Local Nonglobal Minima for Solving Large Scale Extended Trust Region Subproblems

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Abstract

We study large scale extended trust region subproblems (**eTRS**) i.e., the minimization of a general quadratic function subject to a norm constraint, known as the trust region subproblem (**TRS**) but with an additional linear inequality constraint. It is well known that strong duality holds for the **TRS** and that there are efficient algorithms for solving large scale **TRS** problems. It is also known that there can exist at most one local non-global minimizer (**LNGM**) for **TRS**. We combine this with known characterizations for strong duality for **eTRS** and, in particular, connect this with the so-called *hard case* for **TRS**.

We begin with a recent characterization of the minimum for the **TRS** via a generalized eigenvalue problem and extend this result to the **LNGM**. We then use this to derive an efficient algorithm that finds the global minimum for **eTRS** by solving at most three generalized eigenvalue problems.

Keywords: Trust region subproblem, linear inequality constraint, large scale optimization, generalized eigenvalue problem
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Contents

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1 Introduction

We study large scale instances of the *extended trust region subproblem,* ***eTRS***

$$
\begin{array}{rll}
p^* := & \min & q(x) := x^T A x + 2a^T x \\
& \text{s.t.} & g(x) := ||x||^2 - \delta \leq 0, \\
& & \ell(x) := b^T x - \beta \leq 0,
\end{array}
\tag{eTRS}
$$

where $A \in \mathbb{S}^n$ is a real $n \times n$ symmetric matrix, $a, b \in \mathbb{R}^n \backslash \{0\}$ and $\beta \in \mathbb{R}, \delta \in \mathbb{R}_{++}$. Here a linear inequality constraint is added onto the standard *trust region subproblem,* ***TRS***. The **TRS** is an important subproblem in trust region methods for both constrained and unconstrained problems, e.g. [5]. The **eTRS** problem extends the **TRS** and is a step toward solving **TRS** with a general number of inequality constraints. Such problems would be useful for example in the subproblem of finding search directions for sequential quadratic programming (**SQP**) methods for general nonlinear programming, e.g., [3].

It is known that, surprisingly, strong duality holds for **TRS** and the global minimizer can be found efficiently and accurately, even though the objective function is not necessarily convex. The early algorithms for moderate sized problems are based on exploiting the positive semidefinite second order optimality conditions using a Cholesky factorization of the Lagrangian, see e.g., [8, 16]. These methods were extended to the large scale case using a parametrized eigenvalue problem, e.g. [9, 10, 13, 17]. A related problem is finding the local non-global minimizer (**LNGM**) of **TRS** if it exists, see [15]. See [5] for more extensive details, applications, and background for **TRS**.

However, strong duality can fail for the **eTRS**. This is characterized in [2] for the more general two quadratic constraint problem. We show that this is exactly connected to the so-called *hard case* for **TRS**. We use this fact to find an efficient approach for finding the global minimizer for **eTRS**. Recently, a generalized eigenvalue characterization for the **TRS** optimum is derived in Adachi et al [1] based on solving a *single* generalized eigenvalue problem. This algorithm is shown to be extremely efficient for solving the **TRS**. In this paper we extend this result for the **LNGM** optimum using the second largest real generalized eigenvalue of a matrix pencil. This provides an efficient procedure for finding the **LNGM**. From combining the solutions for **TRS** and **LNGM** we derive an efficient algorithm for **eTRS**.

We include a discussion relating strong duality and stability for **eTRS**. Extensive numerical tests show that our new algorithm is accurate and can solve large scale problems efficiently.

Related previous work on strong duality and an eigenvalue approach on **eTRS** appeared in e.g., [11, 12, 18, 19].

1.1 Notation and Preliminaries

We let

$$\lambda_{\min}(A) = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$$

denote the eigenvalues of A in nondecreasing order, and $A = Q\Lambda Q^T$ be the *orthogonal spectral decomposition of* A with the diagonal matrix of eigenvalues $\Lambda = \mathrm{Diag}(\lambda_1, ..., \lambda_n)$. We denote the *orthogonal matrices,* $\mathcal{O}^n$. We let q_i denote the orthonormal columns of the eigenvector matrix $Q \in \mathcal{O}^n$.

For $X \in \mathbb{S}^n$ the space of $n \times n$ real symmetric matrices, we let $X \succeq 0, \succ 0$ denote positive semidefiniteness and definiteness, respectively. In addition, we define the *vector of ones,* e of appropriate size and $\mathrm{Diag}(v)$ be the diagonal matrix formed from the vector v.

It is now well known that, surprisingly, the possibly nonconvex **TRS** problem has the following characterization of optimality with a positive semidefinite Lagrangian Hessian.

Theorem 1.1 (*Characterization of Global Minimum of **TRS***, [8,16])**.** *Define the*

$$\textit{Lagrangian of } \textbf{\textit{TRS}},\ L(x, \lambda) := q(x) + \lambda(\|x\|^2 - \delta). \tag{1.1}$$

*The vector $x^* \in \mathbb{R}^n$ is a global optimum of **TRS** if, and only if, there exists $\lambda \in \mathbb{R}$ such that*

$$\begin{array}{rcrcl} \frac{1}{2}\nabla L(x^*, \lambda^*) & = & (A + \lambda^* I)x^* + a & = & 0, \quad \lambda^* \geq 0 \\ \frac{1}{2}\nabla^2 L(x^*, \lambda^*) & = & A + \lambda^* I & \succeq & 0 \\ & & \|x^*\|^2 - \delta & \leq & 0 \\ & & \lambda(\|x^*\|^2 - \delta) & = & 0 \end{array}$$

□

Now if x^* is a global minimizer of **TRS** and $\nabla^2 L(x^*, \lambda^*)$ is singular, then

$$\lambda^* = -\lambda_1 \text{ and } 0 \neq a \in \mathrm{Range}(A + \lambda^* I) = (\mathrm{Null}(A + \lambda^* I))^{\perp}$$

holds and leads to the following definition.

Definition 1.1 (Hard Case)**.** *The hard case holds for **TRS** if a is orthogonal to the eigenspace corresponding to λ_1,* $\mathrm{Null}(A + \lambda^* I)$.

In addition, the *Slater constraint qualification, **SCQ***, or strict feasibility, can be assumed without loss of generality for feasible instances of **eTRS**.

Lemma 1.1. *The **eTRS** is feasible, respectively strictly feasible, if, and only if*

$$-\sqrt{\delta}\|b\| \leq \beta, \ \textit{respectively} \ -\sqrt{\delta}\|b\| < \beta. \tag{1.2}$$

Moreover, if equality holds on the left in (1.2)*, then **eTRS** has the unique feasible (and so optimal) point* $x^* = -\frac{-\sqrt{\delta}}{\|b\|}b$.

Proof. Consider the problem $\min_x\{x^T b : \|x\|^2 \leq \delta\}$. We can differentiate the Lagrangian to get

$$0 \neq x = \frac{-1}{2\lambda}b, \lambda > 0.$$

Since $x^T b = \frac{-1}{2\lambda} b^T b < 0$, the minimum value is obtained with $0 < \lambda$ small. We now have

$$x^T x = \frac{1}{(2\lambda)^2}\|b\|^2 \leq \delta \implies 2\lambda = \frac{\|b\|}{\sqrt{\delta}}.$$

The result now follows by noting that the linear inequality constraint is

$$x^T b = -\frac{1}{2\lambda}\|b\|^2 \leq \beta$$

and then substituting for the value found for 2λ. □

We note that if the global solution of **TRS** is feasible for **eTRS** then it is clearly optimal. And from the above, we know that it can be found efficiently using a generalized eigenvalue problem. Therefore from this and Lemma 1.1 we can make the following assumption for the theoretical part of the paper. (We do not make this assumption for the algorithmic part.)

