CO350 Linear Programming Chapter 4: Introduction to Duality

25th May 2005

Economic Interpretation of Duality (Section 4.6, Pg 51)

General Production Problem

A factory makes n products from m resources. Each unit of product j requires a_{ij} units of resource i and makes a profit of c_j dollars. Each day, the factory has b_j units of resource i available.

Mathematical Model

$$\begin{array}{lrcll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & (i = 1, 2, \ldots, m) \\ & x_j & \geq & 0 & (j = 1, 2, \ldots, n) \end{array}$$

Suppose the resource constraints are not hard; i.e., we can either buy or sell each resources at certain fixed price.

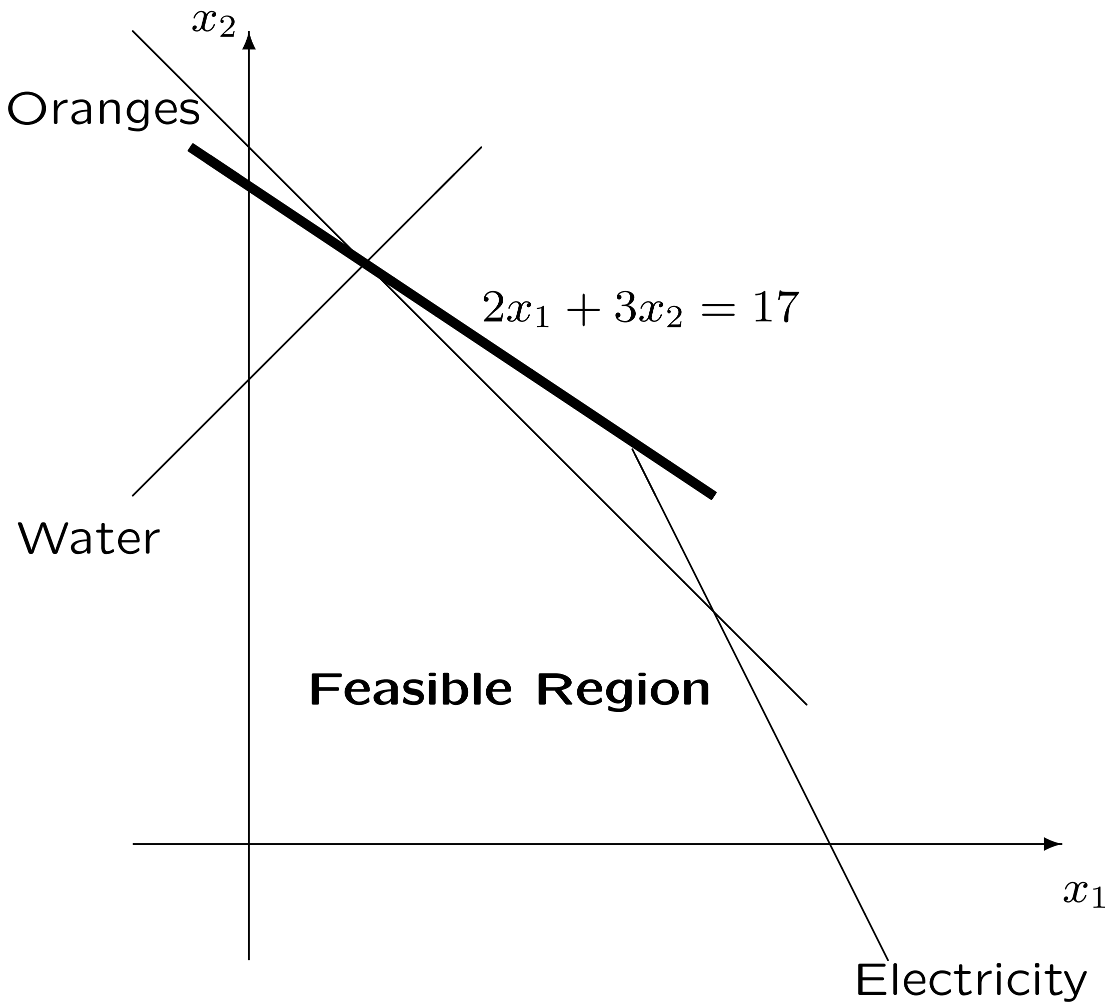
What is the fair price for each resource?

