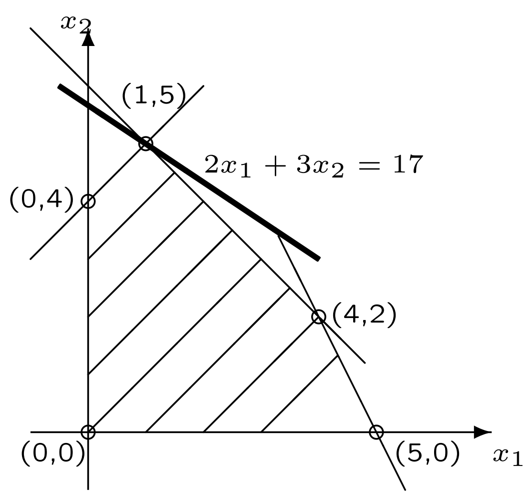
Recap

Terminology

- Objective function
- Constraints
- Feasible solution
- Feasible region
- Objective value
- Optimal solution
- Optimal value

Geometry of LP (Orange Factory Problem)



More Examples: A transportation problem

$$
\begin{array}{lrcll}
\text{minimize} & \sum_{i=1}^{p} \sum_{j=1}^{q} c_{ij} x_{ij} & & & \\
\text{subject to} & \sum_{j=1}^{q} x_{ij} & = & s_i & (i = 1, 2, \ldots, p) \\
& \sum_{i=1}^{p} x_{ij} & = & t_i & (j = 1, 2, \ldots, q) \\
& x_{ij} & \geq & 0 & \begin{pmatrix} i = 1, 2, \ldots, p, \\ j = 1, 2, \ldots, q \end{pmatrix}
\end{array}
$$

Some More Examples

Overdetermined System of Equations

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \ (i = 1, 2, \ldots, m)$$

Suppose the system does not have a solution.

Problem: Find $x \in \mathbf{R}^n$ that is "closest" to solving the system.

Error of solution x is

$$\sum_{i=1}^{m} e_i$$

where

$$e_i := \left| \sum_{j=1}^{n} a_{ij}x_j - b_i \right|$$

Mathematical model

$$\text{minimize} \ \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij}x_j - b_i \right|$$

Not an LP!

Convert to LP

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{m} e_i \\ \text{subject to} & \left| \displaystyle\sum_{j=1}^{n} a_{ij}x_j - b_i \right| = e_i \quad (i = 1, 2, \ldots, m) \end{array}$$

Convert to LP (cont'd)

$$\text{minimize} \quad \sum_{i=1}^{m} \overset{y_i}{\cancel{e_i}}$$

$$\text{subject to} \quad \left| \sum_{j=1}^{n} a_{ij}x_j - b_i \right| \;\overset{\leq}{\cancel{=}}\; \overset{y_i}{\cancel{e_i}} \quad (i = 1, 2, \ldots, m)$$

Observation: (x^*, y^*) optimal $\implies y_i^* = e_i \ (i = 1, 2, \ldots, m)$.

$$\left| \sum_{j=1}^{n} a_{ij}x_j - b_i \right| \leq y_i \iff -y_i \leq \sum_{j=1}^{n} a_{ij}x_j - b_i \leq y_i$$

$$\begin{array}{lrcll} \text{minimize} & \displaystyle\sum_{i=1}^{m} y_i & & & \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij}x_j - b_i - y_i & \leq & 0 & (i = 1, 2, \ldots, m) \\ & \displaystyle\sum_{j=1}^{n} a_{ij}x_j - b_i + y_i & \geq & 0 & (i = 1, 2, \ldots, m) \end{array}$$