Assumption 1.1. *We assume in this paper that **eTRS** is strictly feasible and that the global solution of **TRS** is infeasible for **eTRS**.*

1.2 Outline

We continue in Section 2 with the details on the **LNGM**. This includes known results from [15] and one of the main results of this paper in Theorem 2.4, the necessary conditions for a **LNGM** using the second largest real generalized eigenvalue of a matrix pencil. In Section 3.1 we discuss necessary and sufficient conditions for strong duality to hold for **eTRS**. A discussion on the stability of **eTRS** and resulting stability of our approach is given in Section 3.2.

The various optimality conditions for **eTRS** are applied in Section 4. Included in this section are outlines of the algorithms for an efficient numerical procedure to find the global optimum of **eTRS**. Our numerical results appear in Section 5. We provide concluding remarks in Section 6.

2 On a Local Non-global Minimizer (LNGM) of TRS

2.1 Background on LNGM

Let x^* be a global optimal solution of **eTRS**. Then the linear constraint is either inactive $b^T x^* < \beta$ or active $b^T x^* = \beta$. In the former case, we have x^* must be a local (not global by Assumption 1.1) minimizer of **TRS**, i.e., we can have x^* being a *local non-global minimizer, **LNGM***, of **TRS**. We now provide some background on the **LNGM**.

Lemma 2.1. *If $A \succeq 0$, then no **LNGM** exists.*

Proof. This is immediate since $A \succeq 0$ implies that **TRS** is a convex problem, i.e., a problem where local minima are global minima. (It also follows from Theorem 2.1 below, since $0 \le \lambda^* < -\lambda_1$.) □

Therefore, in this section we assume that $\lambda_1 < 0$. We continue and present some known results related to **LNGM**. Then following the results in [1], we show that the **LNGM** can be computed via a generalized eigenvalue problem.

Theorem 2.1 (*Necessary Conditions for* ***LNGM***, [15])**.** *Let x^* be a **LNGM**. Choose $V \in \mathbb{R}^{n\times(n-1)}$ such that $\left[\begin{array}{c|c} \frac{1}{\|x^*\|}x^* & V \end{array}\right] \in \mathcal{O}^n$. Then there exists a unique $\lambda^* \in (\max\{0, -\lambda_2\}, -\lambda_1)$*[1] *such that*

$$V^T(A + \lambda^* I)V \succeq 0, \qquad (A + \lambda^* I)x^* = -a, \qquad ||x^*||^2 = \delta. \tag{2.1}$$

□

Corollary 2.1. *If the so-called hard case holds for **TRS**, i.e., a is orthogonal to the eigenspace corresponding to λ_1, then no **LNGM** exists.*[2]

Proof. The proof is given in [15, Lemma 3.2]. We include a separate proof to emphasize that a stronger result holds as is given in Corollary 2.2 below.

After a rotation if needed, we can assume for simplicity that $A = \mathrm{Diag}(\lambda)$ is a diagonal matrix. To obtain a contradiction, we assume that $a^T q_1 = 0$. From this assumption we have that the first element $a_1 = 0$. From (2.1) this implies that the first element $x_1^* = 0$ which yields that the first eigenvector given by the first unit vector e_1 satisfies $e_1 = Vu$, for some u. We have $u^T V^T (A + \mu I)Vu = \lambda_1 + \lambda^* < 0$. This contradicts the second order semidefiniteness condition in (2.1). □

Corollary 2.2. *If the weak form of the hard case holds for **TRS**, i.e., a is orthogonal to <u>some</u> eigenvector corresponding to λ_1, then no **LNGM** exists.*

Proof. The proof of Corollary 2.1 just needed one eigenvector orthogonal to a. □

Now let

$$\phi(\lambda) := \|(A + \lambda I)^{-1}a)\|^2.$$

For

$$\lambda \in (\max\{0, -\lambda_2\}, -\lambda_1),$$

[1] We have added the fact that $\lambda^* > 0$ whereas only nonnegativity is given in [15, Theorem 3.1]. Strict complementarity is proved in [14, Prop. 3.4]. In fact, it is easy to see by the second order conditions that strict complementarity holds as well for the global minimum for **TRS** in the $\lambda_1 < 0$ case.

[2] The hard case arises in algorithms for **TRS**. The singularity that can arise requires special treatment, see e.g., [16]. In fact, it can be handled by a shift and deflation step, see [7].

Theorem 2.1 shows that the equation $\phi(\lambda) = \delta$ is a necessary condition for a **LNGM**. Furthermore, using the eigenvalue decomposition of A we have

$$\begin{aligned}
\phi(\lambda) &= \sum_{i=1}^{n} \frac{(q_i^T a)^2}{(\lambda_i + \lambda)^2}, \\
\phi'(\lambda) &= -2 \sum_{i=1}^{n} \frac{(q_i^T a)^2}{(\lambda_i + \lambda)^3}, \\
\phi''(\lambda) &= 6 \sum_{i=1}^{n} \frac{(q_i^T a)^2}{(\lambda_i + \lambda)^4}.
\end{aligned} \tag{2.2}$$

The equations (2.2) imply that the function $\phi(\lambda)$ is strictly convex on $\lambda \in (\max\{0, -\lambda_2\}, -\lambda_1)$ and so it has at most two roots in the interval $(\max\{0, -\lambda_2\}, -\lambda_1)$. The following theorem states that only the largest root can correspond to a **LNGM**.

Theorem 2.2. *([15, Theorem 3.1])*

1. *If x^* is a **LNGM**, then* (2.1) *holds with a unique $\lambda^* \in (\max\{0, -\lambda_2\}, -\lambda_1)$ and with $\phi'(\lambda^*) \geq 0$.*
2. *There exists at most one **LNGM**.*

□

2.2 Characterization using a Generalized Eigenvalue Problem

We now consider the problem of efficiently computing the **LNGM**. Due to the results in Section 2.1 we can make the following two assumptions.

Assumption 2.1. 1. *The smallest two eigenvalues of A satisfy*

$$\lambda_1 < \min\{0, \lambda_2\}.$$

2. *The hard case does* not *hold, i.e., a is* not *orthogonal to the eigenspace corresponding to λ_1 which here is* $\text{span}(q_1)$ *the span of the eigenvector of λ_1, $a^T q_1 \neq 0$.*

To the best of our knowledge, the only algorithm for computing the **LNGM** is the one by Martinez [15] which tries to find the largest root of the equation $\phi(\lambda) = \delta$ for $\lambda \in (\max\{0, -\lambda_2\}, -\lambda_1)$ via an iterative algorithm. Each step of his approach requires solving an indefinite system of linear equations which can be expensive for large scale instances. In what follows, we follow on the ideas of [1] and present a new algorithm that shows that the **LNGM** can be computed efficiently by a generalized eigenvalue problem. Our result is then used to solve large instances of **eTRS**.

