An Eigenvalue Majorization Inequality for Positive Semidefinite Block Matrices: In Memory of Ky Fan

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Abstract

Let $H = \begin{bmatrix} M & K \\ K^* & N \end{bmatrix}$ be a Hermitian matrix. It is known that the eigenvalues of $M \oplus N$ are majorized by the eigenvalues of H. If, in addition, H is positive semidefinite and the block K is Hermitian, then the following reverse majorization inequality holds for the eigenvalues:

$$\lambda\left(\begin{bmatrix} M & K \\ K & N \end{bmatrix}\right) \prec \lambda((M+N) \oplus 0).$$

Interesting corollaries are included.

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1 Introduction

Matrix eigenvalue majorization results have interesting applications in many disciplines of mathematics, e.g. in linear algebra, probability, statistics, combinatorics, etc. It is still a very active research topic that attracts many mathematicians. Recent results on this topic can be found in e.g., [2, 5, 8, 7, 9].

An early result concerning eigenvalue majorization is the fundamental result due to I. Schur (see e.g [1, 6]), which states that the diagonal entries of a Hermitian matrix A are majorized by its eigenvalues, i.e., $\mathrm{diag}(A) \prec \lambda(A)$. This result can be easily extended to block Hermitian matrices. More precisely, if $\begin{bmatrix} M & K \\ K^* & N \end{bmatrix}$ is Hermitian, then

$$\lambda(M \oplus N) \prec \lambda\left(\begin{bmatrix} M & K \\ K^* & N \end{bmatrix}\right). \tag{1.1}$$

Here and throughout, K^* denotes the Hermitian conjugate transpose of K; and $M \oplus N$ denotes the direct sum of M and N, i.e., the block diagonal matrix $\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$.

In this paper, we present the following reverse majorization inequality for a Hermitian positive semidefinite 2×2 block matrix. The proof and some interesting consequences are given in the next Section.

Theorem 1.1. *Let* $H = \begin{bmatrix} M & K \\ K^* & N \end{bmatrix}$ *be a Hermitian positive semidefinite matrix. If, in addition, the block* K *is Hermitian, then the following majorization inequality holds:*

$$\lambda\left(\begin{bmatrix} M & K \\ K & N \end{bmatrix}\right) \prec \lambda((M+N) \oplus 0). \tag{1.2}$$

Here, and throughout the paper, 0 is a zero block matrix of compatible size.

1.1 Preliminary Results

Let $\mathbb{M}^{m\times n}(\mathbb{C})$ be the space of all complex matrices of size $m \times n$ with $\mathbb{M}^{n}(\mathbb{C}) = \mathbb{M}^{n\times n}(\mathbb{C})$. For $A \in \mathbb{M}^{n}(\mathbb{C})$, the vector of eigenvalues of A are denoted by $\lambda(A) = (\lambda_1(A), \lambda_2(A), \cdots, \lambda_n(A))$. If A is Hermitian, we will always arrange the eigenvalues of A in nonincreasing order: $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$.

For two sequences of real numbers arranged in nonincreasing order,

$$x = (x_1, x_2, \cdots, x_n), \qquad y = (y_1, y_2, \cdots, y_n),$$

we say that x is majorized by y, denoted by $x \prec y$ (or $y \succ x$), if

$$\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j \quad (k = 1, \cdots, n-1), \quad \text{and} \quad \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j.$$

We make use of the following lemmas in our proof of Theorem 1.1.

Lemma 1.2. *If $A, B \in \mathbb{M}^n(\mathbb{C})$ are Hermitian, then*

$$2\lambda(A) \prec \lambda(A+B) + \lambda(A-B). \tag{1.3}$$

Proof. The lemma is equivalent to Ky Fan's eigenvalue inequality. The proof can be found in [4, Theorem 4.3.27]; see also [10, Theorem 7.15]. □

Lemma 1.3. *Let $A \in \mathbb{M}^{m\times n}(\mathbb{C})$ with $m \geq n$, then we have*

$$\lambda(AA^*) = \lambda(A^*A \oplus 0). \tag{1.4}$$

2 Proof of Main Result and Corollaries

Before we give the proof of Theorem 1.1, we show by an example that the requirement K being Hermitian is necessary.

