Semidefinite relaxation of the max-cut problem

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Outline

Project Statement

Problem

Find the SDP relaxation of the Max-Cut problem:

1. Solve this relaxation using e.g. SeDumi with MATLAB.
2. Use randomly generated, weighted, undirected graphs, e.g. `W=sprandsym(60,.5)`.
3. Can you strengthen the relaxation by adding additional constraints? (Hint: consider constraints of the type $x_i x_k^2 x_j = x_i x_j$.)

Let $G = (V, E)$ be a graph and let $w : E \to \mathbb{R}$ be an edge weight function on G and set $n = |V|$.

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A Review of Max-cut

$$w_{ij} := \begin{cases} w((i,j)) & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- Let $W = (w_{ij})$, the matrix whose ij-th entry is w_{ij}.
- A cut is a partition of V into two sets $C \subset V$ and $V \setminus C$. Its size is

$$s(C) := \sum_{i \in C, j \in V \setminus C} w_{ij}$$

- A cut C is maximal if $\forall\ \tilde{C} \in 2^V, s(C) \geq s(\tilde{C})$

Max-Cut Problem

For a weighted graph (V, E, w) find a maximal cut C.

Remark

The Max-Cut problem is NP-hard.

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We reforumlate the Max-Cut problem:

Problem: Reformulation of Max-Cut Problem

Given any cut C, define

$$x := (x_1, \ldots, x_n)^T, \quad x_i := \begin{cases} 1 & i \in C \\ -1 & i \notin C \end{cases}$$

$$\begin{array}{rl} \text{Maximize} & t = \frac{1}{2} \sum_{i<j} w_{ij}(1 - x_i x_j) \\ \text{subject to} & x_i \in \{-1, 1\} \ \forall\, i \in V. \end{array}$$

Let $e := (1, 1, \ldots, 1)^T$, where $e \in \mathbb{R}^n$.

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Some Equalities for t

Skip long string of equalities for t

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t &= \frac{1}{2}\sum_{i<j} w_{ij}(1 - x_i x_j) = \frac{1}{4}\left(2\left(\sum_{i<j} w_{ij}\right) - 2\left(\sum_{i<j} w_{ij} x_i x_j\right)\right) \\
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Reformulation and Relaxation of Max-Cut Problem

- $t = x^T Lx = trace(x^T Lx) = trace(x^T(Lx)) = trace((Lx)x^T) = trace(LX) = L \bullet X$, where $X = xx^T$ is a real symmetric matrix, $\implies X \succeq 0$.
- $diag(X) = e \iff x_i^2 = 1 \iff x_i \in \{-1, 1\}$.

Lemma

$X = X^T \succeq 0, \text{rank}(X) = 1 \iff X = xx^T$

W is the edge-weight matrix $\rightsquigarrow L = \frac{1}{4}(Diag(We) - W) \rightsquigarrow$

Problem: Equivalent Reformulation of Max-Cut Problem

$$\begin{array}{ll} \text{Maximize} & L \bullet X \\ \text{subject to} & diag(X) = e, rank(X) = 1, \text{ and } X \succeq 0. \end{array}$$

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W is the edge-weight matrix $\rightsquigarrow L = \frac{1}{4}(Diag(We) - W) \rightsquigarrow$

Problem: Equivalent Reformulation of Max-Cut Problem

$$\begin{array}{ll} \text{Maximize} & L \bullet X \\ \text{subject to} & diag(X) = e, rank(X) = 1, \text{ and } X \succeq 0. \end{array}$$

Reformulation and Relaxation of Max-Cut Problem

- $t = x^T Lx = trace(x^T Lx) = trace(x^T(Lx)) = trace((Lx)x^T) = trace(LX) = L \bullet X$, where $X = xx^T$ is a real symmetric matrix, $\Longrightarrow X \succeq 0$.
- $diag(X) = e \Longleftrightarrow x_i^2 = 1 \Longleftrightarrow x_i \in \{-1, 1\}$.

Lemma

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W is the edge-weight matrix $\rightsquigarrow L = \frac{1}{4}(Diag(We) - W) \rightsquigarrow$

Problem: SDP Relaxation of Max-Cut Problem

$$\begin{array}{ll} \text{Maximize} & L \bullet X \\ \text{subject to} & diag(X) = e \text{ and } X \succeq 0. \end{array}$$

Setting up an Instance

We setup an instance of the Max-Cut problem in MATLAB with the following code:

```
% Fix the dimension (number of vertices in graph)
n=60;

% Generate a weighted graph
W=sprandsym(n,.5); W(1:n+1:n^2)=zeros(1,n);

% Calculate the Laplacian
L=1/4 * (diag(W*ones(n,1))-W);
```

