points are on a manifold of dimension $d-2$ so

the $(4,5)$ entry is yielding. Thus, the submatrix formed by points 1 to 5 is still a proper **EDM**. Thus we can fix the corrupted **EDM** by changing the $(4,6)$ entry instead.

5.3 Empirics for the Hard Case

We now generate random hard problems where some points are generated to be on a linear manifold of dimension less than d. Thus, many points are not in general position. Under the Data specification columns, the fourth column indicates the number of points not in general position. The fifth column indicate the dimension of manifold these points live in. We run the problems with $n = 100$ and d from 2 to 10. For each embedding dimension, we run 50 problems. Our output demonstrate the algorithm works well for the hard problems.

Data specifications					Hard Gale Transform with Multi Blocks		
n	d	noise	# pts not in general position	dim of pts	tol-attained	rel-error	time(s)
100	2	0.016	79	1	100%	1.64e-12	0.058
100	3	0.071	86	1	100%	1.62e-13	0.050
100	4	-0.472	82	3	100%	5.08e-13	0.034
100	5	-0.328	78	3	100%	6.98e-13	0.032
100	6	0.185	89	1	100%	1.08e-12	0.048
100	7	0.334	91	1	100%	3.09e-12	0.045
100	8	-0.193	78	2	100%	3.98e-12	0.041
100	9	0.394	76	8	100%	9.31e-12	0.028
100	10	0.222	83	5	100%	1.91e-11	0.047

Table 5.1: Average of 50 hard problems; hard problems where many point are generated to be on a manifold of dimension less than the embedding dimension d

6 Conclusion

In this paper we have studied a case of error-correction in **EDM** with special structure, i.e., we assume that the **EDM** D has known embedding dimension d and that exactly one distance is in error, is corrupted. We have presented three different strategies for divide and conquer and three different types of facial reduction. Our approaches accurately identify and correct exactly one corrupted distance of an **EDM**. The numerical tests confirm that we can solve huge problems to high precision and quickly. In fact, the tests on random problems with $n = 100,000$ for the best method take approximately 100 seconds to solve to machine precision; and this confirms our analysis of $O(n)$ cost for the best of our three algorithms that we tested. Note that $n = 100,000, d = 3$ means that P has $3(10^6)$ variables and the Gram matrix is dense and has order $5(10^{12})$ variables. Attempting to solve these problems using **SDP** with interior-point or first order methods would not be reasonable and, even if possible, would not obtain high accuracy; whereas we obtain near machine precision.

We include a characterization of when a perturbation of a single element results in a **EDM** with unchanged embedding dimension d. This is equvalent to maintaining the difficult constant rank constraint. Moreover, we provide a characterization for when the **NEDM** problem solves our problem, i.e, this happen if, and only if, the original data element $(D_0)_{ij} = 0$ and the perturbation $\alpha < 0$, a highly degenerate trivial case as the location ij is identified.

In addition, the algorithm extends to any number of corrupted elements that are outside the blocks that we choose. We can also work with a chordal graph and choose overlapping cliques to obtain the principal submatrices.

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