Strong Duality and Minimal Representations for Cone Optimization *

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Abstract

The elegant results for strong duality and strict complementarity for linear programming, LP , can fail for cone programming over nonpolyhedral cones. One can have: unattained optimal values; nonzero duality gaps; and no primal-dual optimal pair that satisfies strict complementarity. This failure is tied to the nonclosure of sums of nonpolyhedral closed cones.

We take a fresh look at known and new results for duality, optimality, constraint qualifications, and strict complementarity, for linear cone optimization problems in finite dimensions. These results include: weakest and universal constraint qualifications, CQs ; duality and characterizations of optimality that hold without any CQ ; geometry of *nice and devious* cones; the geometric relationships between zero duality gaps, strict complementarity, and the facial structure of cones; and, the connection between theory and empirical evidence for lack of a CQ and failure of strict complementarity.

One theme is the notion of *minimal representation* of the cone and the constraints in order to regularize the problem and avoid both the theoretical and numerical difficulties that arise due to (near) loss of a CQ . We include a discussion on obtaining these representations efficiently. A parallel theme is the loss of strict complementarity and the corresponding theoretical and numerical difficulties that arise; a discussion on avoiding these difficulties is included. We include results and examples on the surprising theoretical connection between duality, strict complementarity, and nonclosure of sums of closed cones.

Our emphasis is on results that deal with Semidefinite Programming, SDP .

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Contents

1 Introduction

In this paper we study duality and optimality conditions for the (primal) conic optimization problem

$$v_P := \sup_y \{by : \mathcal{A}^* y \preceq_K c\}, \tag{$\mathbb{P}$}$$

where: $\mathcal{A} : \mathcal{V} \to \mathcal{W}$ is a (onto) linear transformation between two finite dimensional inner-product spaces; $\mathcal{A}^*$ denotes the adjoint transformation; K is a convex cone; and $\preceq_K$ denotes the partial order induced by K, i.e., $c - \mathcal{A}^* y \in K$. We assume that v_P is finite valued. This programming model has been studied for a long time, e.g., in [20] as a generalization of the classical linear program, LP, i.e., the case where K is a polyhedral cone. More recently, many important applications have arisen for more general nonpolyhedral cones. This includes the case when K is the cone of positive semidefinite matrices, $\mathbb{S}^n_+$; then we get semidefinite programming, SDP. Another important case is $K = \mathrm{SOC}_1 \oplus \ldots \oplus \mathrm{SOC}_k$, a direct sum of second order (Lorentz) cones.

We use the subspace form for ($\mathbb{P}$) and derive new characterizations of optimality without any constraint qualification, CQ, as well as new CQs including a *universal* constraint qualification, UCQ, i.e., a CQ that holds independent of the data b, c. In addition, we study the geometry of *nice* and *devious* cones and the relationship that lack of closure has to both strong duality and zero duality gaps; and, we provide characterizations for a zero duality gap. We include a surprising connection between duality gaps and the failure of strict complementarity in the homogeneous problem.

A major theme is showing that using *minimal representations* of the linear transformation $\mathcal{A}$ and/or the cone K regularizes a cone program. This justifies the pleasing paradigm: *efficient modelling provides for a stable program.*

We include a discussion on an efficient auxiliary problem for regularization in the absence of the Slater CQ, and an algorithm that efficiently solves ill-posed problems where the Slater CQ can fail. (This relates to the ongoing work in [53].) In addition, we collect many results on the facial structure of the cone optimization problems.

1.1 Background and Motivation

The research areas related to cone optimization, and in particular to SDP and SOC, remain very active, see e.g. [62, 56, 2, 31, 66, 33, 38, 28] and URL: www-user.tu-chemnitz.de/~helmberg/semidef.html. Optimality conditions and CQs have been studied in e.g., [20, 26, 51] and more recently for both linear and nonlinear problems in e.g., [54]. (See the historical notes in [54, Sect. 4.1.5].) Optimality conditions and strong duality without a CQ have appeared in e.g., [14, 12, 11, 13, 46, 47].

Both strong duality and strict complementarity behave differently for general cone optimization problems, compared to the LP case. First, strong duality for a cone program can fail in the absence of a CQ, i.e., there may not exist a dual optimal solution and there may be a nonzero duality gap. In addition, the (near) failure of the Slater CQ (strict feasibility) has been used in complexity measures, [49, 50]. Moreover, numerical difficulties are well correlated with (near) failure of the Slater CQ, see [22, 23]. Similarly, unlike the LP case, there are general cone optimization problems for which there does not exist a primal-dual optimal solution that satisfies strict complementarity, see e.g. [62] for examples. See [24] for the LP result. Theoretical difficulties arise, e.g., for local convergence rate analysis. Again, we have that numerical difficulties are well correlated with loss of strict complementarity, see [60]. An algorithm for generating SDP problems where strict complementarity fails, independent of whether the Slater CQ holds or not, is also given in [60].

Connections between weakest CQs and the closure of the sum of a subspace and a cone date back to e.g. [26]. We present a surprising theoretical connection between strict complementarity of the homogeneous problem and duality gaps, as well as show that both loss of strict complementarity and strong duality are connected to the lack of closure of the sum of a cone and a subspace.

Examples where no CQ holds arise in surprisingly many cases. For example, Slater's CQ fails for many SDP relaxations of hard combinatorial problems, see e.g. [63, 64, 4]. A unifying approach is given in [57]. Another case is the SDP that arises from relaxations of polynomial optimization problems, e.g. [58]. Current public domain codes for SDP are based on interior-point methods and do not take into account loss of Slater's CQ (strict feasibility) or loss of strict complementarity.

Both of these conditions can result in severe numerical problems, e.g., [60, 22, 58]. A projection technique for handling these cases is given in [53].

1.2 Outline

In Section 2 we present the notation and preliminary results. We introduce: the subspace forms for the cone optimization in Section 2.1.1; the complementarity partition and minimal sets in Section 2.1.3; facial properties in Sections 2.2.1 and 2.2.2; and nice and devious cones in Section 2.2.3. We include many relationships for the facial structure of the cone optimization problems.

The strong duality results, with and without CQs, and the CQs and UCQ, are presented in Section 3, see e.g. Theorem 3.10.

We study the failure of duality and strict complementarity in Section 4. This includes a characterization for a zero duality gap in Section 4.1. The surprising relation between duality gaps and strict complementarity of the homogeneous problem, is given in Section 4.1.2, see e.g. Theorems 4.10 and 4.14.

Our concluding remarks are in Section 5.

Due to the many definitions, we have included an index at the end of the paper; see page 35.

2 Notation and Preliminary Results

The set K is a convex cone if it is a cone, i.e., it is closed under nonnegative scalar multiplication, $\lambda K \subseteq K, \forall \lambda \geq 0$, and, it is also closed under addition $K + K \subseteq K$. The cone K is a proper cone if it is closed, pointed, and has nonempty interior. We use $\overline{S}$ to denote closure, $\operatorname{precl} S = \overline{S} \backslash S$ to denote the preclosure of a set S. We let $\operatorname{conv} S$ denote the convex hull of the set S and $\operatorname{cone} S$ denote the convex cone generated by S. (By abuse of notation, we use $\operatorname{cone} s = \operatorname{cone}\{s\}$, for a single element s. This holds similarly for e.g. $s^\perp = \{s\}^\perp$ and other operations that act on single element sets.) The dual or nonnegative polar cone of a set S is $S^* := \{\phi : \phi s \geq 0, \ \forall s \in S\}$. In this paper we use $\phi s = \langle \phi, s\rangle$ to denote the inner-product of ϕ and s by juxtaposition, if the meaning is clear. In particular, for the space of $n \times n$ symmetric matrices, $\mathbb{S}^n$, we use the trace inner-product $\phi s = \operatorname{trace} \phi s$, i.e., the trace of the product of the matrices ϕ and s. We denote the dimension of $\mathbb{S}^n$ by $\dim \mathbb{S}^n = t(n) := n(n+1)/2$. We let e_i denote the ith unit vector of appropriate dimension, and E_{ij} denote the (i,j)th unit matrix in $\mathbb{S}^n$, i.e., $E_{ii} = e_i e_i^T$ and if $i \neq j$, $E_{ij} = e_i e_j^T + e_j e_i^T$. By abuse of notation, we let x_{ij} denote the ij element of $x \in \mathbb{S}^n$.

The ray generated by $s \in K$ is called an *extreme ray* if $0 \preceq_K u \preceq_K s$ implies that $u \in \operatorname{cone} s$. The subset $F \subseteq K$ is a *face of the cone* K, denoted $F \trianglelefteq K$, if

$$(s \in F, 0 \preceq_K u \preceq_K s) \text{ implies } (\operatorname{cone} u \subseteq F). \tag{2.1}$$

Equivalently, $F \trianglelefteq K$ if $\left(x, y \in K, \frac{1}{2}(x+y) \in F\right) \implies (\operatorname{cone}\{x,y\} \subseteq F)$. If $F \trianglelefteq K$ but is not equal to K, we write $F \triangleleft K$. If $0 \neq F \triangleleft K$, then F is a *proper face* of K. (Similarly, $S_1 \subset S_2$ denotes a proper subset, i.e. $S_1 \subseteq S_2, S_1 \neq S_2$.) For $S \subseteq K$, we let $\operatorname{face} S$ denote the smallest face of K that contains S; equivalently $\operatorname{face} S$ is the intersection of all faces containing S. A face $F \trianglelefteq K$ is an *exposed face* if it is the intersection of K with a hyperplane. The cone K is *facially exposed* if every face $F \trianglelefteq K$ is exposed. If $F \trianglelefteq K$, then the *conjugate face* is $F^c := K^* \cap \{F\}^\perp$. Note that the conjugate face F^c is *exposed* using any $s \in \operatorname{relint} F$, i.e., $F^c = K \cap s^\perp, \forall s \in \operatorname{relint} F$.

We denote by $\preceq_K$ the partial order with respect to K. That is, $x_1 \preceq_K x_2$ means that $x_2 - x_1 \in K$. We also write $x_1 \prec_K x_2$ to mean that $x_2 - x_1 \in \operatorname{int} K$. In particular, $K = \mathbb{S}^n_+$ yields the partial order induced by the cone of positive semidefinite matrices in $\mathbb{S}^n$, i.e., the so-called Löwner partial order.

We consider the following pair of dual conic optimization problems in standard form:

$$v_P := \sup_y \{by : \mathcal{A}^* y \preceq_K c\}, \tag{$\mathbb{P}$}$$

$$v_D := \inf_x \{cx : \mathcal{A}x = b,\ x \succeq_{K^*} 0\}, \tag{$\mathbb{D}$}$$

where the data $(\mathcal{A}, K, b, c)$ are defined above. Throughout, we assume that the optimal value v_P is finite. *Weak duality* holds for any primal-dual feasible solutions y, x, i.e., if $s = c - \mathcal{A}^* y \succeq_K 0, \mathcal{A}x = b, x \succeq_{K^*} 0$, then we get

$$by = (\mathcal{A}x)y = (\mathcal{A}^* y)x = (c - s)x \leq cx. \qquad \text{(Weak Duality)}$$

Denote the feasible solution sets of ($\mathbb{P}$) and ($\mathbb{D}$) by

$$\mathcal{F}^y_P = \mathcal{F}^y_P(c) = \{y : \mathcal{A}^* y \preceq_K c\}, \quad \mathcal{F}^x_D = \mathcal{F}^x_D(b) = \{x : \mathcal{A}x = b,\ x \succeq_{K^*} 0\}, \tag{2.2}$$

respectively. The set of feasible slacks for ($\mathbb{P}$) is

$$\mathcal{F}^s_P = \mathcal{F}^s_P(c) = \{s : s = c - \mathcal{A}^* y \succeq_K 0,\ \text{for some } y\}. \tag{2.3}$$

We allow for the dependence on the parameters b and c. Similarly, the *optimal solution sets* are denoted by $\mathcal{O}^s_P, \mathcal{O}^y_P, \mathcal{O}^x_D$. Moreover, the pair of feasible primal-dual solutions s, x satisfy *strict complementarity* if

$$\begin{array}{c} s \in \operatorname{relint} F_P \text{ and } x \in \operatorname{relint} F^c_P, \quad \text{for some } F_P \trianglelefteq K, \\ \text{or} \\ s \in \operatorname{relint} F^c_D \text{ and } x \in \operatorname{relint} F_D, \quad \text{for some } F_D \trianglelefteq K^*. \end{array} \tag{$\mathbb{SC}$}$$

(Note that this implies $s + x \in \operatorname{int}(K + K^*)$, see Proposition 2.16 part 1, below.)