Interfacing with SeDumi

To be consistent with the notation of SeDumi's documentation and interface, substitute "x" for "z". The SeDumi primal form is

$$\begin{array}{rl} \text{Minimize} & c^T z \\ \text{subject to} & Az = b, \text{ for } z \in \mathcal{K} \end{array}$$

where $c, z, b \in \mathbb{R}^p$, and A is a $p \times p$ matrix. In MATLAB,

```
c=-L(:); % Format the Laplacian to vector form

% Translate constraint diag(X)=e into vector format:
A=sparse(1:n,1:n+1:n^2,ones(1,n),n,n^2);
b=ones(n,1);

% Tell SeDumi that X must be positive semidefinite
K.s=[n];

[X,Y,INFO] = sedumi(A,b,c,K) % Run SeDumi
```

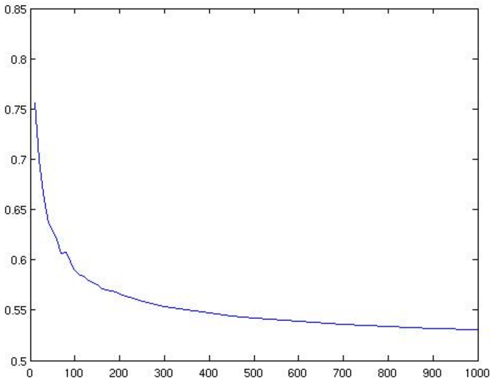
Interfacing with SeDumi

A visual summary of $Az = b$

$$
\left(\begin{array}{cccc|ccccc|ccccc|c|ccc}
1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & & 0 & \dots & 1
\end{array}\right)_{n \times n^2}
\left(\begin{array}{c}
x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \\ \hline x_{21} \\ x_{22} \\ x_{23} \\ \vdots \\ x_{2n} \\ \hline x_{31} \\ x_{32} \\ x_{33} \\ \vdots \\ x_{3n} \\ \hline \vdots \\ \hline x_{n1} \\ \vdots \\ x_{nn}
\end{array}\right)_{n^2 \times 1}
=
\left(\begin{array}{c}
1 \\ 1 \\ 1 \\ \vdots \\ 1
\end{array}\right)_{n \times 1}
$$

Numerical Results: Basic Relaxation $G(n, \frac{1}{2})$

n	**Ratio**
50	6.292239e-01
60	6.202113e-01
70	6.067923e-01
80	6.074731e-01
90	5.980219e-01
100	5.894269e-01
110	5.849831e-01
120	5.837704e-01
130	5.793048e-01
140	5.777026e-01
150	5.749815e-01
160	5.715093e-01
170	5.702395e-01
180	5.686022e-01
190	5.675917e-01
200	5.654331e-01
250	5.591398e-01
300	5.538571e-01
350	5.506719e-01
400	5.472742e-01
450	5.441005e-01
500	5.422019e-01
600	5.390324e-01
700	5.359528e-01
800	5.337012e-01
900	5.320470e-01
1000	5.303538e-01



Triangle Inequalities

An idea proposed by Poljak and Rendl (1995), and further developed by Helmberg, Rendl, Vanderbei and Wolkowicz (1996):

For each triple, $x_i, x_j, x_k \in \{-1, 1\}$ we have

$$\begin{aligned} x_{ij} + x_{jk} + x_{ki} &\geq -1 \quad (1) \\ x_{ij} - x_{jk} - x_{ki} &\geq -1 \quad (2) \\ -x_{ij} + x_{jk} - x_{ki} &\geq -1 \quad (3) \\ -x_{ij} - x_{jk} + x_{ki} &\geq -1. \quad (4) \end{aligned}$$

With (1)-(4), we have the strengthened SDP relaxation:

Problem: Strengthened Relaxation of Max-Cut Problem

$$\begin{aligned} &\text{Maximize} && L \bullet X \\ &\text{subject to} && diag(X) = e, X \succeq 0, \text{ and } (1) - (4). \end{aligned}$$

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Triangle Inequalities

Replacing Inequalities (1)-(4) with Equalities for SeDumi's Input Format

Introduce slack variables $s_{ijk\ell} \geq 0$ to get equalities for SeDumi.

$$\begin{aligned} x_{ij} + x_{jk} + x_{ki} - s_{ijk1} &= -1 \\ x_{ij} - x_{jk} - x_{ki} - s_{ijk2} &= -1 \\ -x_{ij} + x_{jk} - x_{ki} - s_{ijk3} &= -1 \\ -x_{ij} - x_{jk} + x_{ki} - s_{ijk4} &= -1 \end{aligned}$$

- Considering (1)-(4) when a pair of i, j, k are the same is redundant.
- So we are concerned with "only" $N := 2n(n-1)(n-2)/3$ slack variables.