Fair Price: There is no advantage to either buy or sell small amount the resource.

Fair prices are also called shadow prices.

Example (Not in notes)

$$
\begin{array}{lrcrcrl}
\text{maximize} & 2x_1 & + & 3x_2 & & & \\
\text{subject to} & 2x_1 & + & x_2 & \leq & 10 & \text{(Electricity)} \\
 & x_1 & + & x_2 & \leq & 6 & \text{(Oranges)} \\
 & -x_1 & + & x_2 & \leq & 4 & \text{(Water)} \\
 & x_1 & , & x_2 & \geq & 0 &
\end{array}
$$



Question: What is the fair price on water?
Answer: $\frac{1}{2}$

Let $x_j =$ amount of product j produced,

$y_i =$ Fair price of resource i.

1. $y_i < 0 \implies$ advantageous to buy resource i. Contradiction. So $\boxed{y_i \geq 0.}$

2. ONE unit of product j cost $\sum_{i=1}^{m} a_{ij} y_i$.

   $\sum_{i=1}^{m} a_{ij} y_i < c_j \implies$ advantageous to buy resources to make product j. Contradiction. So $\boxed{\sum_{i=1}^{m} a_{ij} y_i \geq c_j.}$

3. $\sum_{j=1}^{n} a_{ij} x_j < b_i$ and $y_i > 0 \implies$ advantageous to sell resource i. Contradiction. So $\boxed{\sum_{j=1}^{n} a_{ij} x_j = b_i \text{ or } y_i = 0.}$

4. $x_j > 0$ and $\sum_{i=1}^{m} a_{ij} y_i > c_j \implies$ advantageous to not make product j and sell the associated resources.

   Contradiction. So $\boxed{x_j = 0 \text{ or } \sum_{i=1}^{m} a_{ij} y_i = c_j.}$

So x_j's and y_i's satisfy

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\leq b_i \quad (i = 1, 2, \ldots, m) \\ x_j &\geq 0 \quad (j = 1, 2, \ldots, n) \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^{m} a_{ij} y_i &\geq c_j \quad (j = 1, 2, \ldots, n) \\ y_i &\geq 0 \quad (i = 1, 2, \ldots, m) \end{aligned}$$

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad &or \quad y_i = 0 \quad (i = 1, 2, \ldots, m) \\ \sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad &or \quad x_j = 0 \quad (j = 1, 2, \ldots, n) \end{aligned}$$

By CS Theorem, x and y are optimal solutions.

Feasibility Theorems (Section 4.7, Pg 52)

Suppose an LP has no feasible solution.

How to we convince someone that the LP is infeasible?

We have seen a very short proof of infeasibility before:

$$\begin{array}{rcrcrcrl} x_1 & - & 2x_2 & + & 2x_3 & = & 2 & \text{— (1)} \\ -x_1 & + & 3x_2 & - & x_3 & = & -3 & \text{— (2)} \\ x_1 & , & x_2 & , & x_3 & \geq & 0 & \text{—(3)} \end{array}$$

$(1) + (2):$ $$x_2 + x_3 = -1$$

(3) implies $$x_2 + x_3 \geq 0$$

Contradiction!

In general:

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_i &= b_i \quad (i = 1, 2, \ldots, m) \\ x_j &\geq 0 \quad (j = 1, 2, \ldots, n) \end{aligned}$$

General idea: Find linear combination of the equations

such that coefficients of x_j's are non-negative

AND right hand side is negative.

Algebraically: Find $y = [y_1, y_2, \ldots, y_m]^T$.

such that $\sum_{i=1}^{m} a_{ij} y_i \geq 0 \ (j = 1, 2, \ldots, n)$

AND $\sum_{i=1}^{m} b_i y_i < 0.$

This idea always work!