Recently, Adachi et al. [1] designed an efficient algorithm for **TRS** which solves just one generalized eigenvalue problem. They consider the following $2n \times 2n$ regular symmetric matrix pencil which has $2n$ finite eigenvalues.[3]

$$M(\lambda) = \begin{bmatrix} -I & A+\lambda I \\ A+\lambda I & -\frac{1}{\delta}aa^T \end{bmatrix}.$$

We can rephrase Theorem 1.1 as x_g^* is a global optimal solution of **TRS** if, and only if, the following system is consistent.

$$(A+\lambda_g^* I)x_g^* = -a, \tag{2.3a}$$
$$A+\lambda_g^* I \succeq 0, \quad \lambda_g^* \geq 0 \text{ and unique}, \tag{2.3b}$$
$$||x_g^*||^2 \leq \delta, \tag{2.3c}$$
$$\lambda_g^*(||x_g^*||^2 - \delta) = 0. \tag{2.3d}$$

Lemma 2.2 (*Generalized Eigenvalue of Pencil*, [1])**.** *For every Lagrange multiplier $\lambda_g^* \neq 0$, satisfying the stationarity condition* (2.3a) *with equality in the quadratic constraint* (2.3c)*, we have* $\det M(\lambda_g^*) = 0$*, i.e., λ_g^* is a generalized eigenvalue of the pencil $M(\lambda)$.*

Proof. The Lemma is proved in [1]. We include a shorter proof.

For simplicity we denote $D = A + \lambda I$ and let $\lambda = \lambda_g^*$ be a Lagrange multiplier satisfying (2.3a). We can rewrite (with $x = x_g^*$)

$$\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} -I & I \\ I & -\frac{1}{\delta}xx^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} -I & D \\ I & -\frac{1}{\delta}xx^T D \end{bmatrix} = M(\lambda). \tag{2.4}$$

The result follows by observing that the vector $0 \neq \begin{pmatrix} x \\ x \end{pmatrix} \in \text{Null}\left(\begin{bmatrix} -I & I \\ I & -\frac{1}{\delta}xx^T \end{bmatrix}\right)$. $\square$

Corollary 2.3. *The set of real generalized eigenvalues of $M(\lambda)$ is nonempty. Moreover, if* $\det M(\lambda) = 0, \lambda \in \mathbb{R}$*, then either $-\lambda$ is an eigenvalue of A or*

$$\det\left(\begin{bmatrix} -I & I \\ I & -\frac{1}{\delta}xx^T \end{bmatrix}\right) = 0, \quad x = -(A+\lambda I)^{-1}a.$$

Proof. This follows immediately from Lemma 2.2 and from (2.4) in its proof. $\square$

The following theorem shows that the global optimal solution of **TRS** can be obtained via computing an eigenpair of the pencil $M(\lambda)$.

Theorem 2.3 (*Eigenvalue Characterization of* ***TRS***, [1])**.** *Let (x_g^*, λ_g^*) be a global optimal solution of* ***TRS*** *with $||x_g^*||^2 = \delta$. Then the multiplier λ_g^* is equal to the largest real eigenvalue of $M(\lambda)$. Furthermore, if $\lambda_g^* > -\lambda_1$, then x_g^* can be obtained by $x_g^* = -\frac{\delta}{a^T y_2} y_1$, where $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}^n$ is an eigenvector for $M(\lambda_g^*)$ and also we have $a^T y_2 \neq 0$.* $\square$

[3] The objective function in [1] is 1/2 our objective function and I in the pencil is a general $B \succ 0$.

Theorem 2.3 establishes that the largest real eigenvalue of $M(\lambda)$ is the Lagrange multiplier associated with the global optimal solution of **TRS**. In the following theorem, we prove that if **TRS** has a **LNGM**, then the corresponding Lagrange multiplier is the second largest real eigenvalue of $M(\lambda)$. This is the main result of this section.

Theorem 2.4 (Eigenvalue Characterization of **LNGM**)**.** *Let x^* be a **LNGM**. Then the corresponding Lagrange multiplier λ^* is equal to the second largest real eigenvalue of $M(\lambda)$. Moreover, x^* can be computed as $x^* = -\frac{\delta}{a^T y_2} y_1$, where $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is an eigenvector for $M(\lambda^*)$ and we also have $a^T y_2 \neq 0$.*

Proof. From Theorem 2.1 we have $\lambda^* \in (\max\{0, -\lambda_2\}, -\lambda_1)$. Moreover, $||x^*||^2 = \delta$ and it follows from Lemma 2.2 that λ^* is an eigenvalue of $M(\lambda)$, i.e. $\det M(\lambda^*) = 0$. We know that the hard case does not hold, see Corollary 2.1. Therefore, by Theorem 2.3 and the optimality conditions in (2.3), we get that the largest real eigenvalue of $M(\lambda)$ is the unique multiplier associated with the global optimal solution of **TRS** and is the unique root of equation $\phi(\lambda) - \delta = 0$ in $(-\lambda_1, \infty)$. Moreover, it follows from Theorem 2.2 that λ^*, the multiplier corresponding to the **LNGM**, is positive and is the largest root of the equation $\phi(\lambda) - \delta = 0$ in $(-\lambda_2, -\lambda_1)$. Next, note that Lemma 2.2 implies that $-\lambda_1$ is not an eigenvalue of $M(\lambda)$. From the interval considerations for the optimum of **TRS** and **eTRS**, this establishes that λ^* is the second largest real eigenvalue of $M(\lambda)$.

Now let $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be an eigenvector for λ^* for $M(\lambda)$. We have

$$(A + \lambda^* I) y_2 = y_1, \tag{2.5}$$

$$(A + \lambda^* I) y_1 = \frac{1}{\delta} a a^T y_2. \tag{2.6}$$

We first show that $a^T y_2 \neq 0$. Suppose by contradiction that $a^T y_2 = 0$. Then, since $(A + \lambda^* I)$ is nonsingular, we obtain first that $y_1 = 0$ from the second equation which then implies $y_2 = 0$ from the first equation, i.e., we have $y_1 = y_2 = 0$, a contraction of the fact that $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ is an eigenvector. Hence, $a^T y_2 \neq 0$. Thus, (2.6) implies that $x^* = \frac{-\delta}{a^T y_2} y_1$ satisfies

$$(A + \lambda^* I) x^* = -a. \tag{2.7}$$

Moreover, we have

$$||x^*||^2 = \frac{\delta^2}{(a^T y_2)^2} y_1^T y_1 = \frac{\delta^2}{(a^T y_2)^2} y_2^T (A + \lambda^* I)(A + \lambda^* I)^{-1} \frac{a a^T}{\delta} y_2 = \delta.$$

□

3 Strong Duality and Stability for eTRS

3.1 Characterization of Strong Duality for eTRS

A necessary and sufficient condition for strong duality of the problem of minimizing a quadratic function over two quadratic inequality constraints, when one of them is strictly convex, is presented in [2]. Since **eTRS** is a special case, we have the following.