The usual constraint qualification, CQ, used for ($\mathbb{P}$) is the Slater condition, i.e., strict feasibility $\mathcal{A}^* \hat{y} \prec c$ ($c - \mathcal{A}^* \hat{y} \in \operatorname{int} K$). If we assume Slater's CQ holds and the primal optimal value is finite, then strong duality holds, i.e., we have a zero duality gap and attainment of the dual optimal value, $\mathcal{O}^x_D \neq \emptyset$,

$$v_P = v_D = cx^*,\ \text{for some } x^* \in \mathcal{F}^x_D. \qquad \text{(Strong Duality)}$$

2.1 Subspace Form, Recession Directions, and Minimal Sets

2.1.1 Subspace Form for Primal-Dual Pair ($\mathbb{P}$) and ($\mathbb{D}$)

Suppose that $\tilde{s}, \tilde{y}$, and $\tilde{x}$ satisfy

$$\mathcal{A}^* \tilde{y} + \tilde{s} = c, \quad \mathcal{A}\tilde{x} = b. \tag{2.4}$$

Then, for any feasible primal-dual triple (x, y, s), where s is, as usual, the primal slack given by $s = c - \mathcal{A}^* y$, we have $c\tilde{x} = (\mathcal{A}^* y + s)\tilde{x} = by + s\tilde{x}$. Therefore, the objective in ($\mathbb{P}$) can be rewritten as

$$\sup_y by = \sup_s (c\tilde{x} - s\tilde{x}) = c\tilde{x} - \inf_s s\tilde{x}.$$

We let $\mathcal{L}$ denote the nullspace $\mathcal{N}(\mathcal{A})$ of the operator $\mathcal{A}$. Then

$$\mathcal{F}_P^s = \mathcal{F}_P^s(c) = (c + \mathcal{L}^\perp) \cap K = (\tilde{s} + \mathcal{L}^\perp) \cap K. \tag{2.5}$$

In addition, for $x \in \tilde{x} + \mathcal{L}$, we get $cx = (\mathcal{A}^*\tilde{y} + \tilde{s})x = \tilde{s}x + \mathcal{A}^*\tilde{y}\tilde{x} = \tilde{s}x + \tilde{y}b$. We can now write the primal and dual conic pair, ($\mathbb{P}$) and ($\mathbb{D}$), in the so-called subspace form (see e.g., [39, Section 4.1]):

$$v_P = c\tilde{x} - \inf_s \{s\tilde{x} : s \in (\tilde{s} + \mathcal{L}^\perp) \cap K\}, \tag{2.6}$$

$$v_D = \tilde{y}b + \inf_x \{\tilde{s}x : x \in (\tilde{x} + \mathcal{L}) \cap K^*\}. \tag{2.7}$$

The symmetry means that we can directly extend results proved for (2.6) to (2.7). Note that we have much flexibility in the choice of $\tilde{s}$ and $\tilde{x}$. In particular, if (2.6) and (2.7) are feasible, we may choose $\tilde{s} \in \mathcal{F}_P^s$ and $\tilde{x} \in \mathcal{F}_D^x$, and in the case that the optimal values are attained, we may choose $\tilde{s} \in \mathcal{O}_P^s$ and $\tilde{x} \in \mathcal{O}_D^x$.

Proposition 2.1 *Let $\tilde{s}, \tilde{y}$, and $\tilde{x}$ satisfy* (2.4). *Then* (2.6) *and* (2.7) *are a dual pair of cone optimization problems equivalent to* ($\mathbb{P}$) *and* ($\mathbb{D}$), *respectively. Moreover,* ($\mathbb{P}$) *(resp.* ($\mathbb{D}$)*) is feasible if and only if $\tilde{s} \in K + \mathcal{L}^\perp$ (resp. $\tilde{x} \in K^* + \mathcal{L}$).* ■

2.1.2 Assumptions

Note that we can shift the data c using any element from $\mathcal{R}(\mathcal{A}^*)$.

Lemma 2.2 *Let $\bar{y} \in \mathcal{W}$. If c is replaced by $\bar{c} = c - \mathcal{A}^*\bar{y}$, then the optimal value is shifted to $v_P - \langle b, \bar{y}\rangle$, and the new feasible set contains $y = 0$. Moreover, the set of optimal solutions is shifted to the set $\mathcal{O}_P^y - \{\bar{y}\}$.*

Proof.
Note that

$$\begin{aligned} v_P &= \sup_y \{by : \mathcal{A}^*y \preceq_K c\} \\ &= \sup_y \{by : \mathcal{A}^*y - \mathcal{A}^*\bar{y} \preceq_K \bar{c} = c - \mathcal{A}^*\bar{y}\} \\ &= \sup_y \{b(y - \bar{y}) + b\bar{y} : \mathcal{A}^*(y - \bar{y}) \preceq_K \bar{c}\} \\ &= b\bar{y} + \sup_w \{bw : \mathcal{A}^*w \preceq_K \bar{c}\}. \end{aligned}$$

■

Corollary 2.3 *For* ($\mathbb{P}$), *we can assume that at least one of the following holds:*

$$c \in \mathcal{N}(\mathcal{A}) \text{ or } c \in K - K.$$

Proof.
From Lemma 2.2, we can shift c with the projection of c onto $\mathcal{R}(\mathcal{A}^*)$, i.e., we get an equivalent problem with $c \leftarrow c - P_{\mathcal{R}(\mathcal{A}^*)} \in \mathcal{N}(A)$. Alternatively, we can shift c with $\mathcal{A}^*y_c$ for y_c that satisfies $\mathcal{A}^*y_c + k_c = c$, for some $k_c \in K - K$. This latter set must be nonempty since the optimal value v_P is finite. We get an equivalent problem with $c \leftarrow c - \mathcal{A}^*y_c = k_c \in K - K$. ■

Assumption 2.4 *In this paper, the following assumptions are made (<u>when needed</u>).*

1. *K is a convex cone, and v_P is finite valued.*

2. $c \in \begin{cases} \mathcal{N}(\mathcal{A}) & \textit{if} \quad \operatorname{int} K \neq \emptyset \\ K - K & \textit{if} \quad \operatorname{int} K = \emptyset \end{cases}$

3. *$\tilde{s}, \tilde{y}, \tilde{x}$ satisfy* (2.4) *in the subspace forms* (2.6) *and* (2.7)*. If b, c are not specified, then we set $c = \mathcal{A}^*\tilde{y} + \tilde{s}$ and $b = \mathcal{A}\tilde{x}$.* ■

2.1.3 Complementarity Partition and Minimal Sets

Denote the minimal faces for the homogeneous problems (recession directions) by

$$f_P^0 := \operatorname{face} \mathcal{F}_P^s(0) = \operatorname{face}\left(\mathcal{L}^\perp \cap K\right), \tag{2.8}$$

$$f_D^0 := \operatorname{face} \mathcal{F}_D^x(0) = \operatorname{face}\left(\mathcal{L} \cap K^*\right). \tag{2.9}$$

(For connections between recession directions and optimality conditions, see e.g., [5, 10, 1].) Note that $f_P^0 \subseteq (f_D^0)^c$ (equivalently, $f_D^0 \subseteq (f_P^0)^c$).

Definition 2.5 *The pair of faces $F_1 \trianglelefteq K, F_2 \trianglelefteq K^*$ form a* complementarity partition *of K, K^* if $F_1 \subseteq F_2^c$. (Equivalently, $F_2 \subseteq F_1^c$.) The partition is* proper *if both F_1 and F_2 are proper faces. The partition is* strict *if $(F_1)^c = F_2$ or $(F_2)^c = F_1$.*

We now see that we can assume $F_1 \trianglelefteq F_2^c$ and $F_2 \trianglelefteq F_1^c$ in Definition 2.5.

Lemma 2.6 *Suppose that $F \trianglelefteq K$, $G \trianglelefteq K$, and $F \subseteq G$. Then $F \trianglelefteq G$.*

Proof.
Suppose that $s \in F$, and $0 \preceq_G u \preceq_G s$. This implies $s \in F$ and $0 \preceq_K u \preceq_K s$, since $G \trianglelefteq K$. We now conclude that $\operatorname{cone} u \subseteq F$, since $F \trianglelefteq K$, i.e., the definition (2.1) is satisfied. ■

The following proposition is well-known. We include a proof for completeness.

Proposition 2.7 *For every linear subspace $\mathcal{L}$, the pair of faces (f_P^0, f_D^0) form a complementarity partition of K, K^*. The partition is strict if K is a polyhedral cone.*

Proof.
That the faces form a complementarity partition is clear from the definitions that use the orthogonal subspaces $\mathcal{L}, \mathcal{L}^\perp$. Next, we apply linear programming duality to the homogeneous primal dual pair, i.e., with data $b = 0$ and $c = 0$. Since both the primal and dual problems are feasible, we know that a strict complementarity optimal primal-dual pair $\bar{s} \in K, \bar{x} \in K^*, \bar{s} + \bar{x} \in \operatorname{int}(K + K^*)$ exists, [24]. E.g., for $K = K^* = \mathbb{R}^n_+$, we get $\bar{s} + \bar{x} > 0$ and necessarily $(f_P^0)^c = f_D^0$. ■

Example 2.8 *Generating examples where we have a proper, not strict, complementarity partition is easy. We use $K = K^* = \mathbb{S}^n_+$ and the algorithm in [60] to generate $\mathcal{A}$ so that we have $(f_P^0)^c \cap (f_D^0)^c \neq \{0\}$. Here the linear transformation $\mathcal{A}^*y = \sum_{i=1}^m y_i A_i$ for given $A_i \in \mathbb{S}^n, i = 1, \ldots, n$. The main idea is to start with $\begin{bmatrix} Q_P & Q_N & Q_D \end{bmatrix}$ an orthogonal matrix; and then we construct one of the m matrices representing $\mathcal{A}$ as*

$$A_1 = \begin{bmatrix} Q_P & Q_N & Q_D \end{bmatrix} \begin{bmatrix} 0 & 0 & Y_2^T \\ 0 & \boxed{Y_1} & Y_3^T \\ Y_2 & Y_3 & Y_4 \end{bmatrix} \begin{bmatrix} Q_P & Q_N & Q_D \end{bmatrix}^T,$$

where $Y_1 \succ 0$, Y_4 symmetric, and $Q_D Y_2 \neq 0$. The other matrices $A_i \in \mathbb{S}^n$ are chosen so that the set $\{A_1 Q_P, \dots, A_m Q_P\}$ is linearly independent.

The *minimal face of* $(\mathbb{P})$ is the face of K generated by the feasible slack vectors; while the *minimal face for* $(\mathbb{D})$ is the face of K^* generated by the feasible set, i.e., we denote

$$f_P := \operatorname{face} \mathcal{F}_P^s, \qquad f_D := \operatorname{face} \mathcal{F}_D^x.$$

Note that both $f_P = f_P(\tilde{s})$ and $f_D = f_D(\tilde{x})$, i.e., they depend implicitly on the points $\tilde{s}, \tilde{x}$ in the subspace formulations (2.6) and (2.7). The primal and dual minimal subspace representations of $\mathcal{L}^\perp$ and of $\mathcal{L}$, respectively, are given by

$$\mathcal{L}_{PM}^\perp := \mathcal{L}^\perp \cap (f_P - f_P), \qquad \mathcal{L}_{DM} := \mathcal{L} \cap (f_D - f_D). \tag{2.10}$$

The *cone of feasible directions at* $\hat{y} \in \mathcal{F}_P^y$ is

$$\mathcal{D}_P^{\leq}(\hat{y}) = \operatorname{cone}\left(\mathcal{F}_P^y - \hat{y}\right). \tag{2.11}$$

We similarly define the cones $\mathcal{D}_P^{\leq}(\hat{s}), \mathcal{D}_D^{\leq}(\hat{x})$. For these three cones, we assume that $\hat{y}, \hat{s}, \hat{x}$ are suitable feasible points in $\mathcal{F}_P^y, \mathcal{F}_P^s, \mathcal{F}_D^x$, respectively. The closures of these cones of feasible directions yield the standard *tangent cones*, denoted $\mathbb{T}_P(\hat{y}), \mathbb{T}_P(\hat{s}), \mathbb{T}_D(\hat{x})$, respectively. (See e.g., [18, 8].) Note that if the primal feasible set is simply K, the cone of feasible directions corresponds to the so-called *radial cone*.

Proposition 2.9 ([59, 55]) *Let K be closed. Then K is a polyhedral cone if and only if at every point $\hat{s} \in K$, the radial cone of K, $\operatorname{cone}(K - \hat{s})$, at $\hat{s}$ is closed.* ■

Example 2.10 *We now look at three examples that illustrate the lack of closure for nonpolyhedral cones, e.g. in each instance we get*

$$K + \operatorname{span} f_P^0 \subsetneq \overline{K + \operatorname{span} f_P^0} = K + ((f_P^0)^c)^\perp. \tag{2.12}$$

The lack of closure in (2.12) can be used to find examples with both finite and infinite positive duality gaps; see Theorem 4.3 below.

1. *First, let $n = 2$ and $\mathcal{L}$ in (2.6) and (2.7) be such that $\mathcal{L}^\perp = \operatorname{span}\{E_{22}, E_{13}\}$. Then $f_P^0 = \operatorname{cone}\{E_{22}\}$ and $f_D^0 = \operatorname{cone}\{E_{11}\}$. Therefore, $f_P^0 = (f_D^0)^c$ and $f_D^0 = (f_P^0)^c$, i.e., this is a strict complementarity partition. Moreover, (2.12) holds e.g. $E_{12} \in (f_P^0)^\perp \cap (f_D^0)^\perp$ and*

$$E_{12} = \lim_{i \to \infty} \left(\begin{bmatrix} 1/i & 1 \\ 1 & i \end{bmatrix} - iE_{22} \right) \in \left(\mathbb{S}_+^2 + (f_D^0)^\perp\right) \setminus \left(\mathbb{S}_+^2 + \operatorname{span} f_P^0\right) = \operatorname{precl}\left(\mathbb{S}_+^2 + \operatorname{span} f_P^0\right).$$

2. *Now, let $n = 3$ and suppose that $\mathcal{L}^\perp = \operatorname{span}\{E_{33}, E_{22} + E_{13}\}$. Then $f_P^0 = \operatorname{cone}\{E_{33}\}$ and $f_D^0 = \operatorname{cone}\{E_{11}\}$. Therefore, $f_P^0 \subsetneq (f_D^0)^c$, i.e., this is not a strict complementarity partition. In addition, note that (2.12) holds and moreover, if we choose $\tilde{s} = \tilde{x} = E_{22} \in (f_P^0)^c \cap (f_D^0)^c$, then $E_{22} \in \operatorname{precl}(\mathcal{L} + (\operatorname{face} \tilde{s})^c) \cap \operatorname{precl}\left(\mathcal{L}^\perp + (\operatorname{face} \tilde{s})^c\right)$. This means that $\bar{s} = \bar{x} = E_{22}$ is both primal and dual optimal, see Theorem 4.3, below.*

3. *Similarly, we can choose* $\mathcal{L}^\perp = \operatorname{span}\{E_{22}, E_{33}, E_{23}, E_{11}+E_{12}\}$*. Then* $f_P^0 = \operatorname{face}\{E_{22}, E_{33}\}$ *and* $f_D^0 = \{0\}$*. Again,* $f_P^0 \subsetneq (f_D^0)^c = S_+^3$*, i.e., this is not a strict complementarity partition. Moreover, similar to part 2, we can choose* $\bar{s}, \bar{x}$ *appropriately, and find points in* $\operatorname{precl}(\mathcal{L} + (\operatorname{face}\bar{s})^c)$ *and* $\operatorname{precl}(\mathcal{L}^\perp + (\operatorname{face}\bar{x})^c)$*.*

■

All instances in the above Example 2.10 have the facial block structure $\begin{bmatrix} f_D^0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f_P^0 \end{bmatrix}$. Viz., the matrices in f_D^0 are nonzero only in the $(1,1)$ block, and the matrices in f_D^0 are nonzero only in the $(3,3)$ block. We now formalize the concept of such block structure for $\mathbb{S}^n$ in the following definition and lemma. (These may be extended to more general cones using appropriate bases.)