Example 2.1. *Let $M = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$, $N = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $K = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Then*

$$\begin{aligned} \lambda((M+N)\oplus 0) &= (4+\sqrt{2}, 4-\sqrt{2}, 0, 0), \\ \lambda\left(\begin{bmatrix} M & K \\ K^* & N \end{bmatrix}\right) &= (4+\sqrt{5}, 4-\sqrt{5}, 0, 0). \end{aligned}$$

Therefore $\lambda(\begin{bmatrix} M & K \\ K^* & N \end{bmatrix}) \not\prec \lambda(M+N)\oplus 0)$.

Proof of Theorem 1.1. Since $H := \begin{bmatrix} M & K \\ K & N \end{bmatrix}$ is positive semidefinite, we may suppose $H \in \mathbb{M}^{2n}(\mathbb{C})$ and write $H = P^*P$, where $P = \begin{bmatrix} X & Y \end{bmatrix}$, for some $X, Y \in \mathbb{M}^{2n\times n}(\mathbb{C})$. Therefore, we have $M = X^*X$, $N = Y^*Y$ and $K = X^*Y = Y^*X$. Note that by Lemma 1.3, we have $\lambda\left(\begin{bmatrix} M & K \\ K & N \end{bmatrix}\right) = \lambda(PP^*)$. The conclusion (1.2) is then equivalent to showing

$$\{X^*Y = Y^*X\} \implies \{\lambda\left((X^*X + Y^*Y)\oplus 0\right) \succ \lambda(XX^* + YY^*)\}. \tag{2.1}$$

First, note that

$$\begin{aligned} (X+iY)^*(X+iY) &= X^*X + Y^*Y + i(X^*Y - Y^*X) \\ &= X^*X + Y^*Y \\ (X-iY)^*(X-iY) &= X^*X + Y^*Y - i(X^*Y - Y^*X) \\ &= X^*X + Y^*Y \\ (X+iY)(X+iY)^* &= XX^* + YY^* - i(XY^* - YX^*) \\ (X-iY)(X-iY)^* &= XX^* + YY^* + i(XY^* - YX^*). \end{aligned}$$

Therefore we see that

$$\begin{aligned} \lambda\left((X^*X + Y^*Y)\oplus 0\right) &= \tfrac{1}{2}\left\{\lambda\left((X+iY)^*(X+iY)\oplus 0\right) + \lambda\left((X-iY)^*(X-iY)\oplus 0\right)\right\} \\ &= \tfrac{1}{2}\left\{(\lambda\left((X+iY)(X+iY)^*\right) + \lambda\left((X-iY)(X-iY)^*\right))\right\} \\ &\succ \lambda(XX^* + YY^*), \end{aligned}$$

where the second equality is by Lemma 1.3 and the majorization follows from applying Lemma 1.2 with $A = (XX^* + YY^*)$, $B = i(XY^* - YX^*)$. □

As we can see from the above proof, a special case of of Theorem 1.1 can be stated as follows.

Corollary 2.2. *Let $X, Y \in \mathbb{M}^n(\mathbb{C})$ with X^*Y is Hermitian. Then we have*

$$\lambda(XX^* + YY^*) \prec \lambda(X^*X + Y^*Y). \tag{2.2}$$

Corollary 2.3. *Let $k \geq 1$ be an integer. If $A, B \in \mathbb{M}^n(\mathbb{C})$ are Hermitian matrices, then we have*

$$\lambda(A^2 + (AB)^k(BA)^k) \succ \lambda(A^2 + (BA)^k(AB)^k). \tag{2.3}$$

Proof. Let $X = A$ and $Y = (BA)^k$. Then $XY = A(BA)^k$ is Hermitian. The result now follows from Corollary 2.2. □

Corollary 2.4. *Let $k \geq 1$ be an integer, and let $A, B \in \mathbb{M}^n(\mathbb{C})$ be Hermitian matrices. Then we have*

1. $\mathrm{trace}[(A^2 + (AB)^k(BA)^k)^p] \geq \mathrm{trace}[(A^2 + (BA)^k(AB)^k)^p]$, *for* $p \geq 1$;
2. $\mathrm{trace}[(A^2 + (AB)^k(BA)^k)^p] \leq \mathrm{trace}[(A^2 + (BA)^k(AB)^k)^p]$, *for* $0 \leq p \leq 1$.

Proof. Since $f(x) = x^p$, is a convex function for $p \geq 1$ and concave for $0 \leq p \leq 1$, corollary follows from Corollary 2.3 and a general property of majorization. (See [6].) □

Remark 2.5. *A key inequality used in [3] to strengthen some Golden-Thompson type inequalities is just a special case of Corollary 2.4 by taking $k = 1$.*

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