CO350 Linear Programming Chapter 5: Basic Solutions

30th May 2005

Recap

Last week, we learn

- Definition of a basis B of matrix A.
- Definition of basic solution x^* of $Ax = b$ determined by basis B.
- x^* is a basic solution if and only if $\{A_j : x_j^* \neq 0\}$ is linearly independent.
- Definition of basic feasible solution of $\{Ax = b,\ x \geq 0\}$.
- Definition of convex sets.
- Definition of extreme points of convex sets.

Relating bfs and extreme points

Theorem 5.3 (Pg 63) Let A be m by n with rank m. Let F be the set $\{x : Ax = b,\ x \geq 0\}$.

$$x^* \text{ is a bfs of } \{Ax = b,\ x \geq 0\}$$

$$\iff \quad x^* \text{ is an extreme point of } F.$$

Proof: "$\Longrightarrow$" [Contradiction]

Suppose x^* is a bfs.
x^* is a feasible solution $\implies$ $x^* \in F$.
x^* is basic solution $\implies$ x^* is determined by some basis B.

Suppose x^* is not an extreme point of F.
$x^* \in F \implies x^*$ lie strictly between two vectors in F.
I.e., there are $x^1, x^2 \in F$ with $x^1 \neq x^2$ and $0 < \lambda < 1$ such that $x^* = \lambda x^1 + (1-\lambda)x^2$.

For each $j \notin B$, $x^*_j = 0$.
So
$$0 = x^*_j = \lambda x^1_j + (1-\lambda)x^2_j$$

$x^1, x^2 \geq 0$ and $0 < \lambda < 1 \implies \lambda x^1_j$ and $(1-\lambda)x^2_j \geq 0$.
So $\lambda x^1_j = (1-\lambda)x^2_j = 0$. Hence $x^1_j = x^2_j = 0$.

Now $x^1_j = x^2_j = 0$ for all $j \notin B$, and also $Ax^1 = Ax^2 = b$. This means that x^1 and x^2 are both basic solutions of $Ax = b$ determined by B.

This contradicts $x^1 \neq x^2$.

Theorem 5.3 (Pg 63) Let A be m by n with rank m. Let F be the set $\{x : Ax = b,\ x \geq 0\}$.

$$x^* \text{ is a bfs of } \{Ax = b,\ x \geq 0\}$$

$$\iff \quad x^* \text{ is an extreme point of } F.$$

Proof: "$\Longleftarrow$" [Contradiction]

Suppose x^* is an extreme point of F.
$x^* \in F \implies Ax^* = b$ and $x^* \geq 0$.

Suppose x^* is not a basic feasible solution.
x^* feasible $\implies$ x^* is not basic.
So $\{A_j : x_j^* \neq 0\}$ is a linearly dependent set (Thm 5.1).

Let $J = \{j : x_j^* \neq 0\}$.
Linearly dependent: there exists d_j $(j \in J)$ such that not all d_j's are zeros and

$$\sum_{j \in J} d_j A_j = 0$$

Fill the remaining d_j's with zeros; i.e. $d_j = 0$ for $j \notin J$.
We now have

$$\sum_{j=1}^{n} d_j A_j = 0$$

i.e., $Ad = 0$.

Theorem 5.3 (Pg 63) Let A be m by n with rank m. Let F be the set $\{x : Ax = b,\ x \geq 0\}$.

$$x^* \text{ is a bfs of } \{Ax = b,\ x \geq 0\}$$
$$\iff \quad x^* \text{ is an extreme point of } F.$$

Proof: "$\Longleftarrow$" (cont'd)

So far we assumed x^* is an extreme point but not basic and obtained vector $d \in \mathbf{R}^n$ such that

$$Ad = 0, \text{ and } d_j = 0 \text{ whenever } x_j^* = 0.$$

Let $x^1 = x^* - \varepsilon d$ and $x^2 = x^* + \varepsilon d$, where $\varepsilon > 0$.
Notice that $x^* = \frac{1}{2}x^1 + \frac{1}{2}x^2$.

For j such that $x_j^* = 0$, we have $d_j = 0$. So $x_j^1 = x_j^2 = 0$.

For j such that $x_j^* > 0$, we have $x_j^1 > 0$ and $x_j^2 > 0$ for ε small enough.

Conclusion: $x^1, x^2 \geq 0$ for $\varepsilon > 0$ small enough.

Also, $\quad Ax^1 = Ax^* - \varepsilon Ad = Ax^* = b$

and $\quad Ax^2 = Ax^* + \varepsilon Ad = Ax^* = b$.

So x^1 and x^2 are both feasible; i.e., $x^1, x^2 \in F$.

Now $\quad x^1 \neq x^2$ (because $x^2 - x^1 = 2\varepsilon d \neq 0$)

and $\quad x^* = \frac{1}{2}x^1 + \frac{1}{2}x^2$

contradicts x^* is an extreme point of F.