Definition 2.11 *The* support of $x \in \mathbb{S}^n$ *is* $\mathcal{S}(x) := \{(ij) : x_{ij} \neq 0\}$*.*

Lemma 2.12 *Let* $K := \mathbb{S}_+^n$ *.*

1. *There exists an orthogonal matrix* Q *and integers* $0 \le k_D < k_P \le n+1$ *such that*

$$x \in f_D^0, (ij) \in \mathcal{S}(Q^T x Q) \implies \max\{i,j\} \le k_D, \tag{2.13}$$

and

$$s \in f_P^0, (ij) \in \mathcal{S}(Q^T s Q) \implies \min\{i,j\} \ge k_P. \tag{2.14}$$

2. *Let* $n \ge 3$ *and suppose the subspace* $\mathcal{L}$ *is such that the complementarity partition* (f_P^0, f_D^0) *is not strict. Then, there exists an orthogonal matrix* Q *and integers* $1 \le k_D < k_P - 1 \le n-1$ *such that* (2.13) *and* (2.14) *hold.*

Proof.
We can choose $x = Q_x D_x Q_x^T \in \operatorname{relint} f_D^0$ and $s = Q_s D_s Q_s^T \in \operatorname{relint} f_P^0$, where Q_x, Q_s has orthonormal columns (of eigenvectors) and both D_x and D_s are diagonal positive definite. Choose Q_r so that $Q := \begin{bmatrix} Q_x & Q_r & Q_s \end{bmatrix}$ is an orthogonal matrix. Then this Q does what we want, since $f_P^0 f_D^0 = 0$.
■

2.2 Facial Properties

We now collect some interesting and useful facial properties for general convex cones K. Further results are given in e.g. [40, 7, 15, 6].

2.2.1 Faces of General Cones

Recall that a nonempty face $F \trianglelefteq K$ is *exposed* if $F = \phi^\perp \cap K$, for some $\phi \in K^*$. Note that the faces of K are closed if K is closed.

Proposition 2.13 *Let* K *be closed and* $\emptyset \neq F \trianglelefteq K$*. Then:*

1. $(F - F) \cap K = (\operatorname{span} F) \cap K = F$*.*

2. $F^{cc} = F$ *if and only if* F *is exposed.*

3. $\overline{K^* + \operatorname{span} F^c} \subseteq \overline{K^* + F^\perp}$*. Moreover, if* K *is facially exposed, then* $\overline{K^* + \operatorname{span} F^c} = \overline{K^* + F^\perp}$*.*

Proof.

1. That $F - F = \operatorname{span} F$ follows from the definition of a cone. Further, suppose $k = f_1 - f_2$ with $k \in K$ and $f_i \in F, i = 1, 2$. Then $k + f_2 = f_1 \in F$. Therefore, $k \in F$, by the definition of a face.

2. The result follows from the fact that the conjugate of $G := F^c$ is exposed by any $x \in \operatorname{relint} G$.

3. That $K^* + \operatorname{span} F^c \subseteq K^* + F^\perp$ is clear from the definition of the conjugate face F^c. To prove equality, suppose that $w = (k + f) \in (K^* + F^\perp) \backslash \overline{K^* + \operatorname{span} F^c}$, with $k \in K^*, f \in F^\perp$. Then there exists ϕ such that $\phi w < 0 \leq \phi(K^* + \operatorname{span} F^c)$. This implies that $\phi \in K \cap (F^c)^\perp = K \cap F$, since K is facially exposed. This in turn implies $\phi w = \phi(k + f) \geq 0$, a contradiction.

■

Proposition 2.14 *Let* $s \in \operatorname{relint} S$ *and* $S \subseteq K$ *be a convex set. Then:*

1. $\operatorname{face} s = \operatorname{face} S$*,*

2. $\operatorname{cone}(K - s) = \operatorname{cone}(K - S) = K - \operatorname{face} s = K - \operatorname{face} S = K + \operatorname{span} \operatorname{face} s = K + \operatorname{span} \operatorname{face} S$*.*

Proof.

1. That $\operatorname{face} s \subseteq \operatorname{face} S$ is clear. To prove the converse inclusion, suppose that $z \in S \subseteq K, z \neq s$. Since $s \in \operatorname{relint} S$, there exists $w \in S$, $0 < \theta < 1$, such that $s = \theta w + (1 - \theta) z$, i.e., $s \in (w, z)$. Since $s \in \operatorname{face} s$, we conclude that both $w, z \in \operatorname{face} s$.

2. That $\operatorname{cone}(K - s) \subseteq K - \operatorname{cone} s \subseteq K - \operatorname{face} s \subseteq K - \operatorname{span} \operatorname{face} s$ is clear. The other inclusions follow from part 1 and $\operatorname{cone}(K - s) \supset \operatorname{cone}(\operatorname{face}(s) - s) = \operatorname{span} \operatorname{face} s$.

■

Proposition 2.15 ([12], Prop 3.2) *Let* $S \subseteq K$*. Then:*

1. there is a unique minimal face, $\operatorname{face} S$*, containing* S*;*

2. there is a unique minimal exposed face, $\operatorname{face}^{\mathrm{ef}} S$*, containing* S*.*

Proof.

1. The intersection of all faces containing S is $\operatorname{face} S$ and it is clearly a face.

2. By [52, Cor. 18.1.3], the dimension of the intersection of two non-nested faces has a lower dimension than either face. Therefore, the face formed from the intersection of all exposed faces containing S can be replaced by the finite intersection $\cap_{i=1}^t \left(\phi_i^\perp \cap K\right), \phi \in K^*, \forall i$. This face is exposed by $\sum_{i=1}^t \phi_i$ and is the desired unique smallest exposed face containing S.

■

The following Proposition 2.16 illustrates some technical properties of faces, conjugates, and closure.

Proposition 2.16 *Let T be a convex cone and $F \trianglelefteq T$.*

1. *Suppose that $\bar{s} \in \operatorname{relint} F$ and $\bar{x} \in \operatorname{relint} F^c$. Then*
$$\bar{s} + \bar{x} \in \operatorname{int}(T + T^*).$$

2. *Suppose that $\bar{s} \in \operatorname{relint} F$. Then*
$$\begin{array}{rl} \overline{\operatorname{cone}(T - \bar{s})} = (F^c)^*; \\ \operatorname{cone}(T - \bar{s}) \supset \operatorname{relint}\ ((F^c)^*) . \end{array} \tag{2.15}$$

3. *Suppose that $F_1 \trianglelefteq T$, $F_2 \trianglelefteq T^*$, are both exposed faces, and $F_2 \subseteq F_1^c$. If $0 \neq \tilde{x} \in F_1^\perp \cap (F_1^c)^\perp \cap F_2^\perp \cap (F_2^c)^\perp$, then*
$$\tilde{x} \in \operatorname{precl}(\overline{T} + \operatorname{span} F_1) \cap \operatorname{precl}(T^* + \operatorname{span} F_1^c) \cap \operatorname{precl}(\overline{T} + \operatorname{span} F_2^c) \cap \operatorname{precl}(T^* + \operatorname{span} F_2).$$

Proof.

1. First, note that if $\operatorname{int}(T + T^*) = \emptyset$, then we have
$$0 \neq (T + T^*)^\perp \subseteq T^{**} \cap T^* = \overline{T} \cap T^* \subseteq T + T^*,$$
a contradiction, i.e., this shows that $\operatorname{int}(T + T^*) \neq \emptyset$.

   Now suppose that $\bar{s} + \bar{x} \notin \operatorname{int}(T + T^*)$. Then we can find a supporting hyperplane $\phi^\perp$ so that $\bar{s} + \bar{x} \in (T + T^*) \cap \phi^\perp \triangleleft T + T^*$ and $0 \neq \phi \in (T + T^*)^* = \overline{T} \cap T^*$. Therefore, we conclude $\phi(\bar{s} + \bar{x}) = 0$ implies that both $\phi\bar{s} = 0$ and $\phi\bar{x} = 0$. This means that $\phi \in \overline{T} \cap \bar{s}^\perp = F^c$ and $\phi \in T^* \cap \bar{x}^\perp = (F^c)^c$, giving $0 \neq \phi \in (F^c) \cap (F^c)^c = \{0\}$, which is a contradiction.

2. The first result follows from: $\overline{\operatorname{cone}(T - \bar{s})} = (T - \bar{s})^{**} = (T^* \cap \bar{s}^\perp)^*$.

   Now suppose that the conclusion in the second statement in (2.15) does not hold. That is, there exists $\bar{d} \in (\operatorname{relint}\ [(F^c)^*]) \setminus (\operatorname{cone}(T - \bar{s}))$. Since T may not be closed, we only have the weak separation $\phi T \geq 0, \phi\bar{s} = 0$, and $\phi\bar{d} \leq 0$. Again $0 \neq \phi \in T^* \cap \bar{s}^\perp = F^c$. Moreover, this contradicts the hypothesis that $\bar{d} \in \operatorname{relint}((F^c)^*)$.

3. First, Lemma 2.6 implies that $F_2 \trianglelefteq F_1^c$. Moreover, using the definition of the conjugate, we get $F_1 \trianglelefteq F_2^c$.

   Now if $\tilde{x} = k + f$, with $k \in T$ and $f \in \operatorname{span} F_1$, then $0 = F_1^c\tilde{x} = F_1^c k + F_1^c f = F_1^c k$. Therefore, $k \in F_1^{cc} = F_1$, i.e., $\tilde{x} \in F_1 + \operatorname{span} F_1$. This implies that $\tilde{x} = 0$, a contradiction. Thus, we have shown that $\tilde{x} \notin T + \operatorname{span} F_1$.

   But,
$$\tilde{x} \in F_1^\perp \cap (F_1^c)^\perp \subseteq F_1^* \cap (F_1^c)^* \subseteq F_1^* \cap (T^* \cap F_1^\perp)^* = F_1^* \cap \overline{(\overline{T} + \operatorname{span} F_1)},$$
i.e., we have shown $\tilde{x} \in \operatorname{precl}(\overline{T} + \operatorname{span} F_1)$. The result for $\tilde{x} \in \operatorname{precl}(T^* + \operatorname{span} F_1^c)$ follows similarly, as do the results for F_2.

∎

Remark 2.17 *From the above Proposition 2.16, part 3, we see that K polyhedral implies that $F^\perp \cap (F^c)^\perp = \{0\}, \forall F \trianglelefteq K$, i.e., the full dimensionality of conjugate faces characterizes polyhedral cones. Moreover, we can combine Proposition 2.9 and Proposition 2.14, and conclude that K is polyhedral if and only if $K + \operatorname{span} F$ is closed, for all $F \trianglelefteq K$.*

2.2.2 Faces for Primal-Dual Pair ($\mathbb{P}$) and ($\mathbb{D}$)

We now present facial properties specific to the primal-dual pair ($\mathbb{P}$) and ($\mathbb{D}$). In particular, this includes relationships between the minimal faces f_P, f_D and the minimal faces for the homogeneous problems, f_P^0, f_D^0. The relationships depend on the specific choices of $\tilde{s}, \tilde{x}$.

Proposition 2.18 *Suppose that both* ($\mathbb{P}$) *and* ($\mathbb{D}$) *are feasible, i.e., equivalently* $\tilde{s} \in K + \mathcal{L}^\perp$ *and* $\tilde{x} \in K^* + \mathcal{L}$. *Let* $\hat{s} \in \mathcal{F}_P(\tilde{s})$ *and* $\hat{x} \in \mathcal{F}_D(\tilde{x})$. *Then the following holds.*

1.
$$f_P^0 \subseteq \text{face}\,(\hat{s} + f_P^0) \subseteq f_P(\tilde{s}), \quad f_D^0 \subseteq \text{face}\,(\hat{x} + f_D^0) \subseteq f_D(\tilde{x}). \tag{2.16}$$

2.
$$\tilde{s} \in (f_D^0)^c + \mathcal{L}^\perp \Leftrightarrow f_P(\tilde{s}) \subseteq (f_D^0)^c; \quad \tilde{x} \in (f_P^0)^c + \mathcal{L} \Leftrightarrow f_D(\tilde{x}) \subseteq (f_P^0)^c.$$

Proof.
Since both problems are feasible, we can assume, without loss of generality, that $\hat{s} = \tilde{s} \in K, \hat{x} = \tilde{x} \in K^*$.

1. Since $\text{cone}\,\tilde{s}$ and f_P^0 are convex cones containing the origin, $\text{cone}\,\tilde{s} + f_P^0 = \text{conv}\,(\text{cone}\,\tilde{s} \cup f_P^0)$; see e.g., [52, Theorem 3.8]. Hence
$$\begin{array}{rcl} f_P^0 & \subseteq & \text{conv}\,(\tilde{s} \cup f_P^0) \\ & \subseteq & \text{conv}\,(\text{cone}\,\tilde{s} \cup f_P^0) \\ & = & \text{cone}\,\tilde{s} + f_P^0 \\ & = & \text{cone}\,(\tilde{s} + f_P^0) \\ & \subseteq & \text{face}\,(\tilde{s} + f_P^0). \end{array}$$
This proves the first inclusion. It is clear that $\tilde{s} + (\mathcal{L}^\perp \cap K) \subseteq (\tilde{s} + \mathcal{L}^\perp) \cap K$. This yields the second inclusion. The final two inclusions follow similarly.