Theorem 3.1 (Characterization Strong Duality **eTRS**). *Strong duality fails for **eTRS** if, and only if, there exist multipliers λ, μ such that the following hold:*

1. $\lambda > 0$ *and* $\mu > 0$*;*
2. $A + \lambda I \succeq 0,$ *and* $\operatorname{rank}(A + \lambda I) = n - 1$*;*
3. *The following system of linear equations is consistent.*

$$2(A+\lambda I)x_i = -2a - \mu b,\ x_i^T x_i = \delta,\ i = 1,2, \qquad (b^T x_1 - \beta)(b^T x_2 - \beta) < 0. \tag{3.1}$$

Proof. This follows immediately from the characterization in [2, Thm 5.2] for two quadratic constraints, since the affine constraint is a special case of a quadratic constraint. □

It is interesting to translate this theorem under our special assumptions and the language of the *hard case.* In fact, we see that loss of strong duality is directly connected to the hard case in **TRS**. Note that the hard case is identified by obtaining a feasible solution that satisfies all the optimality conditions except for complementary slackness.

Corollary 3.1. *Consider the Lagrangian dual of **eTRS** in parametric form.*

$$d^*_{e\boldsymbol{TRS}} := \max_{\mu \geq 0} g(\mu),$$

*where the dual function, $g(\mu)$, with λ implicit in g, is a parametric **TRS**, $\boldsymbol{TRS}_\mu$,*

$$g(\mu) := \max_{\lambda \geq 0} \min_x \left[L(x,\lambda) + \mu b^T x\right] - \mu\beta \tag{$\mathbf{TRS}_\mu$}$$

*Then strong duality fails for **eTRS** if, and only if, there exists $\mu > 0$ such that the parametrized $\boldsymbol{TRS}_\mu$ has a <u>hard case</u> solution x^*_μ that satisfies all the optimality conditions except for complementary slackness, i.e.,*

$$\|x^*_\mu\|^2 < \delta, \quad b^T x^*_\mu = \beta.$$

Proof. Since **eTRS** is a convex problem if $\lambda_1 \geq 0$, without loss of generality we assume that $\lambda_1 < 0$. We conclude that the optimal Lagrange multiplier for $\mathbf{TRS}_\mu$ satisfies $\lambda > 0$ and moreover there exists an optimal solution on the boundary of the trust region.

The three conditions in Theorem 3.1 are equivalent to the optimality conditions for the parametrized problem at μ. And the two points $x_i, i = 1,2$ are on opposite sides of the affine manifold for the linear constraint. We note that necessarily $0 \neq v := x_1 - x_2 \in \text{Null}(A + \lambda I)$. Therefore v is the required eigenvector and this is equivalent to finding the convex combination $x^* = \alpha x_1 + (1-\alpha)x_2, \alpha \in (0,1)$ with $b^T x^* = \beta$ and necessarily $\|x^*\| < \delta$.

Therefore, the parametrized **TRS** has multiple optimal solutions and the hard case holds for the corresponding $\mathbf{TRS}_\mu$, i.e., $2a + \mu b \in \text{Range}(A - \lambda_1 I)$, $\lambda^* = -\lambda_1$.

More details on $\|x^*_\mu\|^2 < \delta$ and the relation with the minimum norm solution $\hat{x} := \frac{1}{2}(A - \lambda_1 I)^\dagger(-2a - \mu b)$ are discussed in Section 4.2.1, where we define the *Moore-Penrose generalized inverse,* $C^\dagger$. In fact, necessarily $\|x^*_\mu\|^2 = \frac{1}{2}(A - \lambda_1 I)^\dagger(-2a - \mu b) + v$ for $v \in \text{Null}(A - \lambda_1 I)$. □

Remark 3.1. *Corollary 3.1 illustrates the geometry of strong duality in terms of the parametrized* $\boldsymbol{TRS}_\mu$. *If we start with* $\mu = 0$ *and increase* $\mu \uparrow$, *then the corresponding optimal solution of* $\boldsymbol{TRS}_\mu$ *moves on the boundary of the trust region. If we encounter the boundary of the linear constraint first then strong duality holds. On the other hand if we encounter the hard case at* $\mu > 0$ *and if we can move using the nullspace* $\bar{x} = x_\mu + v$ *so that* $\|\bar{x}\|^2 < \delta, b^T \bar{x} = \beta$, *then strong duality fails.*

This means that given a ***TRS*** *we can characterize all the* b, β *where strong duality would fail using the characterization of the hard case.*

We know that strong duality fails if the **LNGM** is the optimum for **eTRS**. We now see that it requires a special eigenvalue configuration for strong duality to fail if the linear constraint is active.

Theorem 3.2. *Suppose that* x^* *solves* ***eTRS*** *with* $b^T x^* = \beta$. *Suppose that* $\lambda_2 < 0$. *Then strong duality holds for* ***eTRS***.

Proof. As above, we can construct a full column rank matrix W to represent $\text{Null}(b^T)$. From interlacing of eigenvalues we get that $\lambda_{min}(W^T A W) < 0$. Therefore, there exists an optimal solution on the boundary of the trust region for the projected problem, i.e., complementary slackness holds. This means that the optimum for **eTRS** is also on the boundary of the trust region constraint. We can therefore add a multiple of the identity to the Hessian of the original problem and obtain a convex equivalent problem. This shows that strong duality holds. The dual problem is equivalent to perturbing the Hessian to $Q - \lambda_1 I$ as long as we subtract the constant $\lambda_1 \beta$. □

3.2 Stability for eTRS

We now see that the **eTRS** is stable with respect to perturbations in the data.

Lemma 3.1. *Recall that we have made Assumptions 1.1 and 2.1. Let* x^* *be the optimal solution for* ***eTRS***. *Then the linear independence constraint qualification,* ***LICQ***, *holds at* x^*. *Moreover,* x^* *is the unique optimal solution if the second constraint is inactive. Thus unique Lagrange multipliers* λ_1^*, λ_2^* *exist for the two constraints, respectively.*[4]

[4]We note that the optimum does not have to be unique for the projected problem, i.e., though the hard case does not hold for **TRS**, it can hold for the projected problem.

Proof. Suppose that the second constraint is inactive $b^T x^* < \beta$. First we note that the first constraint is active by the $\lambda_1(A) < 0$ assumption and the gradients of the active constraint is $2x^* \neq 0$. Therefore the **LICQ** holds. This immediately implies that $\lambda_1^* > 0$ exists. Moreover, both x^* and the optimal Lagrange multiplier λ^* are unique by the $\lambda_1(A) < \lambda_2(A)$ assumption.

If the second constraint is active $b^T x^* = \beta$, then x^* is the optimal solution of the projected problem. If the first constraint is inactive, we are done as $\{b\}$ is a linearly independent set. And it is clear from the geometry that if both constraints are active then the gradients $\{b, 2x^*\}$ are linearly dependent only if strict feasibility fails, a contradiction. Therefore, **LICQ** holds and the multipliers are unique. $\square$

Corollary 3.2. *The* ***eTRS*** *is a stable problem with respect to perturbations in the data.*

Proof. This follows from standard results in sensitivity analysis since the Lagrange multipliers are unique, satisfy LICQ and the feasible set is compact, e.g., [6]. $\square$

Remark 3.2. *We note that these results on stability along with standard sensitivity results on eigenvalue algorithms imply that our approach is a robust method for solving* ***eTRS****.*

In addition, strict complementarity can fail for ***eTRS****. If the* ***LNGM*** *is the optimal solution for* ***eTRS****, then one can perturb the linear constraint till it becomes active. It is therefore a redundant constraint illustrating that the corresponding Lagrange multiplier can be zero. This would then be a degenerate problem and perturbing the linear constraint further can make the projected trust region optimal point the optimum for* ***eTRS****, i.e., the result is a jump in the optimal solution.*

4 Algorithm and Subproblems for eTRS

We now describe our proposed method to solve **eTRS** in Algorithm 4.1. This finds the global optimal solution for the general problem of **eTRS**. We include the details about the global minimizer for **TRS** and the details for the subproblems that need to be solved. We do *not* assume that the global minimizer of **TRS** is infeasible in the details of our algorithm, i.e., our algorithm solves the general case.

