2. Eigenvalues and Eigenvectors Chap. 5 of Lay Section 5.1 to 5.6

(a) Definition of eigenvector and eigenvalue (5.1)

i) for an $n \times n$ matrix

ii) for a linear transformation (an operator)
Examples:
A $= I_n$
L = projection in $\mathbb{R}^2$ onto line $x_1 = x_2$
$A^{`}$ = orthogonal matrix, e.g., $A^{`} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix} = \begin{bmatrix} cos30^\circ & - & sin30^\circ \\ sin30^\circ & & cos30^\circ \end{bmatrix}$

$L^{`}$ = rotation by φ in $\mathbb{R}^6$ (Example 3, p. 84).

For an orthogonal matrix, the only eigenvalues are ± 1.

See examples of Linear transformations in Lay, Table 1, 2, pages 85, 86.

(b) Calculating Eigenvalues (5.1, 5.2)

Prop: A scalar λ is an eigenvalue of A iff $\det(A = \lambda I) = 0$ (p. 313)
$c_A(\lambda) \stackrel{def}{=} det(A - \lambda I)$ = characteristic polynomial.
In each of the following matrices find the characteristic polynomial and its roots:

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

Def.: $n \times n$ matrix B is *similar* to matrix A if there exists an invertible matrix P such that $B = P^{-1}AP$.

Prove that similar matrices have the same characteristic polynomial.

Q. Suppose λ is an eigenvalue of A; is it an eigenvalue of A^3? What is true?

Theorem 1 (p. 307): The eigenvalues of a triangular matrix are the entries on its main diagonal.

Proposition (p. 306): 0 is an eigenvalue of A iff A has no inverse.

(c) Eigenvectors and Eigenspaces

Find the eigenvalues and eigenvectors of the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Def: The *algebraic multiplicity* $mult_a(\lambda_i)$ of eigenvalue λ_i if A is the multiplicity of λ_i as a root of the characteristic polynimial $c_A(\lambda)$.

The *geometric multiplicity* $\mathrm{mult}_g(\lambda_i)$ of eigenvalue λ_i of A is the dimension of the eigenspace E_{λ_i} corresponding to λ_i.

The *eigenspace* corresponding to λ_i is the subspace E_{λ_i} of $\mathbb{R}^n$ consisting of the zero vector and all eigenvectors corresponding to λ_i.

Multiplicity Theorem (Theorem 7a, p. 324 Lay): For any eigenvalue λ_i of $n \times n$ matrix A $\mathrm{mult}_g(\lambda_i) \leq \mathrm{mult}_a(\lambda_i)$. Prove this if you have time.

(d) Diagonalization (5.3)

Def: An $n \times n$ matrix is *diagonalizable* if there is an invertible matrix P such that $P^{-1}AP = D$, a diagonal matrix

Example: (p. 321)

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \qquad c_A(\lambda) = -(\lambda - 1)(\lambda + 2)^2$$

$$E_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} \quad E_2 = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Thm. 5 (p. 321): An $n \times n$ matrix is diagonalizable iff A has n linearly independent eigenvectors.

Problem: Is $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ similar to $B = \begin{bmatrix} -3 & 10 \\ -3 & 8 \end{bmatrix}$?

Answer: Yes. Each can be diagonalized to $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Other possible examples:

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(e) Diagonalization

Eigenvectors belonging to distinct eigenvalues are linearly independent:

Theorem 2 (p. 307): If $v_1, \ldots, v_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ in $n \times n$ matrix A, then $\{v_1, \ldots, v_4\}$ is linearly independent.

Proof:

Theorem 6 (p. 323). An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.
More generally
Multiplicity Theorem: (Theorem 7a, p. 324). For each eigenvalue λ_v of $n \times n$ matrix A,

$$mult_g(\lambda_v) \leq mult_a(\lambda_v).$$

Diagonalization Theorem. A square matrix A is diagonalizable iff for each eigenvalue λ_v of $n \times n$ matrix A,

$$mult_g(\lambda_v) = mult_a(\lambda_v).$$

(f) Complex eigenvalues

Examples:
$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

L_A = rotation by 90° clockwise.

Theorem: An $n \times n$ matrix over $\mathbb{R}$ or $\mathbb{C}$ has n eigenvalues if multiplicity is counted.

Examples:

$A = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix}$

$c_A(\lambda) = \lambda^2 - 2\lambda + 2$

$\lambda = 1 \pm i$

Study the dynamical system:

$$x(t+1) = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix} x(t), \quad x_0, = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$