2. Suppose that $\tilde{s} \in (f_D^0)^c + \mathcal{L}^\perp$ and $\tilde{s} + r \in K$ with $r \in \mathcal{L}^\perp$. Then, for all $\ell \in \mathcal{L} \cap K^* \subseteq f_D^0$, we have
$$\begin{array}{rcll} (\tilde{s} + r)\ell & = & \tilde{s}\ell & \\ & = & 0, & \quad \text{since } \tilde{s} \in (f_D^0)^c + \mathcal{L}^\perp. \end{array}$$
This implies that $f_P = \text{face}\,\big((\tilde{s} + \mathcal{L}^\perp) \cap K\big)$ is orthogonal to $f_D^0 = \text{face}\,(\mathcal{L} \cap K^*)$, i.e. the first implication holds.

   For the converse implication, since ($\mathbb{P}$) is feasible, we have $\tilde{s} \in f_P(\tilde{s})$. So if $f_P(\tilde{s}) \subseteq (f_D^0)^c$, then $\tilde{s} \in (f_D^0)^c$. The implication follows since we can add anything in $\mathcal{L}^\perp$ to $\tilde{s}$, while leaving $f_P(\tilde{s})$ unchanged.

   The second equivalence follows similarly.

■

Remark 2.19 *If* $\tilde{s} \in \text{relint}\,\big((f_P^0)^\perp \cap (f_D^0)^c\big)$ *and* $\tilde{x} \in \text{relint}\,\big((f_D^0)^\perp \cap (f_P^0)^c\big)$*; then we conjecture that the following relation between the faces holds as well.*
$$f_P(\tilde{s}) = \text{face}\,(\tilde{s} + f_P^0), \qquad f_D(\tilde{x}) = \text{face}\,(\tilde{x} + f_D^0).$$

Corollary 2.20 $\mathcal{F}_P^s \neq \emptyset \implies f_P^0 \subseteq f_P$. $\mathcal{F}_D^x \neq \emptyset \implies f_D^0 \subseteq f_D$.

Proof.
Since we can always choose $\tilde{s}$ (resp. $\tilde{x}$) as feasible points, the two results follow immediately from (2.16). ■

Additional relationships between the faces follow. But first we need a lemma that is of interest in its own right.

Lemma 2.21 *Let $\tilde{s} \in f_P^0$, and suppose that $s = \tilde{s} + \ell$ is feasible for $(\mathbb{P})$ with $\ell \in \mathcal{L}^\perp$. Then $\ell \in \operatorname{span} f_P^0$.*

Proof.
Let $v \in \operatorname{relint}(\mathcal{L}^\perp \cap K)$. Then $v \in \operatorname{relint} f_P^0$, and since $\tilde{s} \in f_P^0$, we have $v - \epsilon\tilde{s} \in f_P^0$ for some $\epsilon > 0$. Now if ℓ is such that $s = \tilde{s} + \ell$ is feasible for $(\mathbb{P})$, then $\tilde{s} + \ell \in K$, and

$$\frac{1}{\epsilon}v + \ell = \frac{1}{\epsilon}(v - \epsilon\tilde{s}) + (\tilde{s} + \ell) \in f_P^0 + K = K.$$

For convenience, define $\alpha := 1/\epsilon$. Since $\alpha v \in \mathcal{L}^\perp$ and $\ell \in \mathcal{L}^\perp$, we in fact have

$$\alpha v + \ell \in K \cap \mathcal{L}^\perp \subseteq f_P^0. \tag{2.17}$$

Now to obtain a contradiction, suppose that $\ell \notin \operatorname{span} f_P^0$, and write $\ell = \ell_1 + \ell_2$, where $\ell_1 \in \operatorname{span} f_P^0$ and $0 \neq \ell_2 \in (f_P^0)^\perp$. In view of (2.17), we have $0 = \ell_2(\ell + \alpha v) = \ell_2\ell_2 + \ell_2(\ell_1 + \alpha v) = \|\ell_2\|^2 > 0$, a contradiction. ■

Proposition 2.22 1. *$\tilde{s} \in f_P^0 \cup \mathcal{L}^\perp \implies f_P(\tilde{s}) = f_P^0$ and $\tilde{x} \in f_D^0 \cup \mathcal{L} \implies f_D(\tilde{x}) = f_D^0$.*

2. *Let $f_D^0 \lhd K^*$. Then there exists $0 \neq \phi \in K \cap \mathcal{L}^\perp$. Moreover, $(-\phi + \mathcal{L}) \cap K^* = \emptyset$.*

Proof.

1. We begin by proving the first statement. If $\tilde{s} \in \mathcal{L}^\perp$, then $\tilde{s} + \mathcal{L}^\perp = \mathcal{L}^\perp$, so the desired result holds. If instead $\tilde{s} \in f_P^0$, then it follows from Lemma 2.21 that $\ell \in \operatorname{span} f_P^0$ for all feasible points of the form $s = \tilde{s} + \ell$. Hence all feasible s lie in the set $\operatorname{span}(f_P^0) \cap K$, which by Proposition 2.13, part 1, equals f_P^0. So $f_P(\tilde{s}) \subseteq f_P^0$; but, the reverse inclusion holds by Proposition 2.18, part 1.

   The second statement for f_D^0 in proven in a similar way.

2. Existence is by the theorem of the alternative for the Slater CQ; see Lemma 3.14, below.

■

Proposition 2.23 *Let K be closed. If there exists a nonzero $x \in -(K \cap K^*)$ such that*

$$x \in (K \cap \mathcal{L}^\perp)^\perp \cap (K^* \cap \mathcal{L})^\perp, \tag{2.18}$$

then

$$x \in \operatorname{precl}(K + \mathcal{L}^\perp) \cap \operatorname{precl}(K^* + \mathcal{L}).$$

Hence, neither $K + \mathcal{L}^\perp$ nor $K^* + \mathcal{L}$ is closed.

Proof.
To obtain a contradiction, suppose that (2.18) holds but $x \in K + \mathcal{L}^\perp$. Then there exists $w \in K$ such that $x - w \in \mathcal{L}^\perp$. Moreover, $x - w \in -K - K = -K$, so $x - w \in -(K \cap \mathcal{L}^\perp)$. It follows from (2.18) that $\langle x, x - w\rangle = 0$. But $\langle x, x - w\rangle = \langle x, x\rangle + \langle -x, w\rangle > 0$, where we have used the fact that $x \in -K^*$. Hence $x \notin K + \mathcal{L}^\perp$. A similar argument shows that $x \notin K^* + \mathcal{L}$.

Since K and $\mathcal{L}^\perp$ are closed convex cones, we have $(K \cap \mathcal{L}^\perp)^* = \overline{K^* + (\mathcal{L}^\perp)^*} = \overline{K^* + \mathcal{L}}$. It follows from (2.18) that $x \in \overline{K^* + \mathcal{L}}$. Similarly, $x \in \overline{K + \mathcal{L}^\perp}$. This completes the proof. ∎

2.2.3 Nice Cones, Devious Cones, and SDP

Definition 2.24 *A face $F \trianglelefteq K$ is called nice if $K^* + F^\perp$ is closed. A closed convex cone K is called a* nice cone *or a* facially dual-complete cone, FDC*, if*

$$K^* + F^\perp \text{ is closed for all } F \trianglelefteq K. \tag{2.19}$$

The condition in (2.19) was used in [11] to allow for extended Lagrange multipliers in f_P^* to be split into a sum using K^* and $f_P^\perp$. This allowed for restricted Lagrange multiplier results with the multiplier in K^*. The condition (2.19) was also used in [41] where the term *nice cone* was introduced. (In addition, it was shown by Pataki (forthcoming paper) that a FDC cone must be facially exposed.)

Moreover, the FDC property has an implication for Proposition 2.13, part 3. We now see that this holds for SDP .

Lemma 2.25 ([62],[47]) *Suppose that F is a proper face of $\mathbb{S}^n_+$, i.e., $\{0\} \neq F \lhd \mathbb{S}^n_+$. Then:*

$$F^* = \mathbb{S}^n_+ + F^\perp = \overline{\mathbb{S}^n_+ + \operatorname{span} F^c},$$

$$\mathbb{S}^n_+ + \operatorname{span} F^c \text{ is not closed.}$$

∎

From Lemma 2.25, we see that $\mathbb{S}^n_+$ is a nice cone. In fact, as pointed out in [41], many other classes of cones are nice cones, e.g., polyhedral and p-cones. However, the lack of closure property in Lemma 2.25 is *not* a nice property. In fact, from Proposition 2.14, part 2, this corresponds to the lack of closure for radial cones, see [55] which can result in duality problems. Therefore we add the following.

Definition 2.26 *A face $F \trianglelefteq K$ is called* devious *if the set $K + \operatorname{span} F$ is not closed. A cone K is called* devious *if*

$$\text{the set } K + \operatorname{span} F \text{ is } \underline{\text{not}} \text{ closed for all } \{0\} \neq F \lhd K.$$

By Lemma 2.25, $\mathbb{S}^n_+$ is a nice but devious cone. On the other hand, polyhedral cones are nice but not devious, since faces of polyhedral cones are themselves polyhedral and sums of polyhedral sets are closed, e.g., [52, Chapter 9].

The facial structure of $\mathbb{S}^n_+$ is well known, e.g., [47, 62]. Each face $F \trianglelefteq \mathbb{S}^n_+$ is characterized by a unique subspace $S \subseteq \mathbb{R}^n$:

$$F = \left\{x \in \mathbb{S}^n_+ : \mathcal{N}(x) \supset S\right\}; \quad \operatorname{relint} F = \left\{x \in \mathbb{S}^n_+ : \mathcal{N}(x) = S\right\}.$$

The conjugate face satisfies

$$F^c = \left\{x \in \mathbb{S}^n_+ : \mathcal{N}(x) \supset S^\perp\right\}; \quad \operatorname{relint} F = \left\{x \in \mathbb{S}^n_+ : \mathcal{N}(x) = S^\perp\right\}.$$

The description of $\operatorname{span} F$ for $F \trianglelefteq \mathbb{S}^n_+$ is now clear.

Another useful property of SDPs(and the Löwner partial order) is given by the following lemma. This lemma played a critical role in Ramana's explicit description of a dual SDP problem for which strong duality holds.

Lemma 2.27 *Let $\tilde{K} \subseteq \mathbb{S}^n_+$ be a closed convex cone. Then*

$$\left[\left(\operatorname{face} \tilde{K}\right)^c\right]^\perp = \left\{W + W^T : W \in \mathbb{R}^{n\times n}, \begin{bmatrix} I & W^T \\ W & U \end{bmatrix} \succeq 0, \text{ for some } U \in \tilde{K}\right\}.$$

■

Properties 2.28 *The following three properties of the cone $\mathbb{S}^n_+$ are needed for the strong duality approach in Ramana [46]. (The first two also make the Borwein-Wolkowicz approach in [13] behave particularly well.)*

1. *K is facially exposed.*
2. *K is* FDC *.*
3. *Lemma 2.27.*

■

Suppose that the cone K describing the problem ($\mathbb{P}$) is *SDP-representable*. (That is, there exists d and a linear subspace $V \subset \mathbb{S}^d$ such that $V \cap \mathbb{S}^d_{++} \neq \emptyset$ and K is isomorphic to $(V \cap \mathbb{S}^d_+)$.) Then by [17, Cor. 1, Prop. 4], K is facially exposed and FDC , since $\mathbb{S}^d_+$ is. Moreover, by [17, Prop. 3], every proper face of K is a proper face of $\mathbb{S}^d_+$ intersected with the subspace V. Hence, an analogue of Lemma 2.27 is also available in this case. Therefore, SDP-representable cones (which strictly include homogeneous cones, due to a result of Chua [16] and Faybusovich [21]) satisfy all three of the above-mentioned Properties 2.28. For related recent results on homogeneous cones and strong duality, see Pólik and Terlaky [42].

3 Duality and Minimal Representations

In this section we see that *minimal representations* of the problem guarantee strong duality results. We first use the minimal representations and extend the known strong duality results without any constraint qualification that use the *minimal face of* K, see e.g. [12, 11, 13, 61]. Equivalent strong duality results based on an extended Lagrangian are given in [45, 46]. (See [47, 62] for comparison and summaries of the two types of duality results.) By strong duality for ($\mathbb{P}$), we mean that there is a *zero duality gap*, $v_P = v_D$, and the dual optimal value v_D in ($\mathbb{D}$) is attained.

3.1 Strong Duality and Constraint Qualifications

We now present strong duality results that hold with and without CQs. We also present: a weakest constraint qualification (WCQ), i.e., a CQ at a given feasible point $\bar{y} \in \mathcal{F}_P^y(c)$ that is independent of b; and a universal constraint qualification, (UCQ), i.e., a CQ that is independent of both b and c.

Following is the classical, well-known, strong duality result for $(\mathbb{P})$ under the the standard Slater CQ.

Theorem 3.1 (e.g., [36, 54]) *Suppose that Slater's* CQ *(strict feasibility) holds for* $(\mathbb{P})$. *Then strong duality holds for* $(\mathbb{P})$, *i.e.,* $v_P = v_D$ *and the dual value* v_D *in* $(\mathbb{D})$ *is attained. Equivalently, there exists* $\bar{x} \in K^*$ *such that*

$$by + (c - \mathcal{A}^* y)\bar{x} \geq v_P, \quad \forall y \in \mathbb{R}^m.$$

Moreover, if v_P *is attained at* $\bar{y} \in \mathcal{F}_P^y$, *then* $(c - \mathcal{A}^*\bar{y})\bar{x} = 0$ *(complementary slackness holds).* ■

Corollary 3.2 *Suppose that Slater's* CQ *(strict feasibility) holds for* $(\mathbb{P})$ *and* $\bar{y} \in \mathcal{F}_P^y$. *Then,* $\bar{y}$ *is optimal for* $(\mathbb{P})$ *if and only if*

$$b \in \mathcal{A}\left[(K - \bar{s})^*\right], \tag{3.1}$$

where $\bar{s} = c - \mathcal{A}^*\bar{y}$.

