CO367 Nonlinear Optimization
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January 24, 2011

NONLINEAR OPTIMIZATION — Assignment 1

Score composition:

(2.1)	(2.2)	(2.6)	(2.9)	(2.12)	(2.28)	(2.36)	Total	Bonus
3	4	5	5	7	3	3	30	9

Proof by picture

It is necessary to give an algebraic proof; pictorial proof is not accepted.

In general, even if you are providing counter-example to disprove something, it is best to give an algebraic proof (even though this time marks are given to valid counterexample disproving the statement in Q.2.12(e)).

When you are trying to explain your idea with the aid of a diagram, make sure you label the mathematical objects (such as vectors) you are talking about. And don't just draw a picture without any explanation.

(2.2) Many students forgot to mention anything about the trivial case, i.e. when the intersection of 2 sets is a single point or an empty set. Though no marks were deducted, you should be aware of that.

(2.6)

1. To prove that "$\{x \in \mathbb{R}^n : a^T x \leq b\} \subseteq \{x \in \mathbb{R}^n : \tilde{a}^T x \leq \tilde{b}\}$ (where $a, \tilde{a} \in \mathbb{R}^n$ are non-zero and $b, \tilde{b} \in \mathbb{R}$) if and only if there exists $\lambda > 0$ such that $\tilde{a} = \lambda$ and $\tilde{b} \geq \lambda b$", one can argue as follows:

   (a) $\tilde{a}$ must be a scalar multiple of a, that is, there must exists some $\lambda \in \mathbb{R}$ such that $\tilde{a} = \lambda a$.

   (b) $\lambda \neq 0$ (otherwise $\tilde{a} \neq 0$, contradicting the choice of $\tilde{a}$).

   (c) λ cannot be negative.

   (d) Letting $x = \frac{b}{\|a\|^2} a$, since $a^T x = b$, we must have

   $$\lambda b = \lambda a^T x = \tilde{a}^T x \leq \tilde{b}.$$

2. Most people forgot to mention that λ cannot be zero (which is necessary to provide a complete argument).

3. For item 1, you may start by noting that if $\tilde{a} \notin \text{span}(a)$, there exists a non-zero $d \in \mathbb{R}^n$ such that $a^T d = 0$ but $\tilde{a}^T d \neq 0$, since $\tilde{a} \notin \text{span}(a)$ holds if and only if

$$d := \tilde{a} - \frac{\langle \tilde{a}, a \rangle}{\langle a, a \rangle} a \neq 0,$$

and $a^T d = 0$ but $\tilde{a}^T d = \|d\|$.

4. Some people attempted to prove item (i) via the following route:

   (a) a and $\tilde{a}$ are not parallel.

   (b) Let x, x' satisfy $a^T x = b$, $a^T x' = b$.

   (c) Then $a^T(x - x') = 0$, meaning that a and $x - x'$ are perpendicular.

   (d) But a and $\tilde{a}$ are not parallel.

   (e) So $\tilde{a}^T(x - x') \neq 0$.

   The problem is that, in $\mathbb{R}^n$ where $n \geq 3$, given any two non-zero vectors a and $\tilde{a}$, it is *always* possible to pick a non-zero vector $\hat{d}$ such that $a^T \hat{d} = 0$ and $\tilde{a}^T \hat{d} = 0$. In the above procedure, the choice of x' is not specific enough to avoid ending up with $x - x'$ lying in $\{a, \tilde{a}\}^\perp$. Things can get a bit tricky in higher dimension!

(2.9)

1. Given a polyhedron $P = \{x : Ax \leq b\}$, $x_0 \in \int P$ if and only if $Ax_0 < b$.
   This is important to have strict inequality in the question; this ensures that x_i and x_0 would be distinct points.

2. for (c), you can try to construct a counterexample by dividing $\mathbb{R}^2$ into 4 parts by 2 straight lines.

(2.12) Make sure you understand all the notation and what the set is before you try to prove or disprove its convexity.
(2.28) Principal minors can be used to test the positive definiteness of a square matrix.

Submitted on January 24, 2011.