Lemma 4.1. *Strong duality fails for the* ***LNGM****.*

Proof. The Lagrangian of **TRS** is given in (1.1). The Lagrangian dual of **TRS** is $\max_{\lambda \geq 0} \min_x L(x, \lambda)$. Since the inner problem is a minimization of a quadratic, for it to be finite we get the necessary (hidden) condition that the Hessian of the quadratic $A + \lambda I \succeq 0$. This contradicts the Lagrange multiplier condition for **LNGM** given in Theorem 2.2. $\square$

Theorem 4.1. *Suppose that strong duality holds for* ***eTRS*** *and that the optimal solution of* ***eTRS*** *is x^*. Then $b^T x^* = \beta$ and x^* is a global optimal solution of* ***TRS*** *after projection onto the linear manifold of the linear constraint.*

Proof. If $b^T x^* < \beta$, then either x^* is the global minimizer or a **LNGM**. Since strong duality fails for the **LNGM**, we conclude that it must be the global minimizer of the **TRS**. But our Assumption 1.1 means that the global minimizer is infeasible for **eTRS**.

If the linear inequality is active, then we have a **TRS** problem after a projection onto the linear manifold and we obtain the global minimizer on this affine manifold. $\square$

4.1 Main Algorithm

Theorem 4.1 suggests the following Algorithm 4.1 for **eTRS**. Without loss of generality, by Lemma 1.1, we can assume that strict feasibility holds.

In addition, we see that the cost of the algorithm in the worst case is to find λ_1, λ_2 and eigenvector v_1 for λ_1; check for strong duality; find the **TRS** and projected **TRS** optima or the **LNGM** and the projected **TRS** optima.

Recall that, if the **LNGM** exists then we can use Theorem 2.4 and find it efficiently via the second largest real eigenvalue of the matrix pencil. The other subproblems are now discussed.

4.2 Subproblems

4.2.1 Verifying Strong Duality

To specify the value of μ in Theorem 3.1, first notice that, for given μ, system (3.1) is consistent if, and only if, $v^T(2a + \mu b) = 0$ where v is a normalized eigenvector for λ_1. Next, let us consider the following cases:

1. $v^T b = 0$**:** In this case, we show that strong duality holds for **eTRS**. We show this by contradiction. Suppose that strong duality does not hold for **eTRS**. Then system (3.1) has two solutions x_1 and x_2 satisfying $x_i^T x_i = \delta$, $i = 1, 2$, and $(b^T x_1 - \beta)(b^T x_2 - \beta) < 0$ for some $\mu > 0$. Moreover, we know that the solutions x_1 and x_2 necessarily have the form $x_1 = \frac{1}{2}(A - \lambda_1 I)^\dagger(-2a - \mu b) + z_1$ and $x_2 = \frac{1}{2}(A - \lambda_1 I)^\dagger(-2a - \mu b) + z_2$ where z_i, for $i = 1, 2$, is an eigenvector corresponding to λ_1. By the fact that b is orthogonal to the eigenspace of λ_1 (λ_1 has multiplicity one), we have $b^T x_1 - \beta = b^T x_2 - \beta$, a contradiction to the fact that $(b^T x_1 - \beta)(b^T x_2 - \beta) < 0$, i.e., we have strong duality for **eTRS**.

2. $v^T b \neq 0$**:** In this case, consistency of system (3.1), i.e., $v^T(2a + \mu b) = 0$ implies that necessarily $\mu = \frac{-2v^T a}{v^T b}$. If $\mu = 0$, it follows from Theorem 3.1 that **eTRS** enjoys strong duality. If $\mu > 0$, then strong duality does not hold for **eTRS** if, and only if, system (3.1) for $\mu = \frac{-2v^T a}{v^T b}$ has two solutions x_1 and x_2 satisfying $x_i^T x_i = \delta$, $i = 1, 2$, and $(b^T x_1 - \beta)(b^T x_2 - \beta) < 0$.

To verify whether strong duality holds we suppose that x_i, $i = 1, 2$ are as defined in Theorem 3.1. clearly, $x_i = x_p + \alpha_i v$ where v is a normalized eigenvector associated with λ_1, $x_p = \frac{1}{2}(A - \lambda_1 I)^\dagger(-2a - \mu b)$ and α_i, $i = 1, 2$, are roots of the following quadratic equation.

$$\alpha^2 + 2\alpha v^T x_p + x_p^T x_p - \delta = 0.$$

Algorithm 4.1.

INPUT: $A \in \mathbb{S}^n, a, b \in \mathbb{R}^n, \delta \in \mathbb{R}_{++}, \beta \in \mathbb{R}$ *with* $-\sqrt{\delta}\|b\| < \beta$.
INITIALIZATION: *Solve the symmetric eigenvalue problem for* λ_1, λ_2 *and eigenvector* v_1 *for* λ_1.

IF: $\lambda_1 \geq 0$ *or* $\lambda_1 = \lambda_2$, **THEN** *Strong duality holds; solve* ***TRS*** *for* x.

IF: x *is feasible,* **THEN** *it is opt. STOP.*

ELSE: *Solve the projected* ***TRS*** *problem for* x*; it is opt. STOP.*

END:

ELSE: *Check the strong duality condition for* ***eTRS****.*

IF: *strong duality holds,* **THEN** *solve* ***TRS*** *for* x.

IF: x *is feasible,* **THEN** *it is opt. STOP.*

ELSE: *Solve the projected* ***TRS*** *problem for* x*; it is opt. STOP.*

END:

ELSE: *Solve for the projected* ***TRS*** *and the* ***LNGM*** *if it exists; discard* ***LNGM*** *if it is not feasible; choose the* x *as the best of the remaining solutions; it is opt. STOP.*

END:

END:

OUTPUT: x *is optimizer of* ***eTRS****.*

Table 4.1: Algorithm: Solve the General (strictly feasible) **eTRS**

The main task in finding x_i, $i = 1, 2$, is computing x_p. In the sequel, we show that x_p is indeed the solution of a symmetric positive definite linear system. To see this, let us consider the eigenvalue decomposition of A defined as before in which Q contains v as its first column. Noting that $v^T(2a + \mu b) = 0$, we have

$$\begin{aligned}
(A + \gamma vv^T - \lambda_1 I)^{-1}(-2a - \mu b) &= Q(\Lambda + \gamma e_1 e_1^T - \lambda_1 I)^{-1} Q^T(-2a - \mu b) \\
&= Q(\Lambda - \lambda_1 I)^{\dagger} Q^T(-2a - \mu b) \\
&= (A - \lambda_1 I)^{\dagger}(-2a - \mu b),
\end{aligned}$$

where γ is a positive constant and e_1 is the first unit vector. This implies that x_p can be computed efficiently by applying the conjugate gradient algorithm to the following positive definite system.

$$2(A + \gamma vv^T - \lambda_1 I)x_p = (-2a - \mu b).$$

However, we note that the perturbation with γvv^T is *not* required since the right-hand side $(-2a - \mu b) \in \mathrm{Range}(A - \lambda_1 I)$. The MATLAB *pcg* works fine even though the matrix is singular.

