

## 3.2 Properties of Determinants

**THEOREM 3** Let  $A$  be a square matrix.

- If a multiple of one row of  $A$  is added to another row of  $A$  to produce a matrix  $B$ , then  $\det A = \det B$ .
- If two rows of  $A$  are interchanged to produce  $B$ , then  $\det B = -\det A$ .
- If one row of  $A$  is multiplied by  $k$  to produce  $B$ , then  $\det B = k \cdot \det A$ .

**EXAMPLE:** Compute 
$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}.$$

*Solution*

$$\begin{aligned} & \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix} = 5 \begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 10 \\ 2 & 7 & 11 \end{vmatrix} = 5 \begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 2 & 7 & 11 \end{vmatrix} \\ & = 5 \begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -5 \begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{vmatrix} = \underline{-5(1)(2)} = \underline{-10}. \end{aligned}$$

Theorem 3(c) indicates that  $\begin{vmatrix} * & * & * \\ -2k & 5k & 4k \\ * & * & * \end{vmatrix} = k \begin{vmatrix} * & * & * \\ -2 & 5 & 4 \\ * & * & * \end{vmatrix}.$

**EXAMPLE:** Compute  $\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix}$

*Solution*

$$\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -8 & -11 \end{vmatrix}$$

$$= 2(-4) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -8 & -11 \end{vmatrix} = 2(-4) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 2(-4)(1)(1)(5) = -40$$

**EXAMPLE:** Compute  $\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix}$  using a combination of row reduction and cofactor expansion.

*Solution*  $\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix} = -2 \begin{vmatrix} 2 & 3 & 1 \\ 4 & 7 & 3 \\ 1 & 2 & 4 \end{vmatrix} = -2 \begin{vmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}$

$$= 2 \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 0 & -6 \end{vmatrix} = -2(1)(-1)(-6) = -12.$$

Suppose  $A$  has been reduced to  $U = \begin{bmatrix} \blacksquare & * & * & \cdots & * \\ 0 & \blacksquare & * & \cdots & * \\ 0 & 0 & \blacksquare & \cdots & * \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$  by

row replacements and row interchanges, then

$$\det A = \begin{cases} (-1)^r \left( \begin{array}{l} \text{product of} \\ \text{pivots in } U \end{array} \right) & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

**THEOREM 4** A square matrix is invertible if and only if  $\det A \neq 0$ .

**THEOREM 5** If  $A$  is an  $n \times n$  matrix, then  $\det A^T = \det A$ .

**Partial proof** ( $2 \times 2$  case)

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \quad \text{and}$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$$

$$\Rightarrow \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

(3 × 3 case)

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$\det \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = a \begin{vmatrix} e & h \\ f & i \end{vmatrix} - b \begin{vmatrix} d & g \\ f & i \end{vmatrix} + c \begin{vmatrix} d & g \\ e & h \end{vmatrix}$$

$$\Rightarrow \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \det \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}.$$

*Implications of Theorem 5?*

Theorem 3 still holds if the word *row* is replaced with column.

## THEOREM 6 (Multiplicative Property)

For  $n \times n$  matrices  $A$  and  $B$ ,  $\det(AB) = (\det A)(\det B)$ .

**EXAMPLE:** Compute  $\det A^3$  if  $\det A = 5$ .

*Solution:*  $\det A^3 = \det(AAA) = (\det A)(\det A)(\det A)$   
 $= \underline{5(5)(5)} = \underline{125}$ .

**EXAMPLE:** For  $n \times n$  matrices  $A$  and  $B$ , show that  $A$  is singular if  $\det B \neq 0$  and  $\det AB = 0$ .

*Solution:* Since  $(\det A)(\det B) = \det AB = 0$

and

$$\det B \neq 0,$$

then  $\det A = 0$ . Therefore  $A$  is singular.