CO350 Linear Programming Chapter 2: Optimality and Its Alternatives

6th May 2005

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Optimality

Recall our proof that $[1,5]^T$ is an optimal solution for the Orange Factory Problem:

$$\begin{array}{lrcrcrl} \text{maximize} & 2x_1 & + & 3x_2 & & & \\ \text{subject to} & & & & & & \\ & 2x_1 & + & x_2 & \leq & 10 & \text{— (1)} \\ & x_1 & + & x_2 & \leq & 6 & \text{— (2)} \\ & -x_1 & + & x_2 & \leq & 4 & \text{— (3)} \\ & x_1 & , & x_2 & \geq & 0 & \end{array}$$

$x_1 = 1$ and $x_2 = 5$ satisfy all inequalities and achieves a profit of 17.
$(2) + (3) : 2x_2 \leq 10 \implies x_2 \leq 5$ — (4)
So, profit $= 2x_1 + 3x_2 = 2(x_1 + x_2) + x_2 \leq 2 \times 6 + 5 = 17$.
Thus, profit $\leq 12 + 5 = 17 \implies [1,5]^T$ optimal.

In short, profit $= 2x_1 + 3x_2 = 2 \times$ eq.(2) + eq.(4)

$$\begin{aligned} &= 2 \times \text{eq.}(2) + \tfrac{1}{2} \times [\text{eq.}(2) + \text{eq.}(3)] \\ &= \tfrac{5}{2} \times \text{eq.}(2) + \tfrac{1}{2} \times \text{eq.}(3) \\ &\leq \tfrac{5}{2} \times 6 + \tfrac{1}{2} \times 4 = 17 \end{aligned}$$

We shall see in "Chapter 6: The Simplex Method" how to obtain such proof in general.

Infeasibility

An LP problem is said to be infeasible if it does not have any feasible solution.

Example:

$$\begin{array}{ll} \text{maximize} & x_1 \\ \text{subject to} & \end{array}$$

$$\begin{array}{rcrcrcrl} x_1 & - & 2x_2 & + & 2x_3 & = & 2 & \text{— (1)} \\ -x_1 & + & 3x_2 & - & x_3 & = & -3 & \text{— (2)} \\ x_1 & , & x_2 & , & x_3 & \geq & 0 & \text{—(4)} \end{array}$$

$(1)+(2):$ $\quad x_2 + x_3 = -1$

(4) implies $\quad x_2 + x_3 \geq 0$

Contradiction!

> We shall see in "Chapter 7: The Two-Phase Method" how to obtain such proof in general.

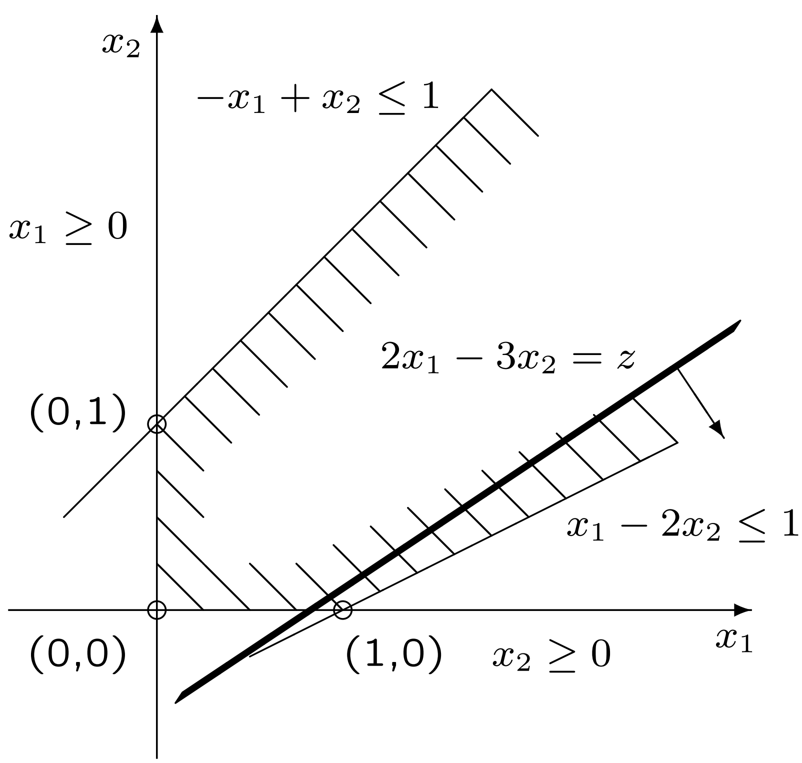
Unboundedness

An LP problem is said to be unbounded if there exist feasible solutions of arbitrarily good objective value.