4.2.2 Solving the TRS Subproblem

The main work of the algorithms lie in solving generalized eigenvalue problems. For the **TRS**, we use the method of [1] that solves the scaled **TRS**

$$\begin{array}{ll} \min & \frac{1}{2}x^T Ax + a^T x \\ & x^T Bx \leq \delta, \end{array} \tag{4.1}$$

where B is a positive definite matrix. The algorithm computes one generalized eigenpair and is able to handle the hard case efficiently. Specifically, it is shown that the optimal Lagrange multiplier corresponding to the solution of (4.1) is the largest real eigenvalue of the $2n \times 2n$ matrix pencil $M_0 + \lambda M_1$, where

$$\tilde{M}(\lambda) = M_0 + \lambda M_1, \qquad M_0 = \begin{bmatrix} -B & A \\ A & -\frac{aa^T}{\delta} \end{bmatrix}, \; M_1 = \begin{bmatrix} O_{n\times n} & B \\ B & O_{n\times n} \end{bmatrix}.$$

As above we have an equivalent result to Lemma 2.2 that every nonzero KKT multiplier is a generalized eigenvalue of the pencil, $\det(\tilde{M}(\lambda)) = 0$.

Lemma 4.2 (*Generalized Eigenvalue of Pencil*, [1, Lemma 3.1])**.** *For every nonzero KKT multiplier $\lambda_g^* \neq 0$ for* (4.1) *with equality in the quadratic constraint we have* $\det \tilde{M}(\lambda_g^*) = 0$, *i.e., λ_g^* is a generalized eigenvalue of the pencil $\tilde{M}(\lambda)$.* □

4.2.3 Solving the Projected TRS Subproblem

We can eliminate the equality constraint $b^T x = \beta$ to solve the projected **TRS**. For ease of exposition, we assume that

$$|b_1| \geq |b_2| \geq \ldots \geq |b_r| > 0 = b_{r+1} = \ldots = b_n.$$

In order to find a basis of $\mathrm{Null}(b^T)$, we define $\bar{b} := \begin{pmatrix} b_2^{-1} & \ldots & b_r^{-1} \end{pmatrix}^T$ and the matrix

$$W := \left[\begin{array}{c|c} -b_1^{-1} e_{r-1}^T & 0_{n-r} \\ \hline \mathrm{Diag}(\bar{b}) & 0 \\ 0 & I_{n-r} \end{array} \right] \in \mathbb{R}^{n\times(n-1)}.$$

Algorithm 4.2. *1. Solve $Ax_0 = -a$ by the conjugate gradient algorithm and keep x_0 if it is feasible, i.e., if $x_0^T B x_0 \leq \delta$.*

2. Compute λ_g^*, the largest generalized eigenvalue of the symmetric regular pencil $M_0 + \lambda M_1$, and a corresponding eigenvector $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, i.e.,

$$\begin{bmatrix} -B & A \\ A & -\frac{aa^T}{\delta} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = -\lambda_g^* \begin{bmatrix} O_{n\times n} & B \\ B & O_{n\times n} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{4.2}$$

3. If $||y_1|| \leq \tau$ (default is $\tau = 10^{-4}$), then the hard case is detected; run Steps 4 to 6. Else go to Step 7.

4. Compute $H := (A + \lambda_g^* B + \alpha \sum_{i=1}^d B v_i v_i^T B)$ where $V = [v_1, ..., v_d]$ is a basis of $\mathrm{Null}(A + \lambda_g^* B)$ that is B-orthogonal, i.e., $V^T B V = I$, $d = \dim(\mathrm{Null}(A + \lambda_g^* B))$ and α is an arbitrary positive scalar.

5. Solve $Hq = -a$ by the conjugate gradient algorithm.

6. Take an eigenvector v computed above, and find η such that $(q + \eta v)^T B (q + \eta v) = \delta$ and return $x^* = q + \eta v$ as global optimal solution of (4.1).

7. Set $x_1 = -sign(a^T y_2)\sqrt{\delta}\frac{y_1}{\sqrt{y_1^T B y_1}}$.

8. The global optimal solution of (4.1) is either x_1 or x_0, whichever gives the smaller objective value.

Table 4.2: Algorithm: Solve scaled **TRS** (4.1), [1, Theorem 3.1]

Define a *particular solution,* $\hat{x}$ satisfying $b^T \hat{x} = \beta, \|\hat{x}\|^2 < \delta$.[5] We choose

$$\hat{x} = \begin{cases} 0, & \text{if } \beta = 0 \\ \frac{\beta}{\|b\|^2} b, & \text{if } \beta \neq 0. \end{cases} {}^{6} \tag{4.3}$$

Then it is clear that

$$b^T x = \beta \iff x = \hat{x} + Wy, \text{ for some } y \in \mathbb{R}^{n-1}.$$

We can now substitute for x into **eTRS** and eliminate the linear equality constraint. The objective function becomes

$$(\hat{x}+Wy)^T A(\hat{x}+Wy) + 2a^T(\hat{x}+Wy) = \left[y^T (W^T A W) y + 2\left(W^T(a + A\hat{x})\right)^T y\right] + \left[(A\hat{x} + 2a)^T \hat{x}\right].$$

[5]Some scaling issues can arise here. It is preferable to take $\hat{x}$ strictly feasible for the trust region constraint.
[6]We note that the choice $\hat{x} = 0$ simplifies the nonhomogeneous **nTRS** below.

The constraint becomes

$$y^T(W^TW)y + 2(W^T\hat{x})^Ty \leq \delta - \hat{x}^T\hat{x}.$$

We get the following equivalent problem in the case that the linear constraint is active.

$$\begin{array}{ll} \min & y^T(W^TAW)y + 2(W^T(a+A\hat{x}))^Ty \\ \text{s.t.} & y^T(W^TW)y + 2(W^T\hat{x})^Ty \leq \delta - \hat{x}^T\hat{x} \end{array} \qquad (\mathbf{TRS}_{proj})$$

We let

$$B := W^TW,\ \hat{A} := W^TAW,\ \hat{a} := W^T(a+A\hat{x}),\ \hat{b} := 2(W^T\hat{x}).,\ \ \hat{\delta} = \delta - \hat{x}^T\hat{x}.$$

Therefore, we need to solve the *nonhomogeneous* ***TRS***, ***nTRS***

$$\begin{array}{ll} \min & x^T\hat{A}x + 2\hat{a}^Tx \\ \text{s.t.} & x^TBx + \hat{b}^Tx \leq \hat{\delta}. \end{array} \text{[7]} \qquad (\mathbf{nTRS})$$

By the change of variables

$$x \leftarrow y + g, \quad \text{with} \quad 2Bg = -\hat{b},$$

we get

$$\begin{array}{rcl} x^T\hat{A}x + 2\hat{a}^Tx & = & (y+g)^T\hat{A}(y+g) + 2\hat{a}^T(y+g) \\ & = & y^T\hat{A}y + 2(\hat{A}g+\hat{a})^Ty + \text{ constant.} \end{array}$$

and

$$\begin{array}{rcl} x^TBx + \hat{b}^Tx & = & (y+g)^TB(y+g) + \hat{b}^T(y+g) \\ & = & y^TBy + (2Bg+\hat{b})^Ty + g^TBg + b^Tg \\ & = & y^TBy + g^TBg + b^Tg \end{array}$$

We write **nTRS** as the *scaled homogeneous* ***TRS***, ***sTRS***,

$$\begin{array}{ll} \min & y^T\hat{A}y + 2(\hat{A}g+\hat{a})^Ty \\ \text{s.t.} & y^TBy \leq \hat{\delta} - g^TBg - \hat{b}^Tg. \end{array} \qquad (\mathbf{sTRS})$$

This means we can directly apply the approach in [1] where the scaled **TRS** is solved using the generalized eigenvalue approach.