Theorem 5.3 (Pg 63) Let A be m by n with rank m. Let F be the set $\{x : Ax = b,\ x \geq 0\}$.

$$\begin{aligned} & x^* \text{ is a bfs of } \{Ax = b,\ x \geq 0\} \\ \iff \quad & x^* \text{ is an extreme point of } F. \end{aligned}$$

A few remarks: (Not in notes)

- The theorem says that bfs and extreme points are algebraic and geometric counterparts.
- The assumption **"A has rank m"** is necessary!
  Without it, we cannot even define a basis of A.
  In this case, the theorem fails because F may still have extreme points but there are no bfs.
- In the proof of the second half, we see that
  if x^* is feasible but not basic, then
  there exists vector $d \in \mathbf{R}^n$ such that
  $$d \neq 0,\ d_j = 0 \text{ whenever } x_j^* = 0, \text{ and } Ad = 0,$$
  and there exists $\varepsilon > 0$ such that
  $$\text{both } x^1 = x^* + \varepsilon d \text{ and } x^2 = x^* - \varepsilon d \text{ are feasible.}$$

Basic feasible solutions for SIF

For the Orange Factory Problem, we used the SEF to discuss basic solutions, but plotted them on a picture that represents the problem in SIF.

(Defn) **Basic feasible solution for SIF**
x^* is a bfs of $\{Ax \le b, x \ge 0\}$ if

$$\begin{bmatrix} x^* \\ s^* \end{bmatrix} = \begin{bmatrix} x^* \\ b - Ax^* \end{bmatrix}$$

is a bfs of $\{Ax + s = b,\ x, s \ge 0\}$.

Theorem 5.4 (Pg 65) Let $F = \{x : Ax \le b,\ x \ge 0\}$.

x^* is a bfs of $\{Ax \le b,\ x \ge 0\}$

$\iff$ x^* is an extreme point of F.

Proof: Exercise

Hint: Relate extreme points of F and

$$\left\{ \begin{bmatrix} x \\ s \end{bmatrix} : Ax + s = b,\ x, s \ge 0 \right\}.$$

This is a very good exercise for you.

Come see me if you need more help/hint for this exercise.

Example

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 0 & -2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find all extreme points of $\{x : Ax \leq b,\ x \geq 0\}$.

By Theorem 5.4, this is equivalent to finding all bfs of

$$\{Ax \leq b,\ x \geq 0\}$$

By definition, this is equivalent to finding all basic feasible solutions of

$$\begin{array}{rcrcrcrcrcl} -x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & & & = & 1 \\ x_1 & & & - & x_3 & & & + & x_5 & = & 1 \\ x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0 \end{array}$$

Let $A' = \begin{bmatrix} -1 & 2 & 2 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 \end{bmatrix}$

Procedure: Go through each possible basis B of A' and decide if the basic solution determined by B is feasible.

$$A' = \begin{bmatrix} -1 & 2 & 2 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Possible candidates for basis: $\{1,2\},\{1,3\},\{1,4\},\{1,5\},$
$\{2,3\},\{2,4\},\{2,5\},$
$\{3,4\},\{3,5\},$
$\{4,5\}$

- $B = \{1,2\}$ is a basis since $A_B = \left[\begin{smallmatrix} -1 & 2 \\ 1 & 0 \end{smallmatrix}\right]$ is nonsingular.
  It determines basic solution $[1,1,0,0,0]^T$ which is feasible.

- $B = \{1,3\}$ is a not basis since $A_B = \left[\begin{smallmatrix} -1 & 2 \\ 1 & -2 \end{smallmatrix}\right]$ is singular.

- $B = \{1,4\}$ is a basis since $A_B = \left[\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is nonsingular.
  It determines basic solution $[1,0,0,2,0]^T$ which is feasible.

- $B = \{1,5\}$ is a basis since $A_B = \left[\begin{smallmatrix} -1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ is nonsingular.
  It determines basic solution $[-1,0,0,0,2]^T$ which is not feasible.

- $B = \{2,3\}$ is a basis since $A_B = \left[\begin{smallmatrix} 2 & 2 \\ 0 & -2 \end{smallmatrix}\right]$ is nonsingular.
  It determines basic solution $[0,1,-\frac{1}{2},0,0]^T$ which is not feasible.

- $B = \{2,4\}$ is a basis since $A_B = \left[\begin{smallmatrix} 2 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ is singular.

$$A' = \begin{bmatrix} -1 & 2 & 2 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Possible candidates for basis: $\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\},$
$\{2,3\}, \{2,4\}, \{2,5\},$
$\{3,4\}, \{3,5\},$
$\{4,5\}$

- $B = \{2,5\}$ is a basis since $A_B = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is nonsingular.
It determines basic solution $[0, \frac{1}{2}, 0, 0, 1]^T$ which is feasible.

- $B = \{3,4\}$ is a basis since $A_B = \left[\begin{smallmatrix} 2 & 1 \\ -2 & 0 \end{smallmatrix}\right]$ is nonsingular.
It determines basic solution $[0, 0, -\frac{1}{2}, 2, 0]^T$ which is not feasible.

- $B = \{3,5\}$ is a basis since $A_B = \left[\begin{smallmatrix} 2 & 0 \\ -2 & 1 \end{smallmatrix}\right]$ is nonsingular.
It determines basic solution $[0, 0, \frac{1}{2}, 0, 2]^T$ which is feasible.

- $B = \{4,5\}$ is a basis since $A_B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is nonsingular.
It determines basic solution $[0, 0, 0, 1, 1]^T$ which is feasible.

In conclusion, the extreme points are

$$[1,1,0]^T, [1,0,0]^T, [0, \tfrac{1}{2}, 0]^T, [0, 0, \tfrac{1}{2}]^T, [0,0,0]^T$$