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Triangle Inequalities

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\left(\begin{array}{c|cccc|cccccc|cccccc|c|ccc}
 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & & 0 & 0 & 0 & \dots & 0 & & & 0 & \dots & 0 \\
 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & & 0 & 0 & 0 & \dots & 0 & & & 0 & \dots & 0 \\
0_{n\times N} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & & 0 & 0 & 1 & \dots & 0 & & \dots & 0 & \dots & 0 \\
 & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & & & \vdots & \ddots & \vdots \\
 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & & 0 & 0 & 0 & \dots & 0 & & & 0 & \dots & 1 \\
\hline
-I_{N\times N} & \multicolumn{20}{c}{tempA_{N\times n^2}}
\end{array}\right)
\left(\begin{array}{c}
s_{1,2,3,1} \\ s_{1,2,4,1} \\ \vdots \\ s_{1,3,4,1} \\ s_{1,3,5,1} \\ \vdots\vdots \\ s_{2,3,4,1} \\ s_{2,3,5,1} \\ \vdots\vdots \\ s_{n-2,n-1,n,1} \\ \hline
s_{1,2,3,2} \\ \vdots\vdots \\ s_{n-2,n-1,n,2} \\ \hline
\vdots\vdots\vdots \\ \hline
s_{1,2,3,4} \\ \vdots\vdots \\ s_{n-2,n-1,n,4} \\ \hline\hline
x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \\ \hline
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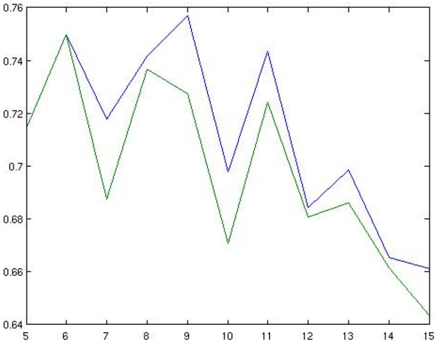
\vdots \\ \hline
x_{n1} \\ \vdots \\ x_{nn}
\end{array}\right)_{N+n^2\times 1}
= \left(\frac{I_{n\times 1}}{-I_{N\times 1}}\right)_{n+N\times 1}
$$

Numerical Results: Triangle Relaxation $G(n, \frac{1}{2})$

n	**Ratio 1**	**Ratio 2**
5	7.142857e-01	7.142857e-01
6	7.500000e-01	7.500000e-01
7	7.177745e-01	6.875000e-01
8	7.417858e-01	7.368421e-01
9	7.570369e-01	7.272727e-01
10	6.978259e-01	6.709667e-01
11	7.433614e-01	7.241379e-01
12	6.845083e-01	6.808511e-01
13	6.986989e-01	6.862745e-01
14	6.655176e-01	6.615384e-01
15	6.613215e-01	6.433655e-01



Quadruple Equalities

An idea of Goemans developed by Anjos and Wolkowicz (2002):

For each quadruple $x_i, x_j, x_j, x_k \in \{-1, 1\}$ we have:

$$x_i x_j x_j x_k = x_i x_j^2 x_k = x_i x_k. \quad (5)$$

With (5), we have the SDP formulation:

Problem: An Equivalent Formulation of Max-Cut Problem

Maximize $L \bullet X$
subject to $diag(X) = e, X \succeq 0,$ and $x_{ij} x_{jk} = x_{ik}$.

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Second lifting/relaxation

After lifting

$$Y = \begin{bmatrix} y & y^T \end{bmatrix}, \text{ where } y = \text{vec}(X)$$

We have the max-cut problem:

$$\begin{array}{lll} \text{Maximize} & \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \bullet Y = L \bullet Y_{11} & \\ \text{subject to} & diag(Y) = e & \text{(S1)} \\ & Y \succeq 0 & \text{(S2)} \\ & Y_{11} = Y_{ii} \;\; \forall i = 2, \ldots, n & \text{(S3)} \\ & \text{diag}(Y_{ij}) = Y_{ii}^{(i,j)} e & \text{(S4)} \\ & Y_{ij} e_j = Y_{11} e_i \;\; \forall j = 2, \ldots n, \; i < j & \text{(S5)} \\ & \text{rank}(Y) = 1. & \text{(S6)} \end{array}$$

Theorem

The second lifting relaxation provides a tighter upper bound.

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Tighter Upper Bound Theorem

- Suppose that X^* is an optimal solution to the first lifting relaxation.
  - If $Rank(X^*) = 1$, then we have the optimal solution to the original max-cut problem.
  - If $Rank(X^*) > 1$, we will show that after lifting X^* to $\overline{Y} = (yy^T)$, where $y = \text{vec } X^*$, $\overline{Y}$ is not feasible to the second relaxation.
    - By contradiction, we assume that $\overline{Y}$ is feasible to the second relaxation, then it satisfies (S4) and (S5), which implies that $X^*_{ik}X^*_{kj} = X^*_{ij}$ for all i, j, k.
    - Hence, $X^* = xx^T$, where $x = (X^*_{1k}, X^*_{2k}, \cdots, X^*_{nk})$, which contradicts Rank $X^* > 1$.
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Numerical Results: Comparison of Algorithms

Comparison of Algorithm Performance

relaxation	n=4	n = 6	n = 8	n = 10
SDP	2.00000000	6.43225787	9.52050490	17.0875592
TRI	1.99999999	5.99999995	8.99999986	16.9999998
QUAD	1.99999999	5.99999988	8.99999900	16.9999996

Thanks!!

Thanks for your attention!

Semidefinite relaxation of the max-cut problem

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