Remark 4.1. *When we solve for the optimimum in* ***sTRS*** *using* (4.2) *we do not form B explicitly but exploit the rank one update structure of W and its inverse. This means we can exploit the original sparsity in A in the objective function and in the, now scaled, I in the original trust region constraint when performing the matrix-vector multiplications needed for* eigs *in MATLAB. Let*

$$\bar{B} := \text{Diag}(\bar{b}), \qquad \bar{e} := \begin{pmatrix} e_{r-1}^T \\ \hline 0_{n-r} \end{pmatrix}.$$

[7] We note again here that if $\beta = 0$ then we can choose $\hat{x} = 0$ and the homogeneous **TRS** is maintained.

Note that

$$
\begin{aligned}
B &= \left[\begin{array}{c|c} \bar{B}^2 & 0 \\ \hline 0 & I_{n-r} \end{array}\right] + b_1^{-2}\bar{e}\bar{e}^T \\
&= \left\{\left[\begin{array}{c|c} \bar{B} & 0 \\ \hline 0 & I_{n-r} \end{array}\right] + ww^T\right\}\left\{\left[\begin{array}{c|c} \bar{B} & 0 \\ \hline 0 & I_{n-r} \end{array}\right] + ww^T\right\} \\
&= B^{1/2}B^{1/2}.
\end{aligned}
$$

We can then find the appropriate rank one update of $\left[\begin{array}{c|c} \bar{B} & 0 \\ \hline 0 & I_{n-r} \end{array}\right]$ *to find the inverse* $B^{-1/2}$. *Therefore we can take a diagonal congruence of both sides of* (4.2) *and obtain a simple right-hand side of the generalized eigenvalue problem.*

5 Numerical Results

We now present our numerical results to illustrate the efficiency of the new algorithm. We compare with the second order cone and semidefinite programming, **SOCP/SDP**, reformulation in [4] on some small instances as this reformulation is not able to handle large instances. Hence, for large instances we just report the solution obtained by our new algorithm.

All computations were done in MATLAB 8.6.0.267246 (R2015b) on a Dell Optiplex 9020 with 16GB RAM with Windows 7. To solve the **SOCP/SDP** reformulation, we used SeDuMi 1.3, [20].

5.1 Four Classes of Test Problems

We divide our tests into *four classes* I,II,III,IV, of test problems.

5.1.1 Class I

In this section, we apply our algorithm and the **SOCP/SDP** reformulation to some **eTRS** instances for which the **LNGM** of the corresponding **TRS** is a good candidate for the global optimal solution of **eTRS**. To generate the desirable random instances of **eTRS**, we proceed as follows. First we construct a **TRS** problem having a local non-global minimizer based on Theorem 2.4. Then we add the inequality constraint $b^Tx \leq \beta$ to enforce that the global minimizer of **TRS** is infeasible but that the **LNGM** remains feasible.

Comparison with the **SOCP/SDP** reformulation is given on some small instances in Table 5.1. We follow [1] and report the relative objective function difference

$$
\frac{|q(x^*) - q(x_{best})|}{|q(x_{best})|} \qquad \text{accuracy measure,}
$$

where x^* is the computed solution by each method and x_{best} is the solution with the smallest objective value among the two algorithms. For each dimension, we have generated 10 **eTRS** instances. We report the dimension n, and the average values of the relative accuracy, the run time in cpu-seconds and we include the time taken for checking the strong duality property of **eTRS** in Algorithm 4.1. Moreover, for each dimension, # **LNGM** denotes the number of test problems among the 10 instances for which our algorithm has detected the **LNGM** of the corresponding **TRS** as a global optimal solution of **eTRS**. It should be noted that the algorithm which gets x_{best} varies from problem to problem and since we are reporting the average of 10 runs, we can have a positive accuracy in both columns of the table.

	Accuracy	Accuracy	CPUsec	CPUsec	CPUsec	# LNGM
	Main Algor.	SOCP/SDP	Main Algor	Str. Dual.	SOCP/SDP	Main Algor.
100	0.0	1.1309e-10	0.043	0.019	1.372e+00	10
200	0.0	2.9945e-10	0.037	0.012	8.440e+00	10
300	0.0	2.7884e-10	0.040	0.012	3.193e+01	10
400	0.0	3.1309e-10	0.049	0.018	9.017e+01	10

;

Table 5.1: Class I: Comparison with **SOCP/SDP** reformulation.

We see in Table 5.1 that our algorithm finds the global optimal solution of **eTRS** significantly faster than the **SOCP/SDP** reformulation and with improved accuracy. The generated matrix A in this the first class of test problems is dense and so we do not perform tests of large size as the aim of our method is solving large sparse **eTRS** instances.

5.1.2 Class II

In this section we test our algorithm on both small and large sparse **eTRS** instances. we take advantage of the following lemma from [15] to generate such **eTRS** instances.

Lemma 5.1 (Lemma 3.4 of [15]). *Consider the **TRS** problem. Suppose that $\lambda_1 < 0$, has multiplicity one, and the **TRS** is an easy case instance. Then there exists $\delta_0 > 0$ such that **TRS** admits a local non-global minimizer for all $\delta > \delta_0$.* □

The second class of test problems are generated as follows. We generate a random sparse symmetric matrix A via `A=sprandsym(n,density)`. Next we generate the vector a via `a=randn(n,1)` and make sure that $v^T a \neq 0$ where v is the eigenvector corresponding to λ_1, i.e., we get the easy case **TRS**. Then we set $\delta = 4000$ following Lemma 5.1. Finally we set $b = 0.9$xopt and $c = ||b||^2$ to cut off xopt, the global optimal solution of the generated **TRS** instance. We have compared our algorithm with the **SOCP/SDP** reformulation on the test problems of small size in both runtime and solution accuracy. For each dimension, we have generated 10 **eTRS** instances and the corresponding numerical results are presented in Table 5.2, where we report the dimension of the problem n, the algorithm run time and the time taken for checking the strong duality property of **eTRS** , and the accuracy at termination averaged over the 10 random instances. Moreover, for each dimension, # **LNGM** denotes the number of test problems among 10 instances for which our algorithm has detected the **LNGM** of the corresponding **TRS** as a global optimal solution of **eTRS**. It should be noted that the algorithm which gets x_{best} varies from problem to problem and since we are reporting the average of 10 runs, we have positive accuracy in the Table. Furthermore, we verified that in all cases, there was a positive duality gap for generated **eTRS** instances. As in the previous test problems the new algorithm finds higher accuracy solutions in significantly shorter time than the **SOCP/SDP** reformulation.