I.e., for maximization (or minimization) objective, there are feasible solutions with value as high (or low) as one wishes.

Example:

$$\begin{array}{lrcrcl} \text{maximize} & 2x_1 & - & 3x_2 & & \\ \text{subject to} & -x_1 & + & x_2 & \leq & 1 \\ & x_1 & - & 2x_2 & \leq & 1 \\ & x_1 & , & x_2 & \geq 0 & \end{array}$$



Example:

$$\begin{array}{llcrcl} \text{maximize} & 2x_1 & - & 3x_2 & & \\ \text{subject to} & -x_1 & + & x_2 & \leq & 1 \\ & x_1 & - & 2x_2 & \leq & 1 \\ & x_1 & , & x_2 & \geq 0 & \end{array}$$

Let
$$\begin{aligned} x_1(t) &= 1 + 2t \\ x_2(t) &= t \end{aligned}$$

Claim: When $t \geq 0$, $[x_1(t), x_2(t)]^T$ is feasible.

Proof:
$$\begin{aligned} -x_1(t) + x_2(t) &= -1 - 2t + t = -1 - t \leq -1 \leq 1 \\ x_1(t) - 2x_2(t) &= 1 + 2t - 2t = 1 \leq 1 \\ x_1(t) &= 1 + 2t \geq 1 \geq 0 \\ x_2(t) &= t \geq 0 \end{aligned}$$

$\Longrightarrow$ $[x_1(t), x_2(t)]^T$ is feasible. □

Objective value of $[x_1(t), x_2(t)]^T$:
$$2x_1(t) - 3x_2(t) = 2(1 + 2t) - 3t = 2 + t$$
can be made as high as one wishes by choosing t large.

Conclusion: The LP is unbounded.

> We shall see in "Chapter 6: The Simplex Method" how to obtain such proof in general.

A Preview: Looking Ahead

The Fundamental Theorem of LP

There are exactly **three** possibilities for each LP problem.

1. It has an optimal solution;
2. It is infeasible;
3. It is unbounded.

Will be proved much later (after mid-term).

Duality Theory

The algebraic arguments that we saw for specific examples can be made general with the help of duality theory so that they apply to any LP problem.

Basic Solutions

The geometric picture of having an optimal solution at some "corner point" is, in some sense, accurate. Algebraically, these "corner points" will be described as feasible solutions that are basic.

Simplex Method

Motivated by the preceding paragraph, we will develop a practical algorithm called the simplex method to solve LP problems.

Matrix Notation

The LP

$$\begin{array}{lrcll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & (i = 1, \ldots, m) \\ & x_j & \geq & 0 & (j = 1, \ldots, n) \end{array}$$

written in matrix notation is

$$\begin{array}{lrcl} \text{maximize} & c^T x & & \\ \text{subject to} & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$c^T = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}$$

Example

$$\begin{array}{lrcrcr} \text{maximize} & 2x_1 & + & 3x_2 & & \\ \text{subject to} & & & & & \\ & 2x_1 & + & x_2 & \leq & 10 \\ & x_1 & + & x_2 & \leq & 6 \\ & -x_1 & + & x_2 & \leq & 4 \\ & x_1 & , & x_2 & \geq & 0 \end{array}$$

in matrix notation is

$$\begin{array}{lrcl} \text{maximize} & c^T x & & \\ \text{subject to} & & & \\ & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

with

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 6 \\ 4 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$c^T = \begin{bmatrix} 2 & 3 \end{bmatrix}$$