Proof.
The result follows from the observation that $(\operatorname{face} \bar{s})^c = K^* \cap \bar{s}^\perp = (K - \bar{s})^*$, i.e., (3.1) is equivalent to dual feasibility and complementary slackness. ■

Strong duality can fail if Slater's CQ does not hold. In [12, 11, 13], an equivalent *regularized* primal problem that is based on the minimal face,

$$v_{RP} := \sup\{by : \mathcal{A}^* y \preceq_{f_P} c\} \tag{3.2}$$

is considered. Its Lagrangian dual is given by

$$v_{DRP} := \inf\{cx : \mathcal{A}x = b,\ x \succeq_{f_P^*} 0\}. \tag{3.3}$$

Theorem 3.3 ([11]) *Strong duality holds for the pair* (3.2) *and* (3.3), *or equivalently, for the pair* $(\mathbb{P})$ *and* (3.3)*; i.e.,* $v_P = v_{RP} = v_{DRP}$ *and the dual optimal value* v_{DRP} *is attained. Equivalently, there exists* $x^* \in (f_P)^*$ *such that*

$$by + (c - \mathcal{A}^* y)x^* \geq v_P, \quad \forall y \in (\mathcal{A}^*)^{-1}(f_P - f_P).$$

Moreover, if v_P *is attained at* $\bar{y} \in \mathcal{F}_P^y$, *then* $(c - \mathcal{A}^*\bar{y})x^* = 0$ *(complementary slackness holds).* ■

Corollary 3.4 *Let* $\bar{y} \in \mathcal{F}_P^y$. *Then* $\bar{y}$ *is optimal for* $(\mathbb{P})$ *if and only if*

$$b \in \mathcal{A}\left[(f_P - \bar{s})^*\right],$$

where $\bar{s} = c - \mathcal{A}^*\bar{y}$.

Proof.
As above in the proof of Corollary 3.2, the result follows from the observation that $f_P^* \cap \bar{s}^\perp = (f_P - \bar{s})^*$. ■

The next result uses the *minimal subspace representation* of $\mathcal{L}^\perp$, introduced in (2.10), $\mathcal{L}_{PM}^\perp = \mathcal{L}^\perp \cap (f_P - f_P)$.

Corollary 3.5 *Let $\tilde{y}, \tilde{s}$, and $\tilde{x}$ satisfy* (2.4) *with $\tilde{s} \in f_P - f_P$ and let*

$$K^* + (f_P)^\perp = (f_P)^*. \tag{3.4}$$

Consider the following pair of dual programs.

$$v_{RP_M} = c\tilde{x} - \inf_s \{s\tilde{x} : s \in (\tilde{s} + \mathcal{L}_{PM}^\perp) \cap K\}, \tag{3.5}$$

$$v_{DRP_M} = \tilde{y}b + \inf_x \{\tilde{s}x : x \in (\tilde{x} + \mathcal{L}_{PM}) \cap K^*\}. \tag{3.6}$$

Then, $v_{RP_M} = v_{RP} = v_P = v_{DRP_M} = v_{DRP}$, and strong duality holds for (3.5) *and* (3.6), *or equivalently, for the pair* ($\mathbb{P}$) *and* (3.6).

Proof.
That $v_P = v_{RP_M} = v_{RP}$ follows from the definition of the minimal subspace representation in (2.10):

$$\begin{array}{rcll} \mathcal{F}_P^s(c) & = & \mathcal{F}_P^s(\tilde{s}) & \\ & = & (\tilde{s} + \mathcal{L}^\perp) \cap f_P, & \text{by definition of } f_P, \\ & = & (\tilde{s} + \mathcal{L}_{PM}^\perp) \cap K, & \text{since } \tilde{s} \in f_P - f_P. \end{array}$$

For the regularized dual, we see that

$$\begin{array}{rcll} v_{DRP_M} & = & \inf_x \left\{ cx : \mathcal{A}x = b, x \succeq_{f_P^*} 0 \right\} & \\ & = & \tilde{y}b + \inf_x \left\{ \tilde{s}x : \mathcal{A}x = b, x = x_k + x_f, x_k \in K^*, x_f \in f_P^\perp \right\}, & \text{by (3.4)} \\ & = & \tilde{y}b + \inf_x \left\{ \tilde{s}x : x = x_k + x_f = \tilde{x} + x_l, x_k \in K^*, x_f \in f_P^\perp, x_l \in \mathcal{L} \right\} & \\ & = & \tilde{y}b + \inf_x \left\{ \tilde{s}x_k : x_k \in (\tilde{x} + \mathcal{L} + f_P^\perp) \cap K^* \right\}. & \end{array}$$

■

Remark 3.6 *The condition in* (3.4) *is equivalent to $K^* + (f_P)^\perp$ being closed, and is clearly true if K is a* FDC *cone.*

Remark 3.7 *Using the minimal subspace representations of $\mathcal{L}$ in* ($\mathbb{D}$), *i.e., replacing $\mathcal{L}$ in* ($\mathbb{D}$) *by $\mathcal{L}_{DM}$ in* (2.10), *we may obtain a result similar to Corollary* (3.5).

Note that if the Slater CQ holds, then the minimal sets (face and subspace) satisfy $f_P = K$ and (2.10). We now see that strong duality holds if at least one of these conditions holds.

Corollary 3.8 *Suppose that* $\operatorname{int} K = \emptyset$ *but the* generalized Slater CQ *(relative strict feasibility) holds for* ($\mathbb{P}$), *i.e.,*

$$\hat{s} := c - \mathcal{A}^*\hat{y} \in \operatorname{relint} K, \text{ for some } \hat{y} \in \mathcal{W}. \qquad \textit{(Generalized Slater CQ)} \tag{3.7}$$

(Equivalently, suppose that the minimal face satisfies $f_P = K$.) Then strong duality holds for ($\mathbb{P}$).

Proof.
The proof follows immediately from Theorem 3.3 after noting that $K = f_P$. ■

The following Corollary illustrates strong duality for a variation of the generalized Slater constraint qualification, i.e., for the case that the minimal subspace satisfies (2.10).

Corollary 3.9 *Let $\tilde{s} \in f_P - f_P$ and K be* FDC *. Suppose that*

$$\mathcal{L}^\perp \subseteq f_P - f_P. \qquad \textit{(Subspace CQ)} \tag{3.8}$$

(Equivalently, suppose that $\mathcal{L}^\perp_{PM} = \mathcal{L}^\perp$.) Then strong duality holds for $(\mathbb{P})$.

Proof.
Follows directly from Corollary 3.5. ■

We summarize the results in the special case that K is FDC (a nice cone). Weakest constraint qualifications for general nonlinear problems are given in [26].

Theorem 3.10 *Let $\tilde{s}, \tilde{x}$ satisfy linear feasibility* (2.4) *with $\tilde{s} \in f_P - f_P$ and let K be* FDC *. Then we have the following conclusions.*

1. *The primal optimal values are all equal, $v_P = v_{RP} = v_{RP_M}$. Moreover, strong duality holds for the primal, where the primal is chosen from the set*

$$\{(2.6), (3.2), (3.5)\} \qquad \textit{(set of primal programs)}$$

   and the dual is chosen from the set

$$\{(3.3), (3.6)\} \qquad \textit{(set of dual programs)}$$

   i.e., the optimal values are all equal and the dual optimal value is attained.

2. *The following are* CQs *for* $(\mathbb{P})$*:*
   (a) *$f_P = K$ (equivalently generalized Slater CQ* (3.7)*);*
   (b) *$\mathcal{L}^\perp \subseteq f_P - f_P$ (equivalently $\mathcal{L}^\perp_{PM} = \mathcal{L}^\perp \cap (K - K)$);*

3. *Let $\bar{y} \in \mathcal{F}^y_P(c)$ and $\bar{s} = c - \mathcal{A}^*\bar{y}$. Then,*

$$\mathcal{D}^{\leq}_P(\bar{y})^* = -\mathcal{A}((K - \bar{s})^*) \quad \textit{is a } \text{WCQ} \textit{ for } (\mathbb{P}) \textit{ at } (\bar{y}, \bar{s}). \tag{3.9}$$

   Equivalently,

$$\mathcal{A}[(f_P - \bar{s})^*] = \mathcal{A}((K - \bar{s})^*) \quad \textit{is a } \text{WCQ} \textit{ for } (\mathbb{P}) \textit{ at } (\bar{y}, \bar{s}). \tag{3.10}$$

Proof.

1. That $v_P = v_{RP}$ follows from the definition of the minimal face f_P. Similarly, $v_P = v_{RP} = v_{RP_M}$ follows from the definition of the minimal subspace representation $\mathcal{L}^\perp_{PM}$.

   That strong duality holds for the regularized pair that uses the minimal face (3.2),(3.3), follows from [11]. The results using the minimal subspace representations follow from Corollary 3.9. More precisely, by the assumptions on $\tilde{s}$, we have

$$\mathcal{F}^s_P(c) = \mathcal{F}^s_P(\tilde{s}) = (\tilde{s} + \mathcal{L}^\perp) \cap K = \left[\tilde{s} + \mathcal{L}^\perp \cap (f_P - f_P)\right] \cap K,$$

   i.e., the feasible set in (3.5) is $\mathcal{F}^s_P(c) = \mathcal{F}^s_P(\tilde{s})$. Moreover therefore, the optimal value in (3.5) is indeed v_P. Since $\mathcal{L}^\perp_{PM} \subseteq f_P - f_P$, we get strong duality from Corollary 3.9.

2. The results follow from Corollaries 3.8, 3.9, respectively.

3. The so-called Rockafellar-Pshenichnyi condition, e.g., [44], [26, Thm 1], states that $\bar{y}$ is optimal if and only if $b \in -\mathcal{D}_P^{\leq}(\bar{y})^*$. From Theorem 3.3, $\bar{y}$ is optimal if and only if $\mathcal{A}\bar{x} = b, \bar{s}\bar{x} = 0$, for some $\bar{x} \in f_P^*$; equivalently, if and only if $b \in \mathcal{A}((f_P - \bar{s})^*)$. The result follows from the fact that strong duality holds if and only if $\mathcal{A}\bar{x} = b, \bar{s}\bar{x} = 0$, for some $\bar{x} \in K^*$; equivalently, $\mathcal{A}\bar{x} = b$, for some $\bar{x} \in (K - \bar{s})^*$.

■

Remark 3.11 *The* WCQ *in* (3.9) *follows the approach in e.g. [26, 19, 65, 34, 35]. Moreover, since for any set S, $\mathcal{A}(S)$ is closed if and only if $S + \mathcal{L}$ is closed (e.g. [29, 9]), we conclude that a necessary condition for a* WCQ *to hold at a feasible $\bar{s} \in \mathcal{F}_P^s$ is that*

$$(K - \bar{s})^* + \mathcal{L} \quad \textit{is closed.} \tag{3.11}$$

(For a detailed study of when the linear image of a closed convex set is closed, see e.g., [41].)

3.1.1 Universal Constraint Qualifications

A universal CQ, denoted UCQ , is a CQ that holds independent of the data b, c.

Theorem 3.12 *Suppose that K is* FDC *, and $\tilde{s} \in K, \tilde{x} \in K^*$ in the primal-dual subspace representation in* (2.6) *and* (2.7). *Then*

$$\mathcal{L}^\perp \subseteq f_P^0 - f_P^0$$

is a UCQ *, i.e., a universal* CQ *for* ($\mathbb{P}$).

Proof.
The result follows from (3.8) and the fact that $\tilde{s} \in K, \tilde{x} \in K^*$ implies $f_P^0 \subseteq f_P$ and $f_D^0 \subseteq f_D$, see Lemma 2.18. ■

Corollary 3.13 *Suppose that $K = \mathbb{S}^n_+$, both v_P and v_D are finite, and $n \leq 2$. Then strong duality holds for at least one of* ($\mathbb{P}$) *or* ($\mathbb{D}$).

Proof.
We have both $\mathcal{F}_P^s \neq \emptyset$ and $\mathcal{F}_D^x \neq \emptyset$. By going through the possible cases, we see that one of the CQs $\mathcal{L}^\perp \subseteq f_P^0 - f_P^0$ or $\mathcal{L} \subseteq f_D^0 - f_D^0$ must hold. ■

3.2 CQ Regularization

In the case that the Slater CQ fails, we can use the paradigm of minimal representation to efficiently regularize ($\mathbb{P}$). This approach is used in [53]. Essentially, the procedure alternates between finding a smaller cone to represent f_P and finding a smaller subspace to represent $\mathcal{L}_{PM}^\perp$.

3.2.1 Theorems of the Alternative

The regularization procedure for $(\mathbb{P})$ is based on the following lemma that specializes [13, Theorem 7.1]. This provides a characterization for Slater's condition.

Lemma 3.14 ([13]) *Suppose that* $\operatorname{int}(K) \neq \emptyset$. *Then exactly one of the following two systems is consistent:*

1. $\mathcal{A}x = 0,\ 0 \neq x \succeq_{K^*} 0,$ *and* $cx = 0.$

2. $\mathcal{A}^* y \prec_K c$ *(Slater* CQ *).*

Proof.
We modify the proof from [13] for our special case. Suppose that $\hat{y} \in \mathcal{F}_P^y$. Define the vector-valued and real-valued functions $g(y) := \mathcal{A}^* y - c$ and $g_x(y) := (\mathcal{A}^* y - c)x$. Note that $\nabla_y g_x(y) = \mathcal{A}x$.