	Accuracy	Accuracy	CPUsec	CPUsec	CPUsec	# LNGM
	Main Algor.	SOCP/SDP	Main Algor	Str. Dual.	SOCP/SDP	Main Algor.
100	0.0	4.2588e-09	0.093	0.028	1.697e+00	9
200	0.0	1.0547e-08	0.128	0.030	1.167e+01	6
300	0.0	9.3557e-09	0.180	0.036	4.694e+01	7
400	0.0	3.3775e-09	0.252	0.042	1.287e+02	5

;

Table 5.2: Class II: Comparison with **SOCP/SDP** reformulation; density 0.1

Now we turn to solving large sparse **eTRS** instances. For this class we just report the results of our algorithm since the **SOCP/SDP** approach could not handle problems of this size. Let x^* be a global optimal solution of **eTRS** and λ^* the corresponding Lagrange multiplier. Depending on the context of the linear constraint being not active or being active, we denote the error in the stationarity condition by: ***KKT1*** $:= ||(A + \lambda^* I)x^* + a||_\infty$ or the corresponding conditions for the scaled active case, respectively; and the error in complementary slackness by ***KKT2*** $:= \lambda^*(||x^*||^2 - \delta)$ or the corresponding condition for the scaled linear active case, respectively. For each dimension, we have generated 10 **eTRS** instances. In both cases the global optimal solution of **eTRS** is obtained from solving generalized eigenvalue problems. Numerical results are presented in Table 5.3.

	Opt. Cond.	Opt. Cond.	CPUsec	CPUsec	# LNGM
	KKT1 eTRS	KKT2 C.S.	Algor Time	Str. dual. Time	Main Algor.
10000	1.4085e-08	-1.3688e-12	1.087	0.168	4
20000	1.3465e-10	-7.7060e-13	2.506	0.294	6
40000	1.9584e-09	-3.8369e-14	10.343	0.963	2
60000	1.9876e-10	1.8024e-14	13.694	1.912	4
80000	1.8937e-10	5.3614e-13	26.768	3.253	5
100000	8.5902e-11	2.8473e-12	29.225	5.415	2

;

Table 5.3: Class II: Large instances; density 0.0001

The following lemma is useful in generating test problems for the next two classes.

Lemma 5.2 (Generating **LNGM**). *Let $A \in \mathbb{S}^n$ and suppose that $\lambda_1 < \min\{0, \lambda_2\}$. Then there exists linear term a for which the eigenvector associated with λ_1 is the* ***LNGM***.

Proof. Let $\mu \in (\max\{0, -\lambda_2\}, -\lambda_1)$. Set $a = -(A + \mu I_n)v_1$ where v_1 is the eigenvector for λ_1 with $||v_1||^2 = \delta$. Then for this choice we have the first order stationary conditions.

Now let Range$(W) = $ Null(v_1^T). Then $W^T(A + \mu I_n)W = \text{diag}(\lambda_2 + \mu, ..., \lambda_n + \mu)$. Due to the choice of μ, the diagonal matrix has all diagonal elements positive. Thus we have the positive definiteness of the reduced Hessian. This implies that v_1 is the **LNGM**. □

5.1.3 Class III

In this section, we consider a class of large sparse **eTRS** instances for which strong Lagrangian duality holds while the corresponding **TRS** has a **LNGM** which is feasible for **eTRS**. We generate the **TRS** using the previous Lemma 5.2 and set $b = (A - \lambda_1 I)x$ where `x=rand(n,1)`. This means that $b^T v_1 = 0$ implying that we have strong duality property for generated **eTRS** instances.

Now let x^* be a global optimal solution of **eTRS**. Then either $b^T x^* < \beta$ or $b^T x^* = \beta$. Since strong duality holds, in the former case, x^* is the global minimizer of the corresponding **TRS**. We define **KKT1** and **KKT2** as the previous section. For each dimension, we have generated 10 **eTRS** instances and the corresponding numerical results are presented in Table 5.4.

	Opt. Cond.	Opt. Cond.	CPUsec	CPUsec
	KKT1 eTRS	KKT2 C.S.	Algor Time	Str. dual. Time
10000	5.4076e-14	5.4076e-14	0.313	0.118
20000	3.1243e-14	3.1243e-14	0.731	0.242
40000	2.0866e-12	2.0866e-12	2.279	0.721
60000	8.9301e-14	8.9301e-14	3.827	1.448
80000	4.5073e-14	4.5073e-14	5.998	2.333
100000	9.7731e-14	9.7731e-14	9.727	3.820

Table 5.4: Class III: density 0.0001

5.1.4 Class IV

For this class also we follow the above Lemma 5.2 to generate **TRS** having **LNGM**. We follow the same procedure as in Section 5.1.3 to obtain A, a, δ and **LNGM** but we set $b = xopt - x_l$ and $\beta = b^T(0.9x_l + 0.1xopt)$ to cut off $xopt$ but leave x_l feasible where $xopt$ and x_l are the global optimal solution and **LNGM** of the corresponding **TRS**, respectively.

	Accuracy	Accuracy	CPUsec	CPUsec	CPUsec	# LNGM
	Main Algor.	SOCP/SDP	Main Algor	Str. Dual.	SOCP/SDP	Main Algor.
100	1.8488e-10	0.0	0.132	0.025	9.170e-01	10
200	2.3815e-10	0.0	0.145	0.025	7.037e+00	10
300	2.1072e-10	0.0	0.230	0.034	2.926e+01	10
400	2.1792e-10	0.0	0.386	0.041	8.877e+01	10

Table 5.5: Class IV: density 0.1

	Opt. Cond.	Opt. Cond.	CPUsec	CPUsec	# LNGM
	KKT1 eTRS	KKT2 C.S.	Algor Time	Str. dual. Time	Main Algor.
10000	1.3807e-13	2.1481e-18	2.564	0.167	10
20000	3.3108e-14	1.2592e-16	2.712	0.314	10
40000	1.9213e-13	-9.3530e-16	10.682	0.981	10
60000	3.8501e-13	7.6124e-16	19.285	2.060	10
80000	6.2677e-14	3.1855e-16	29.587	3.736	10
100000	1.1080e-13	-7.4408e-16	44.761	6.171	10

Table 5.6: Class IV: density 0.0001

6 Conclusion

In this paper we have derived a new necessary condition for the local non-global optimal solution **LNGM** of the **TRS** that is based on the second largest real generalized eigenvalue of a matrix pencil. This is then used to derive an efficient algorithm for finding the global minimizer of the extended **TRS**, the **eTRS**. We have presented numerical tests to show that our method far outperforms current methods for **eTRS**. And our method solves large sparse problems which are too large for current methods to be applied. We have included discussions on a characterization of when strong duality holds for **eTRS** as well as details on the stability of the problem.

It is well known that **TRS** is important for unconstrained trust region methods, restricted Newton methods, for unconstrained minimization; as well it is important for general minimization algorithms such as sequential quadratic programming (**SQP**) methods. For **SQP** methods it is customary to solve a standard quadratic programming problem for the search direction after using something akin to a quasi-Newton method to guarantee convexity of the objective function. The **eTRS** we have studied can be viewed as a step toward solving a **TRS** with multiple linear constraints for the search direction in **SQP** methods.

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