Theorem 4.10 (Farkas' Lemma)

$Ax = b,\ x \geq 0$ has no solution

$\iff$ there is a $y = [y_1, y_2, \ldots, y_m]^T$ such that $A^T y \geq 0,\ b^T y < 0$

Proof:

($\Longleftarrow$) [By contradiction]

Suppose $Ax = b,\ x \geq 0,\ A^T y \geq 0$ and $b^T y < 0$.

$$0 > b^T y = (Ax)^T y = x^T (A^T y) \geq 0$$

Contradiction!

($\Longrightarrow$) [Using Duality Theorem]

Suppose $Ax = b,\ x \geq 0$ has no solution.

$$\begin{array}{llcll} \max & 0^T x & & & \\ \text{s.t.} & Ax & = & b & (P) \\ & x & \geq & 0 & \end{array} \qquad \begin{array}{llcll} \min & b^T y & & & \\ \text{s.t.} & A^T y & \geq & 0 & (D) \end{array}$$

- (P) infeasible $\overset{\text{Duality Thm}}{\Longrightarrow}$ (D) not optimal.
- By Fundamental Thm, (D) is unbounded or infeasible.
- (D) has feasible solution "$y = 0$" $\Longrightarrow$ (D) is unbounded.
- So there is feasible solution y of (D) with arbitrarily small value, say < 0;
  i.e., there is y such that $A^T y \geq 0$ and $b^T y < 0$.

Farkas-type theorems for other forms

Theorem 4.11

$Ax \le b$, $x \ge 0$ has no solution

$\iff$ there is a $y = [y_1, y_2, \ldots, y_m]^T$ such that $y \ge 0,\ A^T y \ge 0,\ b^T y < 0$

Proof: ($\Longleftarrow$) [By contradiction]

Suppose $Ax \le b$, $x \ge 0$, $y \ge 0$, $A^T y \ge 0$ and $b^T y < 0$.

$$0 > b^T y \ge (Ax)^T y = x^T (A^T y) \ge 0$$

Contradiction!

($\Longrightarrow$) [Using Duality Theorem]

Suppose $Ax \le b$, $x \ge 0$ has no solution.

$$\begin{array}{llcll} \max & 0^T x & & & \\ \text{s.t.} & Ax & \le & b & (P') \\ & x & \ge & 0 & \end{array} \qquad \begin{array}{llcll} \min & b^T y & & & \\ \text{s.t.} & A^T y & \ge & 0 & (D') \\ & y & \ge & 0 & \end{array}$$

- (P') infeasible $\stackrel{\text{Duality Thm}}{\Longrightarrow}$ (D') not optimal.
- By Fundamental Thm, (D) is unbounded or infeasible.
- (D') has feasible solution $(y = 0) \Longrightarrow (D')$ is unbounded.
- So there is feasible solution y of (D') with arbitrarily small value, say < 0;
  i.e., there is y such that $A^T y \ge 0$, $y \ge 0$ and $b^T y < 0$.

Theorem 4.12

$Ax \leq b$ has no solution

$\iff$ there is a $y = [y_1, y_2, \ldots, y_m]^T$ such that $y \geq 0$, $A^T y = 0$, $b^T y < 0$

Proof: Exercise

We will learn how to detect an infeasible LP problem, and how to compute the y guaranteed by Farkas's Lemma in Chapter 7.

For now, pay no attention on how to obtain the solutions to Question 10 of "Section 4.8: Exercises".

Summary: Chapter 4

- Definition of dual LP problems (see table on pg 41).
- Weak duality and its corollaries (pg 38).
- Duality Theorem (Thm 4.6, pg 42).
- Duality Thm + Fundamental Thm $\Longrightarrow$ 4 possible scenarios for primal-dual pairs of LP problems.
- Complementary Slackness condition and CS Theorem (pg 47).
- Economic Interpretation of Duality (more to come in Chapter 6).
- Farkas'-type Theorems (more to come in Chapter 7).