Suppose that x satisfies the system in Item 1. Then $\nabla_y g_x(\hat{y}) = \mathcal{A}x = 0$ and $g_x(y) = (\mathcal{A}x)y - cx = 0, \forall y$. Therefore $\hat{y}$ is a global minimizer of the linear function $g_x(y)$, i.e., $(\mathcal{A}^*(\mathbb{R}^m) - c)x = 0$. Since $0 \neq x \succeq_{K^*} 0$, this implies that the Slater CQ fails.

Conversely, suppose that the Slater CQ in Item 2 fails. We have $\operatorname{int} K \neq \emptyset$ and

$$0 \notin (\mathcal{A}^*(\mathbb{R}^m) - c) + \operatorname{int} K.$$

Therefore, we can find $x \neq 0$ to separate the set from 0, i.e., $[(\mathcal{A}^*(\mathbb{R}^m) - c) + \operatorname{int} K]\, x \geq 0$. Therefore, $\hat{y}$ is again a global minimizer of $g_x(y)$, and the optimality conditions imply that this x satisfies the conditions in Item 1. ■

Similarly, we can characterize the generalized Slater CQ for $(\mathbb{D})$.

Corollary 3.15 *Suppose that* $\operatorname{int}(K^*) \neq \emptyset$, *and* $\tilde{x} \in K^* + \mathcal{L}$ *($(\mathbb{D})$ is feasible). Then exactly one of the following two systems is consistent:*

1. $0 \neq \mathcal{A}^* v \succeq_{\bar{K}} 0,$ *and* $bv = 0.$

2. $\mathcal{A}x = b, x \succ_{K^*} 0$ *(generalized Slater* CQ *).*

Proof.
Let $\mathcal{A}^\dagger$ denote the Moore-Penrose generalized inverse of $\mathcal{A}$. Since $(\mathbb{D})$ is feasible, we can assume that $\tilde{x} = \mathcal{A}^\dagger b$ and

$$\mathcal{F}_D^x = \{x \in K^* : x = \mathcal{A}^\dagger b + (I - \mathcal{A}^\dagger \mathcal{A})v, \text{ for some } v \in \mathcal{V}\}.$$

In addition, we can assume that $\tilde{s} = c$ in $(\mathbb{P})$. Therefore, we can apply Lemma 3.14, after exchanging the roles of K and K^* and associating

$$\mathcal{A}^\dagger b \text{ here with } c \text{ in Lemma 3.14 }, \text{ and } (I - \mathcal{A}^\dagger \mathcal{A}) \text{ here with } \mathcal{A}^* \text{ in Lemma 3.14 }. \tag{3.12}$$

From (3.12), the alternative 1 in Lemma 3.14 becomes $(I - \mathcal{A}^\dagger \mathcal{A})^* s = 0, 0 \neq s \geq_{\bar{K}} 0$, and $s(\mathcal{A}^\dagger b) = 0$. After setting $v = (\mathcal{A}^\dagger)^* s$, we get the desired alternative 1 here.

Similarly, using (3.12), the alternative 2 in Lemma 3.14 becomes $(I - \mathcal{A}^\dagger \mathcal{A})x \prec_{K^*} \mathcal{A}^\dagger b$. Equivalently, we have $\tilde{x} - (I - \mathcal{A}^\dagger \mathcal{A})x \in \operatorname{int} K^*$; i.e., we get the desired alternative 2 here. ■

3.2.2 Auxiliary Problem and Regularization

In theory, we can solve the problem in part 1 in Lemma 3.14. Similarly, we could work on the dual and solve the problem in part 1 in Corollary 3.15. However, these problems do not necessarily satisfy an appropriate constraint qualification, so they can be difficult to solve. The following *auxiliary* primal-dual cone optimization problems are used in [53].

$$
\text{(A)} \qquad
\begin{array}{cc}
\inf_{x,\alpha} & \alpha \\
\text{s.t.} & \mathcal{A}x = 0 \\
& \langle c, x\rangle = 0 \\
& \langle e, x\rangle \leq 1 \\
& x + \alpha e \succeq_{K^*} 0 \\
\\
\sup_{y,\beta,\gamma,w} & \gamma \\
\text{s.t.} & \mathcal{A}^* y + \beta c + \gamma e + w = 0 \\
& \langle e, w\rangle = 1 \\
& \gamma \leq 0 \\
& w \succeq_K 0.
\end{array}
\tag{3.13}
$$

It is shown in [53] that both these programs satisfy the generalized Slater CQ . A nonzero x provides a hyperplane $H = x^\perp \supset f_P$, i.e., $f_P \subseteq (x^\perp \cap K) \lhd K$. We then restrict $\mathcal{A}^*$ to a subspace so that the new linear transformation $\mathcal{A}^*_H$ satisfies $\mathcal{R}(\mathcal{A}^*_H) \subseteq H$. We now redefine the problem so that we reduce the dimension of $(\mathbb{P})$. This process is repeated until no $x \neq 0$ can be found. The process must stop in a finite number of iterations, since we work in finite dimensions. The end result is that we have a minimal representation for both the cone and the constraint based on the linear transformation, i.e., we have regularized the problem. (A backward error analysis is done in [53].)

Note that failure of Slater's CQ for $(\mathbb{P})$ can result in failure of strong duality, i.e., we have theoretical difficulties. In addition, it has been shown that near loss of Slater's CQ is closely correlated with the expected number of iterations in interior-point methods both in theory [48, 50] and empirically, [22, 23]. Therefore, a regularization process is an essential preprocessor for SDP solvers. Under exact arithmetic, the auxiliary problem is only of assistance if Slater's CQ fails, so in practice one may solve a "robust" variant of the auxiliary problem that in addition to computing an approximate optimal solution, also computes a measure of the distance to the set of feasible instances for which Slater's CQ fails.

It is also known that lack of strict complementarity for SDP can result in theoretical difficulties. For example, superlinear and quadratic convergence results for interior-point methods depend on the strict complementarity assumption, e.g. [43, 30, 3, 37, 32]. This is also the case for convergence of the central path to the analytic center of the optimal face, [27]. In addition, it is shown empirically in [60] that loss of strict complementarity is closely correlated with the expected number of iterations in interior-point methods. However, one can generate problems where strict complementarity fails independent of whether or not Slater's CQ holds for the primal and/or the dual, [60].

We see below that duality and strict complementarity of the homogeneous problem have a surprising theoretical connection.

4 Failure of Duality and Strict Complementarity

Strong duality for $(\mathbb{P})$ means a zero duality gap, $v_P = v_D$ and dual attainment, $v_D = cx^*$, for some $x^* \in \mathcal{F}_D^x$. The CQs (resp. UCQs), introduced above in Section 3, guarantee that strong duality holds independent of the data b (resp. b and c). Under our assumption that v_P is finite valued, there are three cases of failure to consider: (i) a zero duality gap but with no dual attainment; (ii) an infinite duality gap (dual infeasibility); (iii) a finite positive duality gap.

4.1 Finite Positive Duality Gaps

4.1.1 Positive Gaps and Closure Conditions

We present characterizations for a finite positive duality gap under attainment assumptions in Theorem 4.3 and Corollary 4.4. We first give sufficient conditions for a positive duality gap using well known optimality conditions based on feasible directions.

Proposition 4.1 *Let $\tilde{s} \in \mathcal{F}_P^s$, $\tilde{x} \in \mathcal{F}_D^x$, and $\tilde{s}\tilde{x} > 0$. Suppose that $\tilde{s} \in \mathcal{D}_D^{\leq}(\tilde{x})^*$ and $\tilde{x} \in \mathcal{D}_P^{\leq}(\tilde{s})^*$. Then $\tilde{s}$ is optimal for $(\mathbb{P})$, $\tilde{x}$ is optimal for $(\mathbb{D})$, and $-\infty < v_P < v_D < \infty$.*

Proof.
The optimality of $\tilde{s}$ and $\tilde{x}$ follows immediately from the definition of the cones of feasible directions and the Rockafellar-Pshenichnyi condition, see e.g. the proof of Theorem 3.10. The finite positive duality gap follows from the hypotheses that both $(\mathbb{P})$ and $(\mathbb{D})$ are feasible, and that $\tilde{s}\tilde{x} > 0$. ■

To connect the conditions using the cones of feasible directions with conditions using closure we need the following.

Lemma 4.2 *Suppose that K is closed, $\tilde{y}$ is feasible for $(\mathbb{P})$, with corresponding slack $\tilde{s}$, and that $\tilde{x}$ is feasible for $(\mathbb{D})$. Then*

$$-\mathcal{D}_P^{\leq}(\tilde{y})^* = \mathcal{A}((f_P - \tilde{s})^*) = \overline{\mathcal{A}((K - \tilde{s})^*)}, \quad \mathcal{D}_D^{\leq}(\tilde{x})^* = \mathcal{L}^\perp + (f_D - \tilde{x})^* = \overline{\mathcal{L}^\perp + (K^* - \tilde{x})^*}.$$

Proof.
First note that $b \in \mathcal{A}((K - \tilde{s})^*)$ implies that strong duality holds so that $\tilde{s}$ is optimal for $(\mathbb{P})$. Therefore, $b \in -\mathcal{D}_P^{\leq}(\tilde{y})^* = \mathcal{A}((f_P - \tilde{s})^*)$, by the Rockafellar-Pshenichnyi condition and Theorem 3.3, i.e., we have shown the first containment $\overline{\mathcal{A}((K - \tilde{s})^*)} \subseteq -\mathcal{D}_P^{\leq}(\tilde{y})^* = \mathcal{A}((f_P - \tilde{s})^*)$.

To prove the converse containment, we use Proposition 2.16, part 2, and show that $(\mathcal{A}((K - \tilde{s})^*))^* \subseteq -\mathcal{D}_P^{\leq}(\tilde{y})$. Let $F = \operatorname{face} \tilde{s}$ and let $d \in (\mathcal{A}((K - \tilde{s})^*))^* = (\mathcal{A}(F^c))^*$. Equivalently, $\mathcal{A}^*d \in (F^c)^*$. Then Proposition 2.16, part 2, implies that $-\mathcal{A}^*d$ is a feasible direction at $\tilde{s}$; equivalently, $-d$ is a feasible direction at $\tilde{y}$.

The second containment follows similarly. ■

A well-known characterization for a zero duality gap can be given using the perturbation function. For example, define

$$v_P(\epsilon) := \sup_y \{by : \mathcal{A}^*y \preceq_K c + \epsilon\}, \quad \text{where } \epsilon \in \mathcal{V}.$$

The connection with the dual functional $\phi(x) := \sup_y by + x(c - \mathcal{A}^*y)$ is given in e.g., [36]. Then the geometry shows that the closure of the epigraph of v_P characterizes a zero duality gap. We now use preclosure and present a characterization for a finite positive duality gap in the case of attainment of the primal and dual optimal values.

Theorem 4.3 *Suppose that K is closed, $\tilde{y}$ is feasible for $(\mathbb{P})$, with corresponding slack $\tilde{s}$, and that $\tilde{x}$ is feasible for $(\mathbb{D})$. Then*

$$\tilde{s} \in \mathcal{O}_P^s, \tilde{x} \in \mathcal{O}_D^x, \quad \tilde{s}\tilde{x} > 0,$$

if and only if

$$\tilde{x} \in \operatorname{precl}(\mathcal{L} + (K - \tilde{s})^*), \quad c \in \operatorname{precl}(\mathcal{L}^\perp + (K^* - \tilde{x})^*).$$

Proof.
We have: $\tilde{s} \in \mathcal{O}_P^s$ if and only if $-b \in \mathcal{D}_P^{\leq}(\tilde{y})^*$, and $\tilde{x} \in \mathcal{O}_D^x$ if and only if $c \in \mathcal{D}_D^{\leq}(\tilde{x})^*$. But $\tilde{s}\tilde{x} > 0$ implies that strong duality fails for both the primal and the dual. Therefore, $b \notin \mathcal{A}((K - \tilde{s})^*)$ and $c \notin \mathcal{L}^\perp + (K^* - \tilde{x})^*$. The result now follows from Lemma 4.2 and the fact that $\mathcal{A}\tilde{x} = b \in \operatorname{precl}\mathcal{A}(K - \tilde{s})^*)$ if and only if $\tilde{x} \in \operatorname{precl}(\mathcal{L} + (K - \tilde{s})^*)$. (See e.g., [29, 9].) ∎

It is difficult to apply the closure condition in Theorem 4.3 directly. We now see that we can expand the expression under the closure using the recession cones f_P^0 and f_D^0.

Corollary 4.4 *Suppose that K is closed. Let $\tilde{y} = 0$ be feasible for $(\mathbb{P})$, with corresponding slack $\tilde{s}$, and $\tilde{x}$ be feasible for $(\mathbb{D})$. Let:*

$$F_s := \operatorname{face} \tilde{s}; F_x := \operatorname{face} \tilde{x}; \quad T_P := F_s^c + ((f_D^0)^c)^\perp; T_D := F_x^c + ((f_P^0)^c)^\perp. \tag{4.1}$$

Then:

1. $$\tilde{s} \in \mathcal{O}_P^s, \tilde{x} \in \mathcal{O}_D^x, \quad \tilde{s}\tilde{x} > 0, \tag{4.2}$$

   if and only if

   $$\tilde{x} \notin \mathcal{L} + F_s^c, \quad \tilde{s} \notin \mathcal{L}^\perp + F_x^c, \tag{4.3}$$

   and

   $$\tilde{x} \in \overline{F_s^c + ((\operatorname{face}(F_s^c \cap \mathcal{L}))^c)^\perp + \mathcal{L}}, \quad \tilde{s} \in \overline{F_x^c + ((\operatorname{face}(F_x^c \cap \mathcal{L}^\perp))^c)^\perp + \mathcal{L}^\perp}.$$

2. *Moreover, if $F_s \subseteq (f_D^0)^c$ and $F_x \subseteq (f_P^0)^c$, then (4.2) holds if and only if*

   $$\tilde{x} \in \overline{T_P + \mathcal{L}} \backslash (\mathcal{L} + F_s^c), \quad \tilde{s} \in \overline{T_D + \mathcal{L}^\perp} \backslash (\mathcal{L}^\perp + F_x^c). \tag{4.4}$$

Proof.
We have $F_s \trianglelefteq K$ and $F_s^c \trianglelefteq K^*$. The result follows directly from the characterization in Theorem 4.3 if we use the following equivalences.

$$\begin{array}{rcll} \overline{(K - \tilde{s})^* + \mathcal{L}} & = & \overline{F_s^c + \mathcal{L}} & \\ & = & \overline{F_s^c + \operatorname{span}(F_s^c \cap \mathcal{L}) + \mathcal{L}} & \\ & = & \overline{F_s^c + \operatorname{span}\operatorname{face}(F_s^c \cap \mathcal{L}) + \mathcal{L}}, & \text{by Proposition 2.14, Part 2} \\ & = & \overline{F_s^c + ((\operatorname{face}(F_s^c \cap \mathcal{L}))^c)^\perp + \mathcal{L}}, & \text{by Proposition 2.13, Part 3.} \end{array}$$

Moreover, if $F_s \subseteq (f_D^0)^c$, then $f_D^0 \subseteq \operatorname{face}(F_s^c \cap \mathcal{L})$. We can then apply Proposition 2.13, Part 3 again.

