UBC Math 604 — Assignment 7

Due by 4:00 p.m. Friday 23 March 2001

Consider a linear inverse problem where the function to be determined, unknown to the experimentalist, is the element of $L^2[0,1]$ defined by

$$x_{\text{exact}}(t) = \begin{cases} 1, & \text{if } a \le t < b, \\ 0, & \text{otherwise,} \end{cases} \quad , \qquad \text{for } a = 1/4,\ b = 1/2.$$

The experimentalist knows only the m numbers ("data values")

$$d_k = \int_0^1 g_k(t) x_{\text{exact}}(t)\,dt + n_k, \qquad k = 1, 2, \ldots, m, \tag{$*$}$$

defined in terms of random noises n_k and known kernel functions

$$g_k(t) \stackrel{\text{def}}{=} t^{k-1} e^{-ct} \qquad k = 1, 2, \ldots, m.$$

(In operator notation, $d = Gx + n$, where $(Gx)_k = \langle g_k\,,\, x\rangle$ for $k = 1, \ldots, m$.)

A. For the parameter values $c = 1$, $m = 9$, carry out these steps:

(a) Find $N(x_{\text{exact}})$, recalling that $N(x) \stackrel{\text{def}}{=} \frac{1}{2}\|x\|^2 = \frac{1}{2}\int_0^1 x(t)^2\,dt$.

(b) Assuming all $n_k = 0$, use $(*)$ to compute the (noise-free) data vector $d = (d_1, \ldots, d_m)$. (Accurate numerical computation is acceptable here, even though a tedious hand calculation can provide an exact result.)

(c) Among all functions x satisfying $Gx = d$, find the one for which $N(x)$ is smallest. Report the minimum N-value, and plot the function you find and the function x_{exact} on the same set of axes.

(d) Now treat the data vector d found in (b) as an experimental measurement, imagining that it comes from line $(*)$. In the absence of precise knowledge of the standard deviation σ_k associated with noise process n_k, assume $\sigma_k = |d_k|/10$. (All component errors are independent.) Set up the corresponding covariance matrix S, and define

$$E(x) = \tfrac{1}{2}\|Gx - d\|^2_{S^{-1}}.$$

Then use 30 logarithmically-spaced μ-values (use Matlab's `logspace`), to produce a plot of 30 points on the "tradeoff curve" between model norm and data-fitting accuracy in the (E, N)-plane. Also print a table of values whose rows show μ, cond(LHS), $N(x)$, and $E(x)$, where x is the minimizer. Here cond(LHS) denotes the (2-norm) condition number of the matrix on the left-hand side of the $m \times m$ linear system you must solve as one of the key steps in finding x. Comment (with optional plots) on the relationship between μ and cond(LHS).

(e) Use the results from (d) to find a μ-value for which the minimum gives a vector x with misfit $\|Gx - d\|^2_{S^{-1}}$ between 8 and 9, and a second μ-value for which this misfit lies between 9 and 10. [Recall that the expected value of this quantity equals 9, the number of data components.] Report both these values, and plot the two reconstructed functions x corresponding to them on the same set of axes. Sketch x_{exact} on these axes as well.

(f) Find the point on the tradeoff curve in (d) nearest to the origin. For this point, plot both x and x_{exact} as in parts (c) and (e). (This part will require some combination of mathematical analysis and computation: the desired point lies on the exact tradeoff curve, and is not necessarily one of the points you have already plotted in (e).)

Hint. The integrals you must compute can all be reduced by a change of variable to values of the incomplete Gamma function, defined by

$$\Gamma_{\text{inc}}(r, m) = \frac{1}{\Gamma(m)} \int_0^r t^{m-1} e^{-t}\, dt, \quad \text{where} \quad \Gamma(m) \stackrel{\text{def}}{=} \int_0^\infty t^{m-1} e^{-t}\, dt.$$

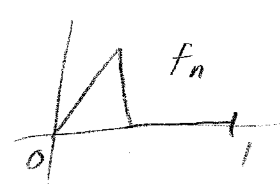
Both $\Gamma(m)$ and $\Gamma_{\text{inc}}(r, m)$ are built-in functions in Matlab.

B. Produce a series of three plots showing the effects of changing the decay rate c and the number of data points m: the values of c on these three plots should be $c = 0.25$, $c = 1$, $c = 4$, respectively. Each plot should include the graphs of x_{exact} and the three minimum-norm functions x that obey $Gx = Gx_{\text{exact}}$ corresponding to $m = 11, 13, 15$. (For c-values where the matrix conditioning is so bad that even Matlab complains, use instead the largest three odd values of m for which no warning is produced.)

C. Explain why the limiting value as $c \to 0^+$ of the matrix

$$\Gamma = GG^* = \begin{bmatrix} \langle g_1\,,\, g_1\rangle & \langle g_1\,,\, g_2\rangle & \langle g_1\,,\, g_3\rangle & \cdots & \langle g_1\,,\, g_m\rangle \\ \langle g_2\,,\, g_1\rangle & \langle g_2\,,\, g_2\rangle & \langle g_2\,,\, g_3\rangle & \cdots & \langle g_2\,,\, g_m\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle g_m\,,\, g_1\rangle & \langle g_m\,,\, g_2\rangle & \langle g_m\,,\, g_3\rangle & \cdots & \langle g_m\,,\, g_m\rangle \end{bmatrix}$$

is the Hilbert matrix of size $m \times m$ ("`hilb(m)`" in Matlab). This observation may be useful in debugging part A.



$f = \lim f_n = 0$

$\int f_n = 1 \neq \int f$

$\lim_{c \to 0^+} \langle g_i, g_j \rangle = \frac{1}{i+j-1}$

$f_n(t) \to 0$
$\forall t \in [0,1]$