The equivalences for F_x follow similarly. ∎

4.1.2 Positive Gaps and Strict Complementarity

In this section we study the relationships between complementarity partitions and positive duality gaps. In particular, we consider cases where the complementarity partition for the pair of faces (f_P^0, f_D^0) fails to be strict.

Example 4.5 *A simple example with data $K = K^* = \mathbb{S}_+^6$ uses*

$$A_1 = -E_{11},\ A_2 = -E_{22},\ A_3 = -E_{34},\ A_4 = -E_{13} - E_{55},$$
$$A_5 = -E_{14} - E_{66},\ b = \begin{pmatrix} 0 & 0 & -2 & 0 & 1 \end{pmatrix}^T,\ c = E_{12} + E_{66}.$$

The faces and recession cones of the primal and dual are

$$f_P^0 = \begin{bmatrix} \mathbb{S}_+^2 & 0 \\ 0 & 0 \end{bmatrix} \trianglelefteq K, \qquad f_D^0 = Q\mathbb{S}_+^2 Q^T \trianglelefteq K^*, \quad \textit{where } Q = [e_3\ e_4],$$
$$f_P = Q\mathbb{S}_+^3 Q^T, \quad \textit{where } Q = [e_1\ e_2\ e_6], \quad f_D = Q\mathbb{S}_+^3 Q^T, \quad \textit{where } Q = [e_3\ e_4\ e_6].$$

The optimal values are $v_P = 0$ and $v_D = 1$. ■

A connection between optimality and complementarity can be seen in the following proposition.

Proposition 4.6 *Suppose that $(\mathbb{P})$ has optimal solution $\tilde{y}$ with corresponding optimal slack $\tilde{s}$, and that $(\mathbb{D})$ has optimal solution $\tilde{x}$. Suppose also that the optimal values $v_P = c\tilde{x} - \tilde{s}\tilde{x}, v_D = \tilde{y}b + \tilde{s}\tilde{x}$ are finite. Then*

$$\tilde{s}\tilde{x} = \inf \left\{ sx : s \in \mathcal{F}_P^s, x \in \mathcal{F}_D^x \right\}.$$

Proof.
The proof is immediate from the subspace form (2.6), (2.7) of the primal-dual pair, i.e., the primal problem shows that $\tilde{s}\tilde{x} \leq s\tilde{x}, \forall s \in \mathcal{F}_P^s$; while the dual problem implies that $\tilde{s}\tilde{x} \leq \tilde{s}x, \forall x \in \mathcal{F}_D^x$. ■

Simple necessary conditions for strict complementarity and sufficient conditions for failure of strict complementarity follow easily from Corollary 4.4.

Corollary 4.7 *Let K be a proper cone. Suppose that $\tilde{s}$ and $\tilde{x}$ are optimal for* (2.6) *and* (2.7)*, respectively, with a positive duality gap $\tilde{s}\tilde{x} > 0$. Moreover, suppose that the complementarity partition for the pair of faces (f_P^0, f_D^0) is strict. Then*

$$\tilde{x} \in \overline{(\text{face}\,\tilde{s})^c + (f_P^0)^\perp + \mathcal{L}} \backslash ((\text{face}\,\tilde{s})^c + \mathcal{L}), \quad \tilde{s} \in \overline{(\text{face}\,\tilde{x})^c + (f_D^0)^\perp + \mathcal{L}^\perp} \backslash ((\text{face}\,\tilde{x})^c + \mathcal{L}^\perp). \quad (4.5)$$

Proof.
The proof follows immediately from Corollary 4.4 after replacing $(f_D^0)^c$ and $(f_P^0)^c$ by f_P^0 and f_D^0, respectively. ■

Corollary 4.8 *Let K be a proper cone. Suppose that $\tilde{s}$ and $\tilde{x}$ are optimal for* (2.6) *and* (2.7)*, respectively, with a positive duality gap $\tilde{s}\tilde{x} > 0$. Moreover, suppose that*

$$\tilde{x} \in \overline{(\text{face}\,\tilde{s})^c + (f_P^0)^\perp + \mathcal{L}} \backslash \overline{(\text{face}\,\tilde{s})^c + ((f_D^0)^c)^\perp + \mathcal{L}}, \quad (4.6)$$

or

$$\tilde{s} \in \overline{(\text{face}\,\tilde{x})^c + (f_D^0)^\perp + \mathcal{L}^\perp} \backslash \overline{(\text{face}\,\tilde{x})^c + ((f_P^0)^c)^\perp + \mathcal{L}^\perp}. \quad (4.7)$$

Then the complementarity partition for the pair of faces (f_P^0, f_D^0) fails to be strict.

Proof.
The proof follows from combining Corollary 4.4 with Theorem 4.7. ■

Example 4.9 *It is possible to have a finite positive duality gap even if the complementarity partition for the pair of faces* (f_P^0, f_D^0) *is strict. Let* $K = K^* = \mathbb{S}_+^5$*, and*

$$A_1 = -E_{11},\ A_2 = -E_{22},\ A_3 = -E_{34},\ A_4 = -E_{13} - E_{45} - E_{55},$$
$$b = \begin{pmatrix} 0 & -1 & -2 & -1 \end{pmatrix}^T,\ c = E_{44} + E_{55}.$$

Then

$$f_P^0 = \begin{bmatrix} \mathbb{S}_+^2 & 0 \\ 0 & 0 \end{bmatrix}, f_D^0 = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{S}_+^3 \end{bmatrix}, \quad f_P = Q\mathbb{S}_+^4 Q^T, f_D = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{S}_+^4 \end{bmatrix},$$

where $Q = [e_1\ e_2\ e_4\ e_5]$*. The primal optimal value is zero and the dual optimal value is* $(\sqrt{5}-1)/2$*, and both are attained. This can be seen using the optimal* $\tilde{s} = c$ *for* $(\mathbb{P})$*, and* $\tilde{x}$ *optimal for* $(\mathbb{D})$ *and applying Corollary 4.7. (Note that the optimal* $x^* = \tilde{x}$ *has values* $1/\sqrt{5}$ *and* $(3-\sqrt{5})/(2\sqrt{5})$ *for the diagonal* $(4,4)$ *and* $(5,5)$ *elements, respectively.)*

One can also have an example without attainment of the optimal values. Consider the SDP *with data* $K = K^* = \mathbb{S}_+^5$*, and*

$$A_1 = E_{11},\ A_2 = E_{22},\ A_3 = E_{34},\ A_4 = E_{13} + E_{45} + E_{55}, b = \begin{pmatrix} 0 & 1 & 2 & 1 \end{pmatrix}^T,\ c = E_{12} + E_{44} + E_{55}.$$

Then

$$f_P^0 = \begin{bmatrix} \mathbb{S}_+^2 & 0 \\ 0 & 0 \end{bmatrix}, f_D^0 = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{S}_+^3 \end{bmatrix}, \quad f_P = Q\mathbb{S}_+^4 Q^T, f_D = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{S}_+^4 \end{bmatrix},$$

where $Q = [e_1\ e_2\ e_4\ e_5]$*. The primal optimal value is zero and the dual optimal value is* 1*, but neither value is attained.* ■

Stronger sufficient conditions for failure of strong duality follow.

Theorem 4.10 *Let* K *be a closed convex cone. Suppose that both* (2.6) *and* (2.7) *are feasible but strong duality fails for either problem. In addition, suppose that*

$$\tilde{s} \in (f_D^0)^c + \mathcal{L}^\perp, \qquad \tilde{x} \in (f_P^0)^c + \mathcal{L}. \tag{4.8}$$

Then the complementarity partition for the pair of faces (f_P^0, f_D^0) *fails to be strict.*

Proof.
Suppose that (4.8) holds. Then by Proposition 2.18, part 2, $f_P \subseteq (f_D^0)^c$ and $f_D \subseteq (f_P^0)^c$. To prove a contradiction, suppose that (f_P^0, f_D^0) forms a strict complementarity partition. Then $(f_D^0)^c = f_P^0$. Since for all feasible problems, $f_P^0 \subseteq f_P$ and $f_D^0 \subseteq f_D$ (Proposition 2.18, part 1), we actually have $f_P^0 = f_P$ and $f_D^0 = f_D$. But then $f_P f_D = 0$, i.e. every feasible point is optimal and we have a zero duality gap. But this contradicts the assumption that strong duality fails for either $(\mathbb{P})$ or $(\mathbb{D})$. ■

Corollary 4.11 *Suppose that both* (2.6) *and* (2.7) *are feasible but strong duality fails either problem. In addition, suppose that all feasible points for* $(\mathbb{P})$ *and* $(\mathbb{D})$ *are optimal. Then the complementarity partition for the pair of faces* (f_P^0, f_D^0) *fails to be strict.*

Proof.
Suppose that all feasible points for $(\mathbb{P})$ are optimal. Then the primal objective function is constant along all primal recession directions. That is, $\langle \tilde{x}, \mathcal{L}^\perp \cap K\rangle = \{0\}$, i.e, $\tilde{x} \in (\mathcal{L}^\perp \cap K)^\perp$. Now by construction, $\tilde{x}$ is dual feasible, i.e, $\tilde{x} \in (\mathcal{L}^\perp \cap K)^\perp \cap K^* = (f_P^0)^\perp \cap K^* = (f_P^0)^c$. Finally, as argued previously, translating $\tilde{x}$ by a point in $\mathcal{L}$ leaves the dual problem unchanged, giving the condition on $\tilde{x}$ in (4.8). In a similar way we can show that if all feasible points for $(\mathbb{D})$ are optimal, then the condition on $\tilde{s}$ in (4.8) holds. The desired result now follows from Theorem 4.10. ∎

We now consider cases when the assumption that the complementarity partition for the pair of faces (f_P^0, f_D^0) fails to be strict implies a finite positive duality gap.

We can apply Corollary 4.4 and obtain a class of SDP problems with a finite positive duality gap.

Corollary 4.12 *Let $K = \mathbb{S}^n_+$ and (after the appropriate orthogonal congruence using Lemma 2.12) let $G := (f_P^0)^c \cap (f_D^0)^c = \operatorname{face}\{E_{ss}, \ldots, E_{tt}\}$. Suppose that $\tilde{s} = \tilde{x} = E_{tt}$ satisfies $\tilde{s} = p + l$, $p \in \mathcal{L}^\perp, l \in \mathcal{L}$, and*

$$p - p_{tt}E_{tt} \in T_P \cap T_D, \tag{4.9}$$

where T_P, T_D are defined in (4.1). Then the underlying problems (2.6) and (2.7) admit a finite nonzero duality gap.

Proof.
From Corollary 4.4 and since $\tilde{s}\tilde{x} > 0$, it is enough to show that $\tilde{s}$ and $\tilde{x}$ satisfy

$$\tilde{x} \in T_P + \mathcal{L}, \quad \tilde{s} \in T_D + \mathcal{L}^\perp.$$

First note that $pl = 0$ and $p_{tt} + l_{tt} = 1$ implies that both $p_{tt} > 0$ and $l_{tt} > 0$ and

$$p_{ij} = \begin{cases} 1 - l_{ij} > 0 & \text{if } i = j = t \\ -l_{ij} & \text{if } i \neq t \text{ or } j \neq t. \end{cases}$$

Therefore, we have $\tilde{s} = p + l = p_{tt}E_{tt} + (p - p_{tt}E_{tt}) + l$. This implies that $(1 - p_{tt})\tilde{s} \in T_P + \mathcal{L}$, i.e. $\tilde{s} \in T_P + \mathcal{L}$. Similarly, $\tilde{s} \in T_D + \mathcal{L}^\perp$. ∎

Remark 4.13 *Let*

$$T := T_P \cap T_D = \left[\begin{array}{cc|c|c} f_D^0 = + & + & 0 & * \\ + & + & 0 & + \\ \hline 0 & 0 & X = 0 & 0 \\ \hline * & + & 0 & f_P^0 = + \end{array}\right],$$

denote the block structure of the cone defined in (4.9), where $*$ denotes free elements, $+$ denotes positive semidefinite principal submatrices, and $X = 0$ denotes the tt position. The structure for p in $\tilde{s} = p + l$ that satisfies (4.9) is therefore

$$p \in \left[\begin{array}{cc|c|c} f_D^0 = + & + & 0 & * \\ + & + & 0 & + \\ \hline 0 & 0 & \mathbb{R}_{++} & 0 \\ \hline * & + & 0 & f_P^0 = + \end{array}\right], \tag{4.10}$$

i.e. the structure in (4.10) shows the support, $\mathcal{S}(p)$.

We can apply Corollary 4.4 to different choices of $\tilde{s}$ and $\tilde{x}$ as long as the inner-product $\tilde{s}\tilde{x} > 0$. This allows us to exploit the structure of $\mathcal{L}$, if it is known.

Our main result for the complementarity relationship follows. For simplicity, we consider the self-dual case.

Theorem 4.14 *Let $K = K^*$ be a proper cone. Suppose that the complementarity partition for the pair of faces (f_P^0, f_D^0) fails to be strict, and $\dim G = 1$, where the face $G := (f_P^0)^c \cap (f_D^0)^c \trianglelefteq K$. Then there exists $\tilde{s} = \tilde{x} \in \operatorname{relint} G$ such that the underlying problems (2.6) and (2.7) admit a finite nonzero duality gap.*

Proof.
First note that $\mathcal{L} \cap K = \{0\}$ if and only if $\mathcal{L}^\perp \cap \operatorname{int} K \neq \emptyset$ if and only if $f_P^0 = K$. Using this and a similar result for $\mathcal{L}^\perp \cap K = \{0\}$, we conclude that both f_P^0 and f_D^0 are proper faces of K.

To complete the proof by contradiction, we assume that $(\operatorname{relint} G) \cap \overline{T_P + \mathcal{L}} = \emptyset$, where $T_P := G^c + ((f_D^0)^c)^\perp$ is defined in Corollary 4.4. Then, see e.g. [52, Thm. 11.3], there exists ϕ such that

$$\phi(\operatorname{relint} G) < 0 \leq \phi\left(G^c + ((f_D^0)^c)^\perp + \mathcal{L}\right).$$

Therefore, since $\dim G = 1$,

$$\phi \in (G^c)^* \cap \operatorname{span}(f_D^0)^c \cap \mathcal{L}^\perp, \quad \phi(G \backslash \{0\}) < 0. \tag{4.11}$$

We now choose $v \in \operatorname{relint} \mathcal{L}^\perp \cap K$ and show that

$$\bar{\alpha} v - \phi \in \mathcal{L}^\perp \cap K, \text{ for some sufficiently large } \bar{\alpha} > 0. \tag{4.12}$$

We now identify the cone $(f_D^0)^c$ with the cone T in Proposition 2.16, part 2. We have $0 \neq v \in \operatorname{relint} F := (f_D^0)^c \cap f_P^0 \subseteq \operatorname{span}(f_D^0)^c$ and $-\phi \in \operatorname{relint}(F^c)^* = \operatorname{relint} G^* \cap (f_D^0)^c \subseteq \operatorname{span}(f_D^0)^c$. We further identify v with $\bar{s}$ and $-\phi$ with $\bar{d}$. Then, by Proposition 2.16, part 2, we get that (4.12) holds.

Now (4.12) contradicts the fact that $G f_P^0 = 0$, i.e. since we now have $\bar{\alpha} v - \phi \in f_P^0$, we let $g \in \operatorname{relint} G$ and get $0 = g(\bar{\alpha} v - \phi) = -g\phi > 0$, a contradiction. Therefore, we conclude that we can choose $\tilde{s} \in (\operatorname{relint} G) \cap \overline{T_P + \mathcal{L}}$. Similarly, we can choose $\tilde{x} \in (\operatorname{relint} G) \cap \overline{T_D + \mathcal{L}^\perp}$. Since we now have $\tilde{s}\tilde{x} > 0$, the main conclusion follows from Corollary 4.4. ■

Remark 4.15 *The above proof of Theorem 4.14 obtains a contradiction by separating two sets and obtaining a feasible direction $\bar{\alpha} v - \phi$ that contradicts the definition of f_P^0. A similar contradiction can be obtained by assuming that $\tilde{s} = \tilde{x} \in \operatorname{relint} G$ and that $\tilde{s}$ is not optimal. This yields a feasible direction $\phi \in \mathcal{L}^\perp$, e.g. $s = \tilde{s} + \phi \in K$, with $\phi(G \backslash \{0\}) < 0$. Using a similar argument to that above, we get that necessarily $\phi \in \mathcal{L}^\perp \cap (f_D^0)^c$ and we can obtain (4.12). The assumption that $\dim G = 1$ was needed to ensure that $\phi(G \backslash \{0\}) < 0$. Thus we see the connection between the separation argument and obtaining feasible directions. In the special case $K := \mathbb{S}_+^n$ (with $\dim G = 1$ still), we can find the descent direction using a a simpler proof based on the Schur complement, rather than the hyperplane separation Theorem.*

To obtain a proof for $\dim G > 1$, it suffices to apply a separation argument and obtain $\phi(G \backslash \{0\}) < 0$. Currently, this is still an open problem.

Example 4.16 *We now see that choosing one of $\tilde{s}, \tilde{x}$ in* relint G *may* not *result in a positive duality gap. Consider the* SDP *with data $K = K^* = \mathbb{S}_+^4$, and*

$$A_1 = E_{44}, \; A_2 = E_{24} + E_{33}, \; A_3 = E_{13} + E_{22}.$$

Then

$$f_P^0 = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{R}_+ \end{bmatrix} \trianglelefteq K, \quad f_D^0 = \begin{bmatrix} \mathbb{R}_+ & 0 \\ 0 & 0 \end{bmatrix} \trianglelefteq K^*,$$

and

$$G := (f_P^0)^c \cap (f_D^0)^c = Q\mathbb{S}_+^2 Q^T,$$

where $Q = [e_2\ e_3]$. If $\tilde{s}$ and $\tilde{x}$ are chosen such that $\tilde{s} \in \operatorname{relint}(G)$ and $\tilde{x} \in G$, with $\tilde{x}_{33} > 0$, then the optimal values are both $x_{33}(s_{33} - s_{23}^2/s_{22})$.

However, there exist matrices $\tilde{s}, \tilde{x} \in G$ that are singular on G such that $(\mathbb{P})$ and $(\mathbb{D})$ admit a positive duality gap. For example, if $\tilde{s} = \tilde{x}$ is the diagonal matrix $\tilde{s}\tilde{x} = \operatorname{Diag}((0\ 0\ s_{33}\ 0))$, then the primal optimal value is zero and the dual optimal value is $x_{33}s_{33}$. [1] *Both values are attained at $\tilde{s} = \tilde{x}$. To apply Corollary 4.4, We can write $\tilde{x}$ as the limit*

$$\begin{aligned}
\tilde{x} &= x_{33} \lim_{i\to\infty} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/i \end{bmatrix} \\
&= x_{33} \lim_{i\to\infty} \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & -2i & 0 & -1/2 \\ i & 0 & 1 & 0 \\ 0 & -1/2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2i & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/i \end{bmatrix} + \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&\in \overline{\mathcal{L} + (\operatorname{face}\tilde{s})^c + ((f_D^0)^c)^\perp}.
\end{aligned}$$

However, one can also find examples where choosing $\tilde{s} = \tilde{x} = E_{tt} \in G$ also results in a zero duality gap. This can be seen using $K = K^* = \mathbb{S}_+^4$, and

$$A_1 = E_{44},\ A_2 = E_{22} - E_{33} + E_{24},\ A_3 = E_{22} + E_{33} + E_{14}.$$

Then

$$f_P^0 = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{R}_+ \end{bmatrix} \trianglelefteq K, \quad f_D^0 = \begin{bmatrix} \mathbb{R}_+ & 0 \\ 0 & 0 \end{bmatrix} \trianglelefteq K^*,$$

and

$$G := (f_P^0)^c \cap (f_D^0)^c = Q\mathbb{S}_+^2 Q^T,$$

where $Q = [e_2\ e_3]$. ■

4.2 Infinite Duality Gap and Devious Faces

As we have already noted, $(\mathbb{D})$ is feasible if and only if $\tilde{x} \in K^* + \mathcal{L}$. Moreover feasiblility of $(\mathbb{D})$ is equivalent to a finite duality gap, recalling our assumption that the primal optimal value v_P is finite. We now see that if a nice cone has a devious face, then it is easy to construct examples with an infinite duality gap.

Proposition 4.17 *Suppose that K is a nice, proper cone and F is a devious face of K, i.e.,*

$$K^* + (F^c)^\perp = \overline{K^* + \operatorname{span} F} \text{ and } (K^* + \operatorname{span} F) \text{ is not closed.}$$

[1] Similarly, we can use the 2, 2 position rather than the 3, 3 position.

Let $\mathcal{L} = \operatorname{span} F$ and choose $c = \tilde{s} = 0$ and $\tilde{x} \in (K^* + (F^c)^\perp)\backslash(K^* + \mathcal{L})$. Then $(\tilde{x} + \mathcal{L}) \cap K^* = \emptyset$ and we get $v_D = +\infty$. Moreover, $\mathcal{L}^\perp = F^\perp$ and, for every feasible $s \in F^\perp \cap K$,

$$\tilde{x}s = \left(\tilde{x}_{K^*} + \tilde{x}_{(F^c)^\perp}\right) s \geq 0,$$

i.e., $0 = v_P < v_D = \infty$.

Proof.
The proof follows from the definitions. ∎

Proposition 4.17 can be extended to choosing any $\mathcal{L}$ that satisfies $K^* + \mathcal{L}$ is not closed and $K^* + \mathcal{L} \subseteq K^* + (F^c)^\perp$.

Example 4.18 *Let $K = \mathbb{S}^2_+$, and suppose that $(\mathbb{P})$ and $(\mathbb{D})$ admit a nonzero duality gap. Then Slater's CQ fails for both primal and dual, i.e., $\{0\} \neq f_P^0 \subsetneq \mathbb{S}^2_+$ and $\{0\} \neq f_D^0 \subsetneq \mathbb{S}^2_+$. After a rotation (see Lemma 2.12) we can assume the problem has the structure*

$$\begin{bmatrix} f_D^0 & 0 \\ 0 & f_P^0 \end{bmatrix},$$

viz., the matrices in f_D^0 are nonzero only in the $(1,1)$ position, and the matrices in f_P^0 are nonzero only in the $(3,3)$ position. There are only three possible options for $\mathcal{L}$: $\operatorname{span}\{E_{11}\}$, $\operatorname{span}\{E_{22}\}$, $\operatorname{span}\{E_{11}, E_{12}\}$, or $\operatorname{span}\{E_{22}, E_{12}\}$. In each case, either $\mathcal{L}$ is one-dimensional and $\mathcal{L}^\perp$ is two-dimensional, or vice versa. So without loss of generality, we may choose $\mathcal{L} = \operatorname{span}\{E_{11}\}$. Now take $\tilde{x} = E_{12} \in \mathbb{S}^2_+ + (f_P^0)^\perp$. Then

$$\tilde{x} \notin \mathbb{S}^2_+ + \mathcal{L} = \mathbb{S}^2_+ + \operatorname{span} f_D^0 \subsetneq \mathbb{S}^2_+ + (f_P^0)^\perp, \tag{4.13}$$

and the dual program $(\mathbb{D})$ is infeasible. But choosing $c = \tilde{s} = E_{22}$ implies that the primal optimal value $v_P = c\tilde{x} - y_1 E_{22}\tilde{x} = 0 < v_D = +\infty$.

Corollary 4.19 *If $K = \mathbb{S}^2_+$, then a finite positive duality gap cannot occur.* ∎

The above Corollary 4.19 also follows from [55, Prop. 4], i.e., it states that a finite positive duality gap cannot happen if $\dim \mathcal{W} \leq 3$.

4.3 Regularization for Strict Complementarity

Suppose that strong duality holds for both the primal and dual SDPs, but strict complementarity fails for every primal-dual optimal solution $(\bar{s}, \bar{x}) \in \mathbb{S}^n_+ \otimes \mathbb{S}^n_+$. Following [60], $(\bar{s}, \bar{x})$ is called a *maximal complementary solution pair* if the pair maximizes the sum $\operatorname{rank}(s) + \operatorname{rank}(x)$ over all primal-dual optimal (s, x). The *strict complementarity nullity*, $g := n - \operatorname{rank}(\bar{s}) - \operatorname{rank}(\bar{x})$.

Let $\mathcal{U} = \mathcal{N}(s) \cap \mathcal{N}(x)$ be the common nullspace of dimension g, and U be the $n \times (n-g)$ matrix with orthonormal columns satisfying $\mathcal{R}(U) = \mathcal{U}$. Let $\begin{bmatrix} U & Q \end{bmatrix}$ be an orthogonal matrix. Then we can regularize so that strict complementarity holds by replacing both primal-dual variables s, x by QsQ^T, QxQ^T, respectively. This is equivalent to replacing the matrices $A_i, i = 1, \ldots, m, c$ that define $\mathcal{A}$ by $Q^T A_i Q, i = 1, \ldots, m, c$. (Note that we would then have to check for possible linear dependence of the new matrices $Q^T A_i Q$, as well as possible loss of Slater CQ .) Finding the common nullspace can be done dynamically during the solution process. This is done by checking the ratios of eigenvalues of s and x between iterates to see if the convergence is to 0 or to $O(1)$. (In the case of LP , this corresponds to identifying nonbasic variables using the so-called Tapia indices, see e.g. [25].)

5 Conclusion

In this paper we have looked at known, and new, duality and optimality results for the (primal) cone optimization problem ($\mathbb{P}$). We have used the subspace formulations of the primal and dual problems, (2.6),(2.7), to provide new CQs and new optimality conditions that hold without any CQ. This includes a UCQ, i.e., a CQ that holds independent of both data vectors b and c. In particular, the optimality characterizations show that a *minimal representation* of the cone and/or the linear transformation of the problem results in regularization, i.e., efficient modelling for the cone K and for the primal and dual constraints results in a stable formulation of the problem.

In addition, we have discussed conditions for a zero duality gap and the surprising relations to complementarity of the homogeneous problem and to the closure of sums of cones.

The (near) failure of Slater's CQ relates to both theoretical and numerical difficulties. The same holds true for failure of strict complementarity. We have discussed regularization procedures for both failures. We hope that these results will lead to preprocessing for current cone